

# Learning, Confidence, and Business Cycles\*

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## Abstract

In this paper we study the amplification feedback between uncertainty and economic activity. We construct a heterogeneous-firm real business cycle model with agents who are averse to ambiguity. Firms face Knightian uncertainty about their own productivity and need to learn it through production. Recessions, caused by either fundamental supply or demand shocks, are periods where the lower production scale implies higher uncertainty in the form of a larger posterior variance. The endogenous increase in uncertainty makes agents less confident and further reduces economic activity. This feedback mechanism generates (i) strong internal propagation with amplified and hump-shaped dynamics, (ii) countercyclical labor wedge and ex-post excess return on capital, and (iii) positive co-movement of aggregate variables in response to demand shocks. We study the feedback effects of time-varying endogenous uncertainty in standard business cycle models using linear methods. Linearity facilitates aggregation in the heterogeneous firm model, where we can additionally analyze the impact of experimentation and firm-level dispersion shocks. We illustrate the main qualitative implications in a stylized model and use a quantitative version to evaluate their magnitudes.

## 1 Introduction

Is firms' confidence about their business conditions important for understanding aggregate fluctuations? How much does this confidence vary over the business cycle? In turn, how do the variations in confidence feed back into the economic activity and affect the propagation of fundamental shocks? To answer these questions, we construct a heterogeneous-firm real business cycle (RBC) model with agents who are averse to ambiguity. Firms face Knightian uncertainty about their own productivity and need to learn it through production. The learning process generates countercyclical endogenous uncertainty that feeds from, and back into, economic activity. We find that our model offers substantial improvements over

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standard RBC models along three dimensions. It generates: (i) strong internal propagation with amplified and hump-shaped dynamics, (ii) countercyclical labor wedge and ex-post excess return on capital, and (iii) positive co-movement of aggregate variables in response to demand shocks. Our model also generates a government spending multiplier that is larger than what standard models would predict.

The corresponding implication of our model is that our mechanism, based on the feedback between economic activity and uncertainty, can replicate some of the major business cycle patterns without requiring: (i) additional frictions to generate persistence and hump-shape responses; (ii) labor wedge shocks, traditionally used as the explanation for the measured countercyclical "wedge" between the marginal rate of substitution of consumption for labor and the marginal product of labor; (iii) aggregate uncertainty shocks, traditionally used to explain the countercyclical excess return of uncertain assets, such as capital, over the risk free rate<sup>1</sup>; (iv) additional rigidities usually required to break the Barro and King (1984) critique and make other types of shocks, besides productivity or intratemporal preference shocks, generate positive co-movements of macro aggregates.

We embed the idea of asymmetric learning (Veldkamp (2005) and van Nieuwerburgh and Veldkamp (2006)) into a heterogeneous-firm setting by subjecting firms to idiosyncratic shocks that they cannot directly observe. Our modeling choice of introducing imperfect information into an idiosyncratic process is motivated by the empirical fact that firm-level volatilities are much larger than fluctuations at the macro level. This fact suggests that uncertainty about idiosyncratic fundamentals may be more important than uncertainty about aggregate shocks. We introduce two unobservable shocks into a firm's production. The first shock is a standard technology shock that affects the marginal return. This shock is persistent. We will refer to it as the 'productivity shock'. The second is a transitory shock that does not scale up with the level of inputs. While what matters for optimal investing and hiring decisions are the realization of the productivity shock, the path of firms' output and inputs is not perfectly revealing about its productivity because it is confounded by the transitory disturbance.<sup>2</sup> Ambiguity averse agents cope with this uncertainty by estimating the underlying productivity process using a standard Kalman filter and considering a set of probability distributions around the estimates.

In the model, the level of inputs endogenously determines the informativeness of output about the idiosyncratic productivity level. Intuitively, when a firm allocates less resources into production, its estimate about its persistent productivity is imprecise because the level

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<sup>1</sup>See Cochrane (2011) for a review of the evidence.

<sup>2</sup>We show that our model is observationally equivalent at the aggregate level to a model where firms learn about their quality or demand of their goods through noisy signals.

of output is largely determined by the realization of the transitory shock. Conversely, its estimate becomes more accurate when it uses more resources because output mostly reflects the realization of productivity. This results in a procyclical signal-to-noise ratio at the firm level. It follows that recessions are periods of high cross-sectional mean of firm-level uncertainty because firms *on average* invest and hire less.

Ambiguity averse agents lack the confidence to assign probabilities to every relevant events. They act *as if* they evaluate plans according to a worst case scenario drawn from a set of multiple beliefs. A wider set of beliefs corresponds to a loss of confidence. In our model, agents have a set of beliefs about the idiosyncratic productivity shock. The belief set is parameterized by an interval of means centered around zero. An increase in uncertainty leads to a wider interval because higher posterior variance makes it harder to distinguish across processes differing in their means. This loss in confidence makes the “worst case” mean worse and thus an agent acts *as if* he received bad news about the future. Importantly, this loss of confidence does not vanish in the aggregate because agents treat as if mean of each firm is on average lower.

We first illustrate the main qualitative predictions of our theory using a stylized example, in which labor is chosen in advance and there is a negative relationship between current uncertainty and labor choices. We then quantitatively evaluate the role of endogenous uncertainty using a calibrated version of the baseline model. In addition to a standard aggregate technology shock, we consider a government spending shock and a firm-level dispersion shock as in Bloom (2009). While our model allows for time-varying uncertainty, we can use standard log-linear methods to solve the model. In turn, log-linear decision rules facilitate aggregation and allow us to study confidence about the idiosyncratic TFP process.

First, we find that endogenous uncertainty is a powerful propagation mechanism. A positive shock that raises economic activity increases the level of confidence, which in turn further affects economic activity, leading to an amplified and hump-shaped impulse response. Second, consistent with the data, our model generates countercyclical labor wedge and ex-post excess return. During recessions, ambiguity increases and thus leads to an unusually low equilibrium labor supply. The increase in ambiguity also makes capital less attractive to hold and thus investors holding an ambiguous asset are compensated by the higher excess return. Third, the model generates positive co-movements in response to demand shocks. Consider, for example, a positive shock to government spending. In standard RBC models, consumption sharply declines due to the income effect and the government spending multiplier is small. In our model, an increase in hours due to a positive government spending shock raises the level of confidence, which feeds back and raises the level of the economic activity. Because of this amplification effect, government spending multiplier is larger and

the negative response of consumption is mitigated.

We point out two additional features of the model. First, we can easily study the implications of experimentation using linear methods. This is in contrast to experimentation in Bayesian setting, which requires non-linear tools. We show that experimentation has a first-order effect in our setup. For example, we find that equilibrium solved under passive learning substantially overstates the impact of dispersion shocks. Second, in contrast to other business cycle theories with news and time-varying uncertainty, our model does not have to rely on real or nominal rigidities to generate positive co-movement.<sup>3</sup> This is because the level of input (such as labor supply) is chosen before the realization of idiosyncratic productivity. This timing naturally arises from imperfect information about the underlying productivity process.

## 2 A stylized business cycle example

To illustrate the role of endogenous uncertainty in business cycles, we consider a stylized model. We focus on the qualitative features implied by the feedback between uncertainty and economic activity. In this simple model we make two key assumptions: (1) labor is chosen before productivity is known and (2) there is a negative relationship between current uncertainty and past labor choice. Both of these features arise endogenously in a model with imperfect information about persistent productivity, which we develop later in the paper, where the inputs for current production are chosen based on past information. There imperfect information about current productivity makes labor an intertemporal decision for which uncertainty matters, thus implying the first assumption. In turn, there we formalize the feedback from activity to uncertainty by considering a filtering problem where a larger previous scale of production implies a stronger current signal-to-noise ratio about the time-varying, hidden state of productivity. Instead, in this stylized model, we simply postulate the reduced form feedback effect as the second assumption. We use this setup to develop intuition on the novel economic mechanisms implied by this uncertainty-activity feedback. For that purpose we also abstract here from additional internal propagation such as capital accumulation.

A representative agent has the following per-period utility function

$$U(C_t, H_t) = \frac{C_t^{1-\gamma}}{1-\gamma} - \beta \frac{H_t^{1+\eta}}{1+\eta}.$$

where  $C_t$  is consumption,  $H_t$  is the amount of hours worked,  $\gamma$  is the coefficient of relative

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<sup>3</sup>See, however, Angeletos et al. (2014) for an important exception.

risk aversion, and  $\eta$  is the inverse of the Frisch labor elasticity. We simplify algebra below by multiplying the disutility of labor by the discount factor  $\beta$ .

Output is produced according to

$$Y_t = Z_t H_{t-1}. \quad (2.1)$$

The  $t - 1$  subscript on hours reflects the assumption that labor input is chosen before the realization of productivity  $Z_t$ , which is random. The resource constraint is given by

$$C_t + G_t = Y_t, \quad (2.2)$$

where government spending,  $G_t$ , follows an AR(1) process

$$\ln G_{t+1} = (1 - \rho) \ln \bar{G} + \rho \ln G_t + u_{g,t+1}, \quad (2.3)$$

where  $u_{g,t+1}$  is distributed *i.i.d.*  $N(0, \sigma_g^2)$ . We use upper bars to denote the steady states. Hence,  $\bar{G}$  is the steady-state level of government spending.

The productivity process takes the form

$$\ln Z_{t+1} = \mu_t^* + u_{z,t+1}, \quad (2.4)$$

where  $u$  is an iid sequence of shocks, normally distributed with mean zero and variance  $\sigma_u^2$ . The sequence  $\mu$  is deterministic and unknown to agents – its properties are discussed further below. For simplicity, we assume that the realization of the current productivity level does not affect future productivity. We relax this assumption in the quantitative model we introduce later.

Agents perceive the unknown component  $\mu_t$  to be ambiguous. We parametrize their one-step-ahead set of beliefs at date  $t$  by a set of means  $\mu_t \in [-a_t, a_t]$ . Here  $a_t$  captures agent's lack of confidence in his probability assessment of productivity  $Z_{t+1}$ . We allow confidence itself to change over time, and in particular, we assume that  $a_t$  is negatively related to past labor supply:

$$a_t = \bar{a} - \zeta \hat{H}_{t-1}, \quad \zeta > 0, \quad (2.5)$$

where we use hats to denote log-deviations from the steady states (and hence  $\hat{H}_{t-1} = \ln H_{t-1} - \ln \bar{H}$ ). If the agent works less last period, he is less confident in his probability assessments. In the quantitative model, we derive the relationship between  $a_t$  and  $H_{t-1}$  in equation (2.5) as an endogenous outcome of a learning process. Intuitively, in a model with random additive shocks to the production in (2.1), lower  $H_{t-1}$  increases the posterior

variance around the estimate of the persistent, hidden state component of productivity  $Z_t$ .

The representative household has recursive multiple priors utility. Collect the exogenous state variables  $Z_t$  and  $G_t$  in a vector  $s_t \in S$ . A household consumption plan  $C$  gives, for every history  $s^t$ , the consumption of the final good  $C_t(s^t)$  and the amount of hours worked  $H_t(s^t)$ . For a given consumption plan  $C$ , utility is defined recursively by

$$U_t(C; s^t) = U(C_t, H_t) + \beta \min_{\mu_t \in [-a_t, a_t]} E^{\mu_t} [U_{t+1}(C; s^t, s_{t+1})], \quad (2.6)$$

where  $\beta$  is the subjective discount factor and  $[-a_t, a_t]$  is the set of conditional probabilities about the ambiguous component of the next period's state, namely here the productivity  $Z_{t+1}$ . We restricted attention only to ambiguity about productivity and assumed that the agent has full confidence in the conditional probability distribution for  $G_{t+1}$  as given by the process in (2.3). The recursive formulation ensures that preferences are dynamically consistent. Details and axiomatic foundations are in Epstein and Schneider (2003b). If  $a_t = 0$ , we obtain standard separable log utility with those conditional beliefs. If  $a_t > 0$ , then agents are not willing to integrate over the beliefs indexed by  $\mu_t$  and narrow down the set to a singleton. In response, households take a cautious approach to decision making and act *as if* the true data generating process is given by the worst-case conditional mean.

We now solve the social planner's problem, for which the Bellman equation is

$$V(H_{-1}, Z, G) = \max_H \left[ U(C, H) + \beta \min_{\mu \in [-a, a]} E^\mu V(H, Z', G') \right],$$

where the constraints are given by the production function (2.1) and resource constraint (2.2). The conditional distribution of  $Z'$  under belief  $\mu$  is given by (2.4), where ambiguity evolves according to the law of motion (2.5). The transition law of the exogenous state  $G$  is given by (2.3).

The worst-case belief can be easily solved for at the equilibrium consumption plan: the worst case expected productivity is low. It follows that the social planner's problem is solved under the worst case belief  $\mu = -a$ . Denoting conditional moments under the worst case belief by stars and combining the first order condition for labor with the envelope condition, we obtain

$$H^\eta = E^* [C'^{-\gamma} Z']. \quad (2.7)$$

The optimality conditions equates the current marginal disutility of working with its expected benefit, formed under the worst-case belief. The latter is given by the marginal product of labor weighted by the marginal utility of consumption. In deriving the optimal hours worked, we assume that the agent does not internalize the effect of hours on the evolution of

confidence, a form of what is usually referred in the learning literature as 'passive learning'. This assumption will be relaxed in the quantitative model we introduce later.

To characterize dynamics we use a log-linear approximation of decision rules around the steady state. As detailed in Ilut and Schneider (2014), a log-linear solution method still maintains the role of time-varying uncertainty, here manifested in the form of Knightian uncertainty, since ambiguity is over conditional means and the worst-case mean is linear in the state variables.

We take logs of the optimality condition with respect to hours in (2.7) and substitute the log-linearized constraints (2.1) and (2.2). The log-linearized decision rule of hours around the steady state relates current hours worked with the worst-case exogenous variables as

$$\hat{H}_t = \varepsilon_Z \hat{a}_t + \varepsilon_G \rho \hat{G}_t.$$

We use the method of undetermined coefficients to find the elasticities  $\varepsilon_Z$  and  $\varepsilon_G$ . They are equal to  $(1 - \gamma\lambda_Y) / (\eta + \gamma\lambda_Y)$  and  $\gamma\lambda_G / (\eta + \gamma\lambda_Y)$ , respectively, with  $\lambda_Y \equiv \bar{Y}/\bar{C}$  and  $\lambda_G \equiv \bar{G}/\bar{C}$ .

The response of optimal hours to news about expected productivity is affected by the intertemporal elasticity of consumption (IES), which here also equals the inverse of coefficient of relative risk aversion. When the IES is large enough, so that  $\gamma^{-1} > \lambda_Y$  and thus  $\varepsilon_Z > 0$ , an increase in expected productivity raises hours. In that case the intertemporal substitution effects dominates the wealth effect that would lower hours through the effect on marginal utility.

Since expected productivity is formed under the worst-case conditional mean, and the latter is a function of past hours as in (2.5), we have

$$\hat{H}_t = \zeta \varepsilon_Z \hat{H}_{t-1} + \varepsilon_G \rho \hat{G}_t \quad (2.8)$$

Substituting the laws of motion for  $\hat{G}_t$  together with rewriting optimal hours in (2.8) for period  $t - 1$ , we have

$$\hat{H}_t = (\zeta \varepsilon_Z + \rho) \hat{H}_{t-1} - \zeta \varepsilon_Z \rho \hat{H}_{t-2} + \varepsilon_G \rho u_{g,t}. \quad (2.9)$$

Equilibrium output and consumption follow immediately as

$$\hat{Y}_t = \hat{Z}_t + \hat{H}_{t-1}, \quad (2.10)$$

$$\hat{C}_t = \lambda_Y \hat{Y}_t - \lambda_G \hat{G}_t. \quad (2.11)$$

The dependence of ambiguity on labor supply (2.5) gives rise to three key properties. First, the endogenous ambiguity could be an important propagation mechanism. When there is no ambiguity ( $\zeta = 0$ ), hours and output simply trace the movement of the exogenous government spending. In contrast, with endogenous ambiguity there is an additional AR(2) term that could potentially generate hump-shaped and persistent dynamics.

Second, the model can generate countercyclical labor wedge and excess returns. To see how the model generates countercyclical labor wedge, note that an increase in ambiguity due to a reduction in labor supply looks like an increase in the labor income tax from the econometrician. To show this, define the labor wedge as the implicit labor tax that equates the marginal rate of substitution of consumption for labor with the marginal product of labor. Because labor is chosen in advance, the econometrician will measure the current tax from

$$\frac{H_{t-1}^\eta}{C_t^{-\gamma}} = (1 - \tau_t) Z_t$$

The key insight is that labor is chosen based on the worst-case probability distribution, so using (2.7)

$$1 - \tau_t = \frac{E_{t-1}^* [C_t^{-\gamma} Z_t]}{C_t^{-\gamma} Z_t}$$

In a model with rational expectations, the labor wedge will not be predictable, and  $E_{t-1}\tau_t = 0$ , where the  $E_t$  is formed under the econometrician's data generating process which uses  $\mu = 0$ . Instead here there is a systematic difference between the worst-case distribution and the average realization under the econometrician's data generating process. In log-linear deviations, the labor wedge is proportional to the time-varying ambiguity, which using (2.5), makes it predictable based on past labor supply as :

$$E_{t-1}\hat{\tau}_t = -\zeta\varepsilon_Z\hat{H}_{t-2}$$

The intuition is that when there is ambiguity ( $\zeta > 0$ ) and the substitution effect is strong enough so that  $\varepsilon_Z > 0$ , labor supply at  $t - 1$  is lower as  $t - 1$  confidence is lower. From the perspective of the econometrician measuring at time  $t$  labor and consumption choices, together with measured productivity, the low labor supply is surprisingly low and can be rationalized as a high labor income tax at  $t - 1$ . In turn, the low time  $t - 1$  confidence is due to the low lagged labor supply, so the econometrician will find a systematic negative relationship between lagged hours and the labor income tax. This countercyclical labor wedge does not arise from separate labor supply shocks but instead is generated by any underlying shock that moves labor supply.

To understand how the model generates countercyclical excess returns, we analyze a



decentralized version of our economy. There, one can show [details to be added] that, in log-linear terms, the excess return on a price of a claim to consumption next period measured by an econometrician is given by

$$E_t \hat{x}_{t+1}^e = -\zeta \hat{H}_{t-1}$$

As in Ilut and Schneider (2014) and Bianchi et al. (2014), the conditional expected excess return depends positively on the amount of ambiguity. In this model this conditional premium arises endogenously from the fluctuations in the economic activity.

Third, output multiplier to government spending may be above one and consumption may increase in response to an increase in government spending. To see this consider again the case of no ambiguity ( $\zeta = 0$ ). From (2.9) and (2.10), the initial impact of a unit-increase in government spending to hours and output are given by  $\rho \varepsilon_{h,G}$  and then monotonically decreases.<sup>4</sup> The government spending multiplier is given by

$$\frac{dY_t}{dG_t} \approx \frac{\lambda_Y \hat{Y}_t}{\lambda_G \hat{G}_t},$$

which, given that  $\rho \varepsilon_{h,G} < \lambda_G / \lambda_Y$ , is less than one. Using (2.11) the output increase cannot offset the increase in government spending so consumption declines.

When instead the level of ambiguity is affected by past labor supply, an increase in hours due to an increase in government spending leads to an increase in confidence, which further raises hours over time. Because of this amplification effect, government spending multiplier could be above one and consumption need not decline.

We illustrate the dynamics of this stylized model in Figure 1. We choose parameters as follows: a ratio of government spending to output of  $g = 0.2$ , based upon we have  $\lambda_Y = 1/(1-g)$  and  $\lambda_G = g/(1-g)$ ;  $\gamma = 0.5$  so the IES=2 and we pick  $\eta = 0.5$  so the Frisch elasticity of labor supply=2; a persistence of the government spending shock of  $\rho = 0.95$ ; and for the ambiguity model a feedback effect  $\zeta = 1.5$ .

Figure 1 plots the response of endogenous variables to a 1 percent increase in government spending and compares the economy with ambiguity (black solid line) to that with rational expectations (RE, blue dashed line), in which  $\zeta = 0$ . In the RE model, output and hours simply track the AR(1) evolution of exogenous government spending and consumption decreases. The expected labor wedge is zero. When ambiguity is present, output and hours show more variability and a hump-shaped response. This comes from the AR(2) solution for hours worked, as shown by formula (2.9). With ambiguity the increase in hours is

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<sup>4</sup>As can be seen from (2.10), the impact of output arrives with a one-period lag.

large enough so that consumption actually increases after several periods. The government spending multiplier, which is always lower than one under RE, is larger with ambiguity, to the point that it can be above one and have a net stimulative effect on output. At the same time, the measured labor wedge is countercyclical.

To summarize, this stylized model endogenously generates two types of countercyclical wedges: the labor and the premium wedge. These wedges have been taken as exogenous sources of fluctuations in standard quantitative business cycle models. In our model they are simply manifestations of the changing confidence brought upon by the response of the economy to the fundamental shock. In turn, the endogenous confidence amplifies and prolongs the response of endogenous variables, giving rise to hump-shape responses. It thus does not require additional internal sources of persistence, such as habit formation in consumption.

In the quantitative model, we extend the simple model described above in three ways. First, we derive the relationship between  $a_t$  and  $H_{t-1}$  in equation (2.5) as an endogenous outcome of a learning process. Here we build on van Nieuwerburgh and Veldkamp (2006), where they assume that production is subject to unobserved persistent marginal productivity shocks and to unobserved i.i.d additive productivity shocks. In that setup, a larger scale of production results in a larger signal-to-noise ratio about the hidden state of TFP and thus to lower uncertainty about the next-period productivity. We then use the methods in Bianchi et al. (2014) to map the time-varying uncertainty into time-varying confidence using linear methods. Second, we introduce capital accumulation and variable utilization. This will turn the stylized model into a relatively standard RBC model. Third, we extend the model to a heterogeneous firm setting and study ambiguity about the idiosyncratic TFP process. The aggregation of this economy is facilitated by our use of linearization methods. This extension not only allows us to consider large changes in confidence, but also to analyze the impact of experimentation and firm-level dispersion shocks, as those emphasized for example by Bloom (2009).

### 3 The model

We now introduce our quantitative model, which is a real business cycle model augmented with two key features: Agents are ambiguity averse and face Knightian uncertainty about the firm-level TFP processes. After presenting the environment, we discuss in detail the information friction that gives rise to equilibrium fluctuations in confidence.

### 3.1 Environment

#### Households

As in the stylized business cycle model presented earlier, the representative household has recursive multiple priors utility:

$$U_t(C; s^t) = \ln C_t - \varphi \frac{H_t^{1+\eta}}{1+\eta} + \beta \min_{p \in \mathcal{P}_t(s^t)} E^p[U_{t+1}(C; s^t, s_{t+1})], \quad (3.1)$$

where  $\varphi$  is a scaling parameter that determines hours worked, and  $\eta$  is the inverse of Frisch labor supply elasticity.  $\mathcal{P}_t(s^t)$  is a set of conditional probabilities about next period's state  $s_{t+1} \in S_{t+1}$ . They maximize utility subject to the budget constraint given by

$$C_t + B_t + \int P_{l,t}^e \theta_{l,t} dl \leq W_t H_t + R_{t-1} B_{t-1} + \int (D_{l,t} + P_{l,t}^e) \theta_{l,t-1} dl + T_t,$$

where  $B_t$  is the one-period risk-free bond,  $W_t$  is the real wage,  $R_t$  is the risk-free interest rate, and  $T_t$  is a transfer.  $D_{l,t}$  and  $P_{l,t}^e$  are the dividend payout and the price of a unit of share  $\theta_{l,t}$  of firm  $l$ , respectively.

#### Firms

There is a continuum of firms, indexed by  $l \in [0, 1]$ , which act in a perfectly competitive manner. They use capital  $K_{l,t-1}$ , which is utilized at rate  $U_{l,t}$ , and hire labor  $H_{l,t}$  to produce goods  $Y_{l,t}$  according to the production function

$$Y_{l,t} = A_t \{z_{l,t} (U_{l,t} K_{l,t-1})^\alpha (\gamma^t H_{l,t})^{1-\alpha} + \gamma^t \nu_{l,t}\}, \quad \nu_{l,t} \sim N(0, \sigma_{\nu,t}^2), \quad (3.2)$$

where  $\gamma$  is the growth rate of labor augmenting technical progress. The scale of the idiosyncratic i.i.d. shock  $\nu_{l,t}$  grows at rate  $\gamma$ , which ensures that the shock does not vanish along the balanced growth path.  $z_{l,t}$  is an idiosyncratic technology shock that follows

$$z_{l,t} = (1 - \rho_z) \bar{z} + \rho_z z_{l,t-1} + \epsilon_{z,l,t}, \quad \epsilon_{z,l,t} \sim N(0, \sigma_{z,t}^2), \quad (3.3)$$

and  $A_t$  is an aggregate technology shock that follows

$$\ln A_t = \rho_A \ln A_{t-1} + \epsilon_{A,t}, \quad \epsilon_{A,t} \sim N(0, \sigma_A^2).$$

We assume that the idiosyncratic shock  $z_{l,t}$  follows a normal process instead of a log-normal process. This technical assumption will be useful in solving the learning problem because

it makes the information friction linear. We also consider dispersion shocks that affect the volatility of idiosyncratic shocks:

$$\ln \sigma_t = \rho_\sigma \ln \sigma_{t-1} + \epsilon_{\sigma,t}, \quad \epsilon_{\sigma,t} \sim N(0, \sigma_\sigma^2),$$

and

$$\ln \sigma_{z,t} = \ln \sigma_t + \ln \bar{\sigma}_z,$$

$$\ln \sigma_{\nu,t} = \ln \sigma_t + \ln \bar{\sigma}_\nu.$$

Firms cannot directly observe the realizations of idiosyncratic shocks  $z_{l,t}$  and  $\nu_{l,t}$ . This informational assumption leads to a non-invertibility problem: Firms cannot tell whether an unexpectedly high realization of output is due to an increase in individual technology or a favorable transitory disturbance. Instead, they need to form the estimates using all other available information, including the path of output and inputs. In contrast, they perfectly observe the aggregate shocks  $A_t$  and  $\sigma_t$ .

Different from other papers in the asymmetric learning literature (Veldkamp (2005), van Nieuwerburgh and Veldkamp (2006), Fajgelbaum et al. (2013), Ordoñez (2013), and Saijo (2014)), we introduce imperfect information into an idiosyncratic process. Our formulation has two advantages. First, firm-level volatilities are empirically much larger than fluctuations at the aggregate. This fact suggests that uncertainty about idiosyncratic fundamentals may be larger and hence more important than uncertainty about the macro-level process. Second, because firms learn about their individual-specific characteristics, the impact of imperfect information is unlikely to be substantially affected by an introduction of market for information or a release of official statistics.

Firms choose  $\{U_{l,t}, K_{l,t}, H_{l,t}, I_{l,t}\}$  to maximize shareholder value

$$E_0^* \sum_{t=0}^{\infty} M_0^t D_{l,t},$$

where random variables  $M_0^t$  denotes prices of  $t$ -period ahead contingent claims based on conditional worst case probabilities and is given by

$$M_0^t = \beta^t \lambda_t,$$

where  $\lambda_t$  is the marginal utility of consumption at time  $t$  by the representative household.

$D_{l,t}$  is the dividend payout given by

$$D_{l,t} = Y_{l,t} - W_t H_{l,t} - I_{l,t},$$

where  $I_{l,t}$  is investment.

Note that the shareholder value depends on the worst case expectations  $E_0^*$ . This is because state prices reflect the representative household's ambiguity. An important feature of ambiguity aversion is that, unlike the case of risk, idiosyncratic uncertainty does not vanish under diversification. Uncertainty affects ambiguity-averse household's utility by lowering the worst-case mean and hence the household acts as if the mean of each individual firm's technology is lower. As a result, ambiguity is not diversified away in the aggregate and uncertainty lowers the mean of the aggregate technology<sup>5</sup>.

Their capital stock is follows the law of motion

$$K_{l,t} = (1 - \delta(U_{l,t}))K_{l,t-1} + I_{l,t},$$

where the depreciation rate is positively related to the intensity of utilization

$$\delta(U) = \delta_0 + \delta_1(U - 1) + \frac{\delta_2}{2}(U - 1)^2.$$

### Interpretation of the additive shock

Our specification follows van Nieuwerburgh and Veldkamp (2006) by generating a procyclical signal-to-noise ratio by adding an unobservable additive shock  $\nu_{l,t}$  to the production function. We offer two interpretations of the additive shock. First, in the Appendix, we argue that at the aggregate level our baseline model is observationally equivalent to a model where firms learn about the demand of their goods through noisy signals. In this version of the model, firms are subject to unobservable idiosyncratic shocks to the weight attached to their goods in the CES aggregator for final goods.<sup>6</sup> It is natural to interpret the shock as a shock to the quality or demand of goods produced by an individual firm  $l$ . The additive shock, which vanish in the aggregate due to the law of large numbers, is replaced with an i.i.d. observation error of the underlying idiosyncratic shock; agents observe noisy signals about the demand, whose precision is increasing in the level of individual production.

Second, using the argument made by van Nieuwerburgh and Veldkamp (2006), it can

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<sup>5</sup>See Marinacci 1999 or Epstein and Schneider 2003a for formal treatments of the law of large numbers for i.i.d. ambiguous random variables. There they show that sample averages must (almost surely) lie in an interval bounded by the highest and lowest possible mean, and these bounds are tight in the sense that convergence to a narrower interval does not occur.

<sup>6</sup>We thank Yan Bai for helpful discussions that led to this formulation.

be shown that the procyclical signal-to-noise ratio arises from aggregation of production units with common and idiosyncratic shocks within a firm. Suppose that a firm's output is the sum of production units operating that produce one unit of output using (utilization adjusted) capital and labor. In turn, the output of each production unit is determined by the sum of a common firm-specific component  $z_{l,t}$  and an idiosyncratic component  $\tilde{v}_{l,t}^i$ , so that  $y_{l,t}^i = z_{l,t} + \tilde{v}_{l,t}^i$ . Denoting  $N_{l,t}$  the number of production units operating at period  $t$ , a firm's output is given by  $Y_{l,t} = \sum_{i=1}^{N_{l,t}} y_{l,t}^i = z_{l,t}N_{l,t} + \sum_{i=1}^{N_{l,t}} \tilde{v}_{l,t}^i$ . Proposition 1 in van Nieuwerburgh and Veldkamp (2006) shows that, as long as the idiosyncratic shocks are not perfectly correlated across production units, the signal-to-noise ratio (defined as the variance of signal  $z_{l,t}N_{l,t}$  divided by the variance of noise  $\sum_{i=1}^{N_{l,t}} \tilde{v}_{l,t}^i$ ) is increasing in  $N_{l,t}$ .<sup>7</sup> Intuitively, this is because the effect of idiosyncratic shocks to each production unit becomes smaller when we aggregate more production units.

## Market clearing and resource constraint

We impose the market clearing conditions for the labor market and the bond market:

$$H_t = \int_0^1 H_{l,t} dl, \quad B_t = 0.$$

The resource constraint is given by

$$C_t + I_t + G_t = Y_t$$

where  $I_t \equiv \int_0^1 I_{l,t} dl$ ,  $Y_t \equiv \int_0^1 Y_{l,t} dl$ , and  $G_t$  is the government spending. We assume that the government balances budget each period ( $G_t = -T_t$ ). We also assume  $G_t = g_t Y_t$  where  $g_t$  follows

$$\ln g_t = (1 - \rho_g)\bar{g} + \rho_g \ln g_{t-1} + \epsilon_{g,t}, \quad \epsilon_{g,t} \sim N(0, \sigma_g^2).$$

## Timing

The timing of the event at period  $t$  is as follows:

Stage 1 : Pre-production stage

- Agents observe the realization of aggregate shocks ( $A_t, \sigma_t$ , and  $g_t$ ).

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<sup>7</sup>We assume that  $\tilde{v}_{l,t}^i$  is independently distributed across time so that tracking individual production does not reveal additional information.

- Given forecasts about the idiosyncratic technology and its associated worst-case scenario, firms make utilization decision and hire labor ( $U_{l,t}$  and  $H_{l,t}$ ). The household supply labor  $H_t$  and the labor market clears at the wage rate  $W_t$ .

Stage 2 : Post-production stage

- Idiosyncratic shocks  $z_{l,t}$  and  $\nu_{l,t}$  realize (but are unobservable) and production takes place.
- Given output and input, firms update estimates about their idiosyncratic technology and use it to form forecasts for production next period.
- Firms make investment  $I_{l,t}$  and pay out dividends  $D_{l,t}$ . The household makes consumption and asset purchase decisions ( $C_t$ ,  $B_t$ , and  $\theta_{l,t}$ ).

### 3.2 Learning about idiosyncratic productivity under ambiguity

Firms form estimates about the idiosyncratic shock  $z_{l,t}$  from the observables. Since the problem is linear and Gaussian, Bayesian updating using Kalman filter is optimal from the statistical perspective of minimizing the mean square error of the estimates. With ambiguity, firms are not fully confident in a unique probabilistic description of the underlying data generating process. The agents use the observed data to learn about the hidden technology, and they do so by using the Kalman filter to obtain a benchmark probability distribution. Ambiguity is modeled as the one-step ahead set of conditional beliefs  $\mathcal{P}_t(s^t)$  in (3.1), which here consists of alternative probability distribution surrounding the benchmark controlled by a bound on the relative entropy distance. Thus, our ambiguity-averse agents continue to use the ordinary Kalman filter to estimate the latent technology and evaluate plans according to the worst-case means that are implied by the posterior estimates.

To ease notation, we set the trend growth rate  $\gamma$  to zero. We denote learning problem of the model with positive growth is provided in the appendix along with other equilibrium conditions. We denote  $F_{l,t} \equiv (U_{l,t}K_{l,t-1})^\alpha H_{l,t}^{1-\alpha}$ . After production at period  $t$ , the measurement equation of the Kalman filter is given by

$$Y_{l,t}/A_t = F_{l,t}z_{l,t} + \nu_{l,t},$$

and the transition equation is given by

$$z_{l,t} = (1 - \rho_z)\bar{z} + \rho_z z_{l,t-1} + \epsilon_{z,l,t}.$$

Note that, unlike standard time-invariant Kalman filter, the coefficient in the measurement equation  $F_{l,t}$  is time-varying.<sup>8</sup> The key property of our filtering system is that the signal-to-noise ratio is procyclical, which follows from the fact that input  $F_{l,t}$  is procyclical. The flip side implication of this property is that uncertainty is countercyclical; The posterior variance of idiosyncratic technology  $z_{l,t}$  rises during recessions. Intuitively, when a firm puts less resources into production, its estimate about its productivity is imprecise because the level of output is largely determined by the realization of the transitory shock. Conversely, its estimate is accurate when it uses more resources because output mostly reflects the realization of productivity.

To characterize the filtering problem, we start by deriving the one-step-ahead prediction from the period  $t - 1$  estimate  $\tilde{z}_{l,t-1|t-1}$  and its associated error variance  $\Sigma_{l,t-1|t-1}$ . We have

$$\begin{aligned}\tilde{z}_{l,t|t-1} &= (1 - \rho_z)\bar{z} + \rho_z\tilde{z}_{l,t-1|t-1}, \\ \Sigma_{l,t|t-1} &= \rho_z^2\Sigma_{l,t-1|t-1} + \sigma_{z,t}^2.\end{aligned}$$

Then, given observables (output  $Y_{l,t}$  and aggregate productivity  $Z_t$ ) firms update their estimates according to

$$\tilde{z}_{l,t|t} = \tilde{z}_{l,t|t-1} + \text{Gain}_{l,t}(Y_{l,t}/Z_t - \tilde{z}_{l,t|t-1}F_{l,t}), \quad (3.4)$$

where  $\text{Gain}_{l,t}$  is the Kalman gain and is given by

$$\text{Gain}_{l,t} = \left[ \frac{F_{l,t}^2 \Sigma_{l,t|t-1}}{F_{l,t}^2 \Sigma_{l,t|t-1} + \sigma_{\nu,t}^2} \right] F_{l,t}^{-1}.$$

The term inside the bracket is the informativeness of the observation and the term outside is the adjustment term due to the fact that the signal is multiplied by  $F_{l,t}$ . The updating rule for variance is

$$\begin{aligned}\Sigma_{l,t|t} &= (1 - \text{Gain}_{l,t}F_{l,t})\Sigma_{l,t|t-1}, \\ &= \left[ \frac{\sigma_{\nu,t}^2}{F_{l,t}^2 \Sigma_{l,t|t-1} + \sigma_{\nu,t}^2} \right] \Sigma_{l,t|t-1}.\end{aligned} \quad (3.5)$$

The first line says that the error shrinks as we learn more from the observation; the error is decreasing in the size of the Kalman gain. The second line says that the error variance is increasing in the un-informativeness of the observation, which is the variance of noise divided

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<sup>8</sup>However, also note that, after production, the coefficient and the variance of shocks are pre-determined, which allows us to use Kalman filter.



by the total variance. We can see that, holding  $\Sigma_{l,t|t-1}$  and  $\sigma_{\nu,t}^2$  constant, the posterior variance  $\Sigma_{l,t|t}$  increases as input  $F_{l,t}$  decreases.

We now describe the dynamics of the idiosyncratic technology  $z_{l,t}$  from the perspective of a firm.

$$\begin{aligned} z_{l,t+1} &= (1 - \rho_z)\bar{z} + \rho_z z_{l,t} + \epsilon_{z,l,t+1} \\ &= (1 - \rho_z)\bar{z} + \rho_z(\tilde{z}_{l,t|t} + u_{l,t}) + \epsilon_{z,l,t+1}, \end{aligned}$$

where  $u_{l,t}$  is the estimation error of  $z_{l,t}$  and  $u_{l,t} \sim N(0, \Sigma_{l,t|t})$ . A firm  $l$  is not confident in the estimate  $\tilde{z}_{l,t}$ . It considers a set of probability distributions, of the form

$$z_{l,t+1} = (1 - \rho_z)\bar{z} + \rho_z \tilde{z}_{l,t|t} + \mu_{l,t+1}^* + \rho_z u_{l,t} + \epsilon_{z,l,t+1}, \quad (3.6)$$

where  $\mu_{l,t+1} \in [-a_{l,t}, a_{l,t}]$ . From the perspective of the firm, a change in posterior variance translates into a change in uncertainty about the one-step-ahead realization of technology. As in Bianchi et al. (2014), the change in uncertainty, in turn, affects the set of possible  $\mu_{l,t+1}$  and thus the worst-case mean.

More precisely, agents only consider the conditional means  $\mu_{l,t+1}^*$  that are sufficiently close to the long run average of zero in the sense of relative entropy:

$$\frac{(\mu_{l,t+1}^*)^2}{2\rho_z^2 \Sigma_{l,t|t}} \leq \frac{1}{2}\eta_a^2, \quad (3.7)$$

where the left hand side is the relative entropy between two normal distributions that share the same variance  $\rho_z^2 \Sigma_{l,t|t}$  but have different means ( $\mu_{l,t+1}^*$  and zero) and  $\eta_a$  is a parameter that determines the size of the entropy constraint. Agents compare the normal distributions with variance  $\rho_z^2 \Sigma_{l,t|t}$  because we assume that they only treat the estimation error  $u_t$  as ambiguous; They are fully confident in the law of motion  $z_{l,t+1} = (1 - \rho_z)\bar{z} + \rho_z z_{l,t} + \epsilon_{z,l,t+1}$  and treat the technology shock  $\epsilon_{z,l,t+1}$  as risk. (3.7) implies that the worst-case mean is given by

$$-a_{l,t} = -\eta_a \rho_z \sqrt{\Sigma_{l,t|t}}. \quad (3.8)$$

The relative entropy can be thought of as a measure of distance between the two distributions. When uncertainty  $\Sigma_{l,t|t}$  is high, it becomes difficult to distinguish between different processes. As a result, agents become less confident and contemplate wider sets  $\mu_{l,t+1}$  of conditional probabilities.

In Bayesian decision making, experimentation is valuable because it raises expected utility by improving posterior precision. Ambiguity-averse agents also value experimentation since

it affects utility by tightening the set of conditional probability considered. In our model, firms take into account the impact of the level of input on worst-case mean when they make decisions. Although we allow active learning by firms, our model can still be solved using standard linear methods. This is in contrast to experimentation in a Bayesian setting, which requires non-linear tools. When we present our quantitative results, we assess the contribution of experimentation by comparing our baseline results with those under passive learning.

## 4 Equilibrium characterization and solution

We start by discussing the recursive representation of the model. This clarifies the sequence of events and the information sets that agents base their action on. We then build on the framework to describe the solution method we use to solve for the equilibrium law of motion.

### 4.1 Recursive competitive equilibrium

As in Angeletos et al. (2014), it is useful to divide the agents' problem into two stages; stage 1 (pre-production stage) and stage 2 (post-production stage). To ease exposition, we abstract from utilization momentarily. We collect exogenous aggregate state variables (such as aggregate TFP) in a vector  $X$  with a cumulative transition function  $F(X'|X)$ . The endogenous aggregate state is the distribution of firm-level variables. A firm's type is identified by the posterior mean estimate of productivity  $\tilde{z}_l$ , the posterior variance  $\Sigma_l$ , and its capital stock  $K_l$ . The worst-case TFP is not included because it is implied by the posterior mean and variance. We denote the cross-sectional distribution of firms' type by  $\xi_1$  and  $\xi_2$ .  $\xi_1$  is a stage 1 distribution over  $(\tilde{z}_l, \Sigma_l, K_l)$  and  $\xi_2$  is a stage 2 distribution over  $(\tilde{z}'_l, \Sigma'_l, K_l)$ .  $\xi'_1$ , in turn, is a distribution over  $(\tilde{z}'_l, \Sigma'_l, K'_l)$  at stage 1 in the next period.<sup>9</sup>

First, consider the household's problem. The household's wealth can be summarized by a portfolio  $\vec{\theta}_l$  which consists of share  $\theta_l$  for each firm and the risk-less bond holdings  $B$ . We use  $V_1^h$  and  $V_2^h$  to denote the household's value function at stage 1 and stage 2, respectively. We use  $m$  to summarize the income available to the household at stage 2. The household's

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<sup>9</sup>See also Senga (2015) for a recursive representation of an imperfect information heterogeneous-firm model with time-varying uncertainty.

problem at stage 1 is

$$\begin{aligned}
V_1^h(\vec{\theta}_l, B; \xi_1, X) &= \max_H \left\{ -\varphi \frac{H^{1+\eta}}{1+\eta} + E^*[V_2^h(\hat{m}; \hat{\xi}_2, X)] \right\} \\
\text{s.t. } \hat{m} &= WH + RB + \int (\hat{D}_l + \hat{P}_l)\theta_l dl
\end{aligned} \tag{4.1}$$

where we momentarily use the *hat* symbol to indicate random variables that will be resolved at stage 2. The household's problem at stage 2 is

$$\begin{aligned}
V_2^h(m; \xi_2, X) &= \max_{C, \vec{\theta}_l', B'} \left\{ \ln C + \beta \int V_1^h(\vec{\theta}_l', B'; \xi_1', X') dF(X'|X) \right\} \\
\text{s.t. } C + B' + \int P_l \theta_l' dl &\leq m \\
\xi_1' &= \Gamma(\xi_2, X)
\end{aligned} \tag{4.2}$$

In problem (4.1), households choose labor supply based on the worst-case stage 2 value (recall that we use  $E^*$  to denote worst-case conditional expectations). The problem (4.2), in turn, describes the household's consumption and asset allocation problem given the realization of income and aggregate states. In particular, they take as given the law of motion of the next period's distribution  $\xi_1' = \Gamma(\xi_2, X)$ , which in equilibrium is consistent with the firm's policy function. Importantly, in contrast to the stage 2 problem, a law of motion that describes the evolution of  $\xi_2$  from  $(\xi_1, X)$  is absent in the stage 1 problem. Indeed, if there is no ambiguity in the model, agents take as given the law of motion  $\xi_2 = \Upsilon(\xi_1, X)$ , which in equilibrium is consistent with the firm's policy function and the true data generating process of the firm-level TFP. Since agents are ambiguous about each firm's TFP process, they cannot settle on a single law of motion about the distribution of firms. Finally, the continuation value at stage 2 is governed by the transition density of aggregate exogenous states  $X$ .

Next, consider the firms' problem. We use  $v_1^f$  and  $v_2^f$  to denote the firm's value function at stage 1 and stage 2, respectively. Firm  $l$ 's problem at stage 1 is

$$\begin{aligned}
v_1^f(\tilde{z}_l, \Sigma_l, K_l; \xi_1, X) &= \max_{H_l} E^*[v_2^f(\hat{z}_l', \Sigma_l', K_l; \hat{\xi}_2, X)] \\
\text{s.t. } &\text{Updating rules (3.4) and (3.5)}
\end{aligned} \tag{4.3}$$

and firm  $l$ 's problem at stage 2 is

$$\begin{aligned}
v_2^f(\tilde{z}'_l, \Sigma'_l, K_l; \xi_2, X) &= \max_{I_l} \left\{ \lambda(Y_l - WH_l - I_l) + \beta \int v_1^f(\tilde{z}'_l, \Sigma'_l, K'_l; \xi'_1, X') dF(X'|X) \right\} \\
\text{s.t. } K'_l &= (1 - \delta)K_l + I_l \\
\xi'_1 &= \Gamma(\xi_2, X)
\end{aligned} \tag{4.4}$$

where we simplify the exposition by expressing a firm's value in terms of the marginal utility  $\lambda$  of the representative household. Similar to the household's problem, a firm's problem at stage 1 is to choose the labor demand so as to maximize the worst-case stage 2 value. Note that the posterior mean  $\tilde{z}'_l$  will be determined by the realization of output  $Y_l$  at stage 2 while the posterior variance  $\Sigma'_l$  is determined by  $\Sigma_l$  and the input level at stage 1. In problem (4.4), the firm then chooses investment taking as given the realization of output and the updated estimates of its productivity. Note that, as in the household's problem, firms take as given the (equilibrium) law of motion of the distribution of firms in the stage 2 problem but not in the stage 1 problem.

The discussion above highlights one of the key features of our model; the level of labor input is chosen before the realization of firm-level productivity and that this timing arises naturally from imperfect information about the underlying productivity process. This labor-in-advance feature allows us to circumvent the Barro and King (1984) critique and hence generate feedback effects of time-varying uncertainty consistent with business cycle co-movement without additional rigidities.

We conclude this subsection by providing a brief definition of the recursive competitive equilibrium of our model. The recursive competitive equilibrium is a collection of value functions, policy functions, and prices such that

1. Households and firms optimize; (4.1) – (4.4).
2. The labor market, goods market, and asset markets clear.
3. The law of motion  $\xi'_1 = \Gamma(\xi_2, X)$  is induced by the firms' policy function  $I_l(\tilde{z}'_l, \Sigma'_l, K_l; \xi_2, X)$ .

## 4.2 Log-linearized solution

We solve for the equilibrium law of motion using standard log-linear methods. This is possible for two reasons. First, since the filtering problem firms face is linear, the law of motion of the posterior variance can be characterized analytically (Saijo 2014). Because the level of inputs has first-order effects on the level of posterior variance, linearization accurately captures the impact of economic activity on confidence. Second, as in Ilut and Schneider (2014),

we consider ambiguity about the mean and hence the feedback from confidence to economic activity is also well-approximated by linearization. In turn, log-linear decision rules facilitate aggregation because the cross-sectional mean becomes a sufficient statistic for the dynamics of the distribution of firms.

We follow Ilut and Schneider (2014) and solve for the equilibrium dynamics using a guess-and-verify approach:

- (a) guess the worst case beliefs  $p^0$ .
- (b) solve the model assuming that agents have agents have expected utility and beliefs  $p^0$ .
- (c) compute the agent's value function  $V$ .
- (d) verify that the guess  $p^0$  indeed achieves the minima.

In what follows we explain step (b) by deriving log-linearized expressions for the expected worst-case output at stage 1 and the realized output at stage 2.<sup>10</sup> We use the example to illustrate that uncertainty about the firm-level TFP has a first-order effect at the aggregate and generates a countercyclical labor wedge.

As in Ilut and Schneider (2014), we first find the worst-case steady state by evaluating a deterministic version of the filtering problem and standard first-order conditions under the guessed worst-case belief. Potential complications arise because the worst-case technology depends on the level of economic activity. Since the worst-case technology, in turn, determines the level of economic activity, there could be multiple steady states. We circumvent this multiplicity by treating the posterior variance of the level of idiosyncratic TFP as a parameter and by focusing on the steady state that is implied by that posterior variance.

Next, we log-linearize the model around the worst-case steady state. To do this, we first log-linearize the expected worst-case output of individual firm  $l$  at stage 1:

$$E_t^* \hat{Y}_{l,t}^0 = \hat{Z}_t^0 + E_t^* \hat{z}_{l,t}^0 + \hat{F}_{l,t}^0, \quad (4.5)$$

and the realized output of individual firm  $l$  at stage 2:

$$\hat{Y}_{l,t}^0 = \hat{Z}_t^0 + \hat{z}_{l,t}^0 + \hat{F}_{l,t}^0, \quad (4.6)$$

where we use  $\hat{x}_t^0 = x_t - \bar{x}^0$  to denote log-deviations from the worst-case steady state and set the trend growth rate  $\gamma$  to zero to ease notation. The worst-case individual output (4.5) is the sum of three components: the current level of aggregate TFP, the worst-case

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<sup>10</sup>We provide a general description of the procedure in the Appendix.

individual TFP, and the input level. The realized individual output (4.6), in turn, is the sum of aggregate TFP, the *realized* individual TFP, and the input level.

We then aggregate the log-linearized individual conditions (4.5) and (4.6) to obtain the cross-sectional mean of worst-case individual output:

$$E_t^* \hat{Y}_t^0 = \hat{A}_t^0 + E_t^* \hat{z}_t^0 + \hat{F}_t^0, \quad (4.7)$$

and the cross-sectional mean of realized individual output:

$$\hat{Y}_t^0 = \hat{A}_t^0 + \hat{z}_t^0 + \hat{F}_t^0, \quad (4.8)$$

where we simply eliminate subscript  $l$  to denote the cross-sectional mean, i.e.,  $\hat{x}_t^0 \equiv \int_0^1 \hat{x}_{l,t}^0 dl$ .

So far we have characterized the dynamics of output under the worst-case scenario. Our final step is to characterize the dynamics under the true data generating process (DGP). To do this, we feed in the cross-sectional mean of individual TFP, which is constant under the true DGP, into (4.7) and (4.8). Using (4.7), the cross-sectional mean of worst-case output is given by

$$E_t^* \hat{Y}_t = \hat{A}_t + E_t^* \hat{z}_t + \hat{F}_t, \quad (4.9)$$

where we use  $\hat{x}_t = x_t - \bar{x}$  to denote log-deviations from the steady-state under the true DGP. Using (4.8), the realized aggregate output is given by

$$\hat{Y}_t = \hat{A}_t + \hat{F}_t, \quad (4.10)$$

where we used  $\hat{z}_t = 0$  under the true DGP. Importantly,  $E_t^* \hat{z}_t$  in (4.10) is not necessarily zero outside the steady state. To see this, combine (3.6) and (3.8) and log-linearize to obtain an expression for  $E_t^* \hat{z}_{l,t}$ :

$$E_t^* \hat{z}_{l,t} = \varepsilon_{z,z} \hat{\tilde{z}}_{l,t-1|t-1} - \varepsilon_{z,\Sigma} \hat{\Sigma}_{l,t-1|t-1}, \quad (4.11)$$

From (3.5), the posterior variance is negatively related to the level of input  $F$ :

$$\hat{\Sigma}_{l,t-1|t-1} = \varepsilon_{\Sigma,\Sigma} \hat{\Sigma}_{l,t-2|t-2} - \varepsilon_{\Sigma,F} \hat{F}_{l,t-1}, \quad (4.12)$$

where for simplicity we assumed there is no dispersion shock. The elasticities  $\varepsilon_{z,z}$ ,  $\varepsilon_{z,\Sigma}$ ,  $\varepsilon_{\Sigma,\Sigma}$ , and  $\varepsilon_{\Sigma,F}$  are functions of structural parameters and are all positive. We combine (4.11) and (4.12) to obtain

$$E_t^* \hat{z}_{l,t} = \varepsilon_{z,z} \hat{\tilde{z}}_{l,t-1|t-1} - \varepsilon_{z,\Sigma} \varepsilon_{\Sigma,\Sigma} \hat{\Sigma}_{l,t-2|t-2} + \varepsilon_{z,\Sigma} \varepsilon_{\Sigma,F} \hat{F}_{l,t-1}. \quad (4.13)$$

Finally, we aggregate (4.13) across all firms:

$$E_t^* \hat{z}_t = -\varepsilon_{z,\Sigma} \varepsilon_{\Sigma,\Sigma} \hat{\Sigma}_{t-2|t-2} + \varepsilon_{z,\Sigma} \varepsilon_{\Sigma,F} \hat{F}_{t-1},$$

where we used  $\int_0^1 \hat{z}_{l,t-1|t-1} dl = 0$ .<sup>11</sup> Thus, the level of economic activity  $\hat{F}_{t-1}$  has a first-order effect on the cross-sectional average of the worst-case mean. During recessions, firms on average produce less, which leads to lower confidence about their firm-level TFP. This endogenous reduction in confidence further reduces equilibrium hours worked and other economic activity. From the perspective of the econometrician who measures labor productivity from (4.10), this manifest itself as an increase in the labor wedge.

## 5 Main results

We are interested in studying the role of endogenous firm-level uncertainty in business cycles. In this section, we evaluate the empirical performance of the calibrated version of our model and contrast its quantitative implications with those of a standard RBC model.

### 5.1 Parameterization

Table 1 summarizes the parameters used in our exercise. In order to facilitate comparison of our model with a standard RBC model, we use common values used in the literature whenever possible. The annual rate of labor augmenting technological progress is set to 1.6 percent and the discount factor is set to 0.99. The capital share of income is 0.3. The Frisch elasticity of labor supply is equal to 2 and the annual steady state depreciation rate is set to 10 percent. The parameter that relates utilization rate to depreciation ( $\delta_2/\delta_1$ ) implies a fairly elastic capital utilization in equilibrium. Schmitt-Grohé and Uribe (2012), for example, found a similar value in their estimation. The autocorrelation parameters for aggregate shocks are broadly in line with the previous literature.<sup>12</sup>

We choose  $\rho_z = 0.7$  and  $\sigma_z = 0.4$  for the idiosyncratic TFP process. Idiosyncratic TFP is less persistent than the aggregate, which is in line with the finding in Kehrig (2015). The values imply a cross-sectional standard deviation of TFP of 0.56 and is in line with the estimates found in Bloom et al. (2012) using the establishment-level data. We would like to bound the size of ambiguity  $\eta_a$  by the variability of the data. Based on this concern, Ilut and

<sup>11</sup>This follows from aggregating the log-linearized version of (3.4) and evaluating the equation under the true DGP. Intuitively, since the cross-sectional mean of idiosyncratic TFP is constant, the cross-sectional mean of posterior mean should be constant as well.

<sup>12</sup>For example, autocorrelation of the dispersion shock  $\rho_\sigma$  is consistent with the time series of cross-sectional TFP dispersion reported in Bloom et al. (2012).

Schneider (2014) argue that a reasonable upper bound for  $\eta_a$  is 2. We choose a conservative value of  $\eta_a = 1$ . Recall from the discussion in the earlier section that we re-parameterize the model so that we take the worst-case steady state posterior variance  $\bar{\Sigma}^0$  of idiosyncratic TFP as a parameter. This posterior variance, together with  $\rho_z$  and  $\sigma_z$ , will pin down the standard deviation of additive shock  $\sigma_\nu$ . David et al. (2015) estimate the posterior variance of a firm-specific shock (in the context of our model, a TFP shock) to be around 8–13%. We choose the worst-case steady state posterior variance so that at the zero-risk steady state the posterior variance  $\bar{\Sigma}$  is 12%.

Our parameterization implies that the cross-sectional mean of the worst-case individual TFP at the zero-risk steady state is about 82 percent of the actual realized level. The Kalman gain at the zero-risk steady state, normalizing the level of input to one, is 0.45. To put this in perspective, the gain implies that an observation from quarters ago will receive a weight  $(1 - 0.45)^4 \approx 0.09$ . Thus, learning is fairly precise and quick under our parameterization. Finally, the ex-post excess return on capital at the zero-risk steady state is 6.4 annual percentage point. We obtain a sizable excess return because our model emphasizes uncertainty generated by idiosyncratic shocks and firm-level learning.

## 5.2 Impulse response analysis

Figure 2 plots the impulse response to a positive technology shock. In addition to the response from the baseline model (labeled ‘Baseline’), we also report the responses from the model with passive learning (labeled ‘Passive’) and the standard rational expectation (RE) RBC model (labeled ‘RE’). The solution to the RE model is obtained by simply setting the entropy constraint  $\eta_a$  to zero. In this case, agents think in terms of single probabilities and the model reduces to a rational expectation model. Note also that when  $\eta_a = 0$ , firm-level learning cancels out in the aggregate due to linearization and the law of large numbers.

Compared to the RE version, our model generates amplified and hump-shaped response in output, investment, and hours. These dynamics are due to the endogenous variation in firms’ confidence. In response to a positive technology shock, firms (on average) increase their inputs, such as hours and the capital utilization rate. The increase in inputs lowers uncertainty which implies that firms contemplate a narrower set of conditional probabilities; the worst-case scenarios become less worse. As a result, agents act *as if* their idiosyncratic productivities are higher and thus further increase their economic activity. At the same time, from the perspective of the econometrician, the labor supply and the demand for capital is surprisingly high. Thus, both labor wedge and ex-post excess return on capital decline.

Finally, we compare our baseline impulse response with the response from the passive



learning model. Initially the output and hours responses of the baseline model with active learning are larger than those of passive learning. In the medium run, however, the responses of passive learning become larger. This is due to a dynamic interaction of two opposing forces. On one hand, higher production during booms increases the value of experimentation because it raises the marginal benefit of an increase in expected worst-case technology. On the other hand, there is an offsetting effect coming from a reduction in posterior variance. Since the level of posterior variance is downward convex in the level of inputs, the marginal reduction in posterior variance due to an increase in inputs is smaller during booms. During the initial period of a positive technology shock, the first effect dominates the second. As the economy slows down, the second effect becomes more important.

Figure 3 reports the impulse responses to a dispersion shock that increases firm-level volatilities. An increase in volatilities generates a brief increase in output and hours followed by a large and persistent decline in economic activity. The initial response is driven purely by the wealth effect due to bad “news”; households anticipate a future decline in worst-case technology so they cut consumption and increase hours. In the medium run, the increase in volatilities reduces firms’ confidence and thus generates a contraction in the economic activity and an increase in wedges. Compared to the model with passive learning, the effect of a dispersion shock is significantly smaller under our baseline model with active learning. Intuitively, agents try to counteract an exogenous increase in uncertainty by increasing production and by experimenting more. Under passive learning, this effect is absent so it overstates the effect of a dispersion shock.

Figure 4 shows the impulse response to an increase in government spending. In the standard RBC model, due to the negative wealth effect hours increase but consumption and investment decline. In our model, an increase in government spending, due to its effects of raising hours worked, also raises firms’ confidence, which further stimulates economic activity. As a result, output and hours increase are larger and investment increases as well. The negative response of consumption is somewhat mitigated. Figure 5 reports the government spending multiplier for output. Under the baseline calibration, the government spending multiplier is always below one in the RE model. Our model generates a larger multiplier than the RE model and the multiplier becomes larger than one after the initial period.<sup>13</sup>

As emphasized by Hall (2009), a more elastic labor supply would increase the value of government spending multiplier in standard models. In our model, there is an additional effect coming from a larger amplification: Since firms increase their labor input more

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<sup>13</sup>Note that, in a model with capital, government spending multiplier could be above one even with a negative response of consumption if it is offset by an increase in investment.

aggressively, their confidence increases much more. To illustrate the effect of higher labor supply elasticity, we also report the government spending multiplier when we set  $\eta = 0.12$ .<sup>14</sup> The labor supply elasticity implied by this value is close to that implied by the indivisible labor model – see Hansen (1985) and Rogerson (1988). Figure 5 shows that the government spending multiplier of our model is much larger, reaching 2.5 at the peak. The effect of labor supply elasticity is also important for understanding the co-movement in response to a government spending shock. Figure 6 reports the consumption response to a government spending shock in our model and the RE version. Again, we show the responses under both the baseline parameterization and high labor supply elasticity. As we saw earlier, consumption declines in both models under the baseline parameterization. When the labor supply is more elastic, the consumption response remains negative in the RE model. In our model, however, consumption continues to increase after the initial drop and remains positive after six quarters; the increase in confidence is large enough that it overturns the negative wealth effect induced by the government spending.

### 5.3 Business cycle moments

Table 2 reports the HP-filtered second moments. To facilitate comparison with the standard frictionless RBC model, we assume that the only source of aggregate disturbance is the technology shock. We choose the standard deviation of the aggregate technology shock so that the output standard deviation in our baseline model matches the data ( $100\sigma_A = 0.273$ ).

First, our endogenous uncertainty mechanism generates output amplification of  $(1.11/0.88 \Rightarrow)$  26 percent. The baseline model is also successful in generating a larger standard deviation of hours relative to output. The low volatility of hours has been a major shortcoming of RBC theories. Our model is less successful in reproducing the volatilities of consumption and investment. Second, our model can replicate the strong negative correlation with economic activity and the labor wedge. Third, our model gives a closer match in terms of autocorrelations. The baseline model generates higher autocorrelations in levels and, more importantly, positive autocorrelations in growth rates of output and other variables. As pointed out by Cogley and Nason (1995) and Rotemberg and Woodford (1996), a standard RBC model cannot generate persistence in output growth due to its weak internal propagation mechanism.

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<sup>14</sup>We do not change other parameter values.

## 6 Estimation and extensions

In this section, we further explore the implications of our model by conducting Bayesian estimation. (Details to be added.)

## 7 Appendix

### 7.1 A model of learning about firm-specific demand

In this section, we show that the baseline model with additive shock in the production function (3.2) can be reinterpreted as a model where firms learn about their demand from noisy signals.

There is a continuum of firms, indexed by  $l \in [0, 1]$ , which produce intermediate goods and sell them to a large representative “conglomerate”. The conglomerate, who holds shares of the intermediate firms, acts in a perfectly competitive manner and combine the intermediate goods to produce final goods. To ease exposition, we momentarily abstract from all aggregate shocks, labor-augmenting technological growth, and utilization.

The conglomerate combines intermediate output  $Y_{l,t}$  according to the following CES aggregator:

$$Y_t = \left[ \int_0^1 z_{l,t}^{\frac{\sigma-1}{\sigma}} Y_{l,t}^{\frac{\sigma-1}{\sigma}} dl \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1$$

where  $z_{l,t}$  follows an AR(1) as in (3.3). In turn, intermediate goods  $Y_{l,t}$  are produced according to

$$Y_{l,t} = K_{l,t-1}^\alpha H_{l,t}^{1-\alpha}.$$

We assume that all agents, including the conglomerate and households, cannot observe the realization of  $z_{l,t}$ . Instead, after the production of final and intermediate goods they observe noisy signals

$$s_{l,t} = Y_{l,t} z_{l,t} + \tilde{v}_{l,t}, \quad \tilde{v}_{l,t} \sim N(0, \sigma_{\tilde{v}}^2) \quad (7.1)$$

where  $\tilde{v}_{l,t}$  is an observation error. Agents use all available information, including the path of signals  $s_{l,t}$  and intermediate output  $Y_{l,t}$ , to form estimates about the realization of  $z_{l,t}$ .

In this context, it is natural to think  $z_{l,t}$  as a “quality” of intermediate good  $l$  that is difficult to observe. Alternatively, since the final good could directly be used for consumption,  $z_{l,t}$  can be interpreted as an unobservable demand for a variety  $l$ . Crucially, (7.1) implies that the signal-to-noise ratio is increasing in the level of output  $Y_{l,t}$ . It is plausible that firms learn more about the quality or demand,  $z_{l,t}$ , of their goods when they produce and sell more. For example, when a restaurant serves more customers it generates more website reviews and hence people learn more about the quality of their meals.

Intermediate firms  $l$  choose price  $P_{l,t}$  and inputs to maximize the shareholder value

$$E_0^* \sum_{t=0}^{\infty} M_0^t D_{l,t}.$$

$D_{l,t}$  is the dividend payout to the conglomerate given by

$$D_{l,t} = \frac{P_{l,t}}{P_t} Y_{l,t} - W_t H_{l,t} - I_{l,t},$$

where  $P_t$  is the price of the final good given by

$$P_t = \left[ \int_0^1 P_{l,t}^{1-\sigma} dl \right]^{\frac{1}{1-\sigma}}.$$

The conglomerate, in turn, choose intermediate inputs  $Y_{l,t}$  and shares  $\theta_{l,t}$  to maximize the shareholder value

$$E_0^* \sum_{t=0}^{\infty} M_0^t D_t,$$

where  $D_t$  is the dividend payout to the households

$$D_t = Y_t + \int (D_{l,t} + P_{l,t}^e) \theta_{l,t-1} dl - \int P_{l,t}^e \theta_{l,t} dl.$$

The household side of the economy is the same as in the baseline model except that the households hold shares  $\theta_t$  of conglomerates instead of shares  $\theta_{l,t}$  of intermediate firms.

We now reintroduce aggregate shocks and utilization and describe the timing of the event at period  $t$ .

Stage 1 : Pre-production stage

- Agents observe the realization of aggregate shocks ( $Z_t, \sigma_t$ , and  $g_t$ ).
- Given forecasts about  $z_{l,t}$  and its associated worst-case scenario, firms make utilization decision, hire labor, and choose price ( $U_{l,t}, H_{l,t}$ , and  $P_{l,t}$ ). The household supply labor  $H_t$  and the labor market clears at the wage rate  $W_t$ .
- Firms produce intermediate output  $Y_{l,t}$  and sells it to the conglomerates at price  $P_{l,t}$ .

Stage 2 : Post-production stage

- $z_{l,t}$  realize (but are unobservable) and production of the final goods  $Y_t$  takes place. Agents observe noisy signals  $s_{l,t}$ .
- Firms and conglomerates update estimates about  $z_{l,t}$  and use it to form forecasts for production next period.

- Firms make investment  $I_{l,t}$  and pay out dividends  $D_{l,t}$  to the conglomerates. The conglomerates make asset purchase decisions  $\theta_{l,t}$  and pay out dividends  $D_t$  to the households. Finally, households make consumption and asset purchase decisions ( $C_t$ ,  $B_t$ , and  $\theta_t$ ).

In the perfect competition limit ( $\sigma \rightarrow \infty$ ), this version of the model is observationally equivalent to the baseline model at the aggregate level. The introduction of the conglomerate is important for two reasons. First, it prohibits households from inferring  $z_{l,t}$  from utility by directly consuming intermediate goods  $Y_{l,t}$ . Second, it generates countercyclical ex-post excess return on equity held by the household. This is because dividend payout by the conglomerate to the household ( $D_t$ ) is based on the realized return on capital. Note that the dividend payout by the intermediate firms to the conglomerate ( $D_{l,t}$ ) is not based on the realized return since the production and market clearing of  $Y_{l,t}$  happens before the realization of  $z_{l,t}$ .

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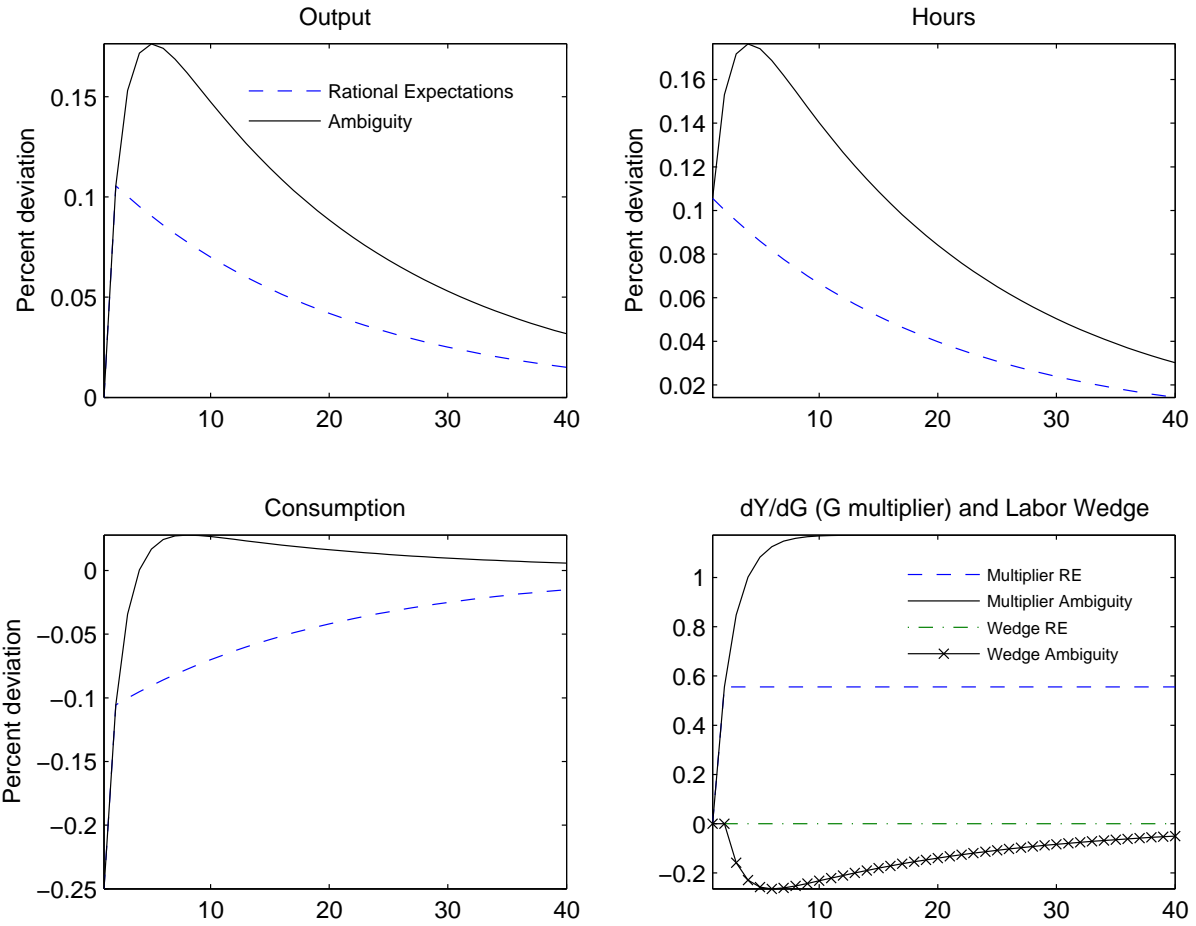


Figure 1: Impulse response for a 1% increase in government spending (the stylized model). Blue dashed line is the model with rational expectations, black solid line with ambiguity.

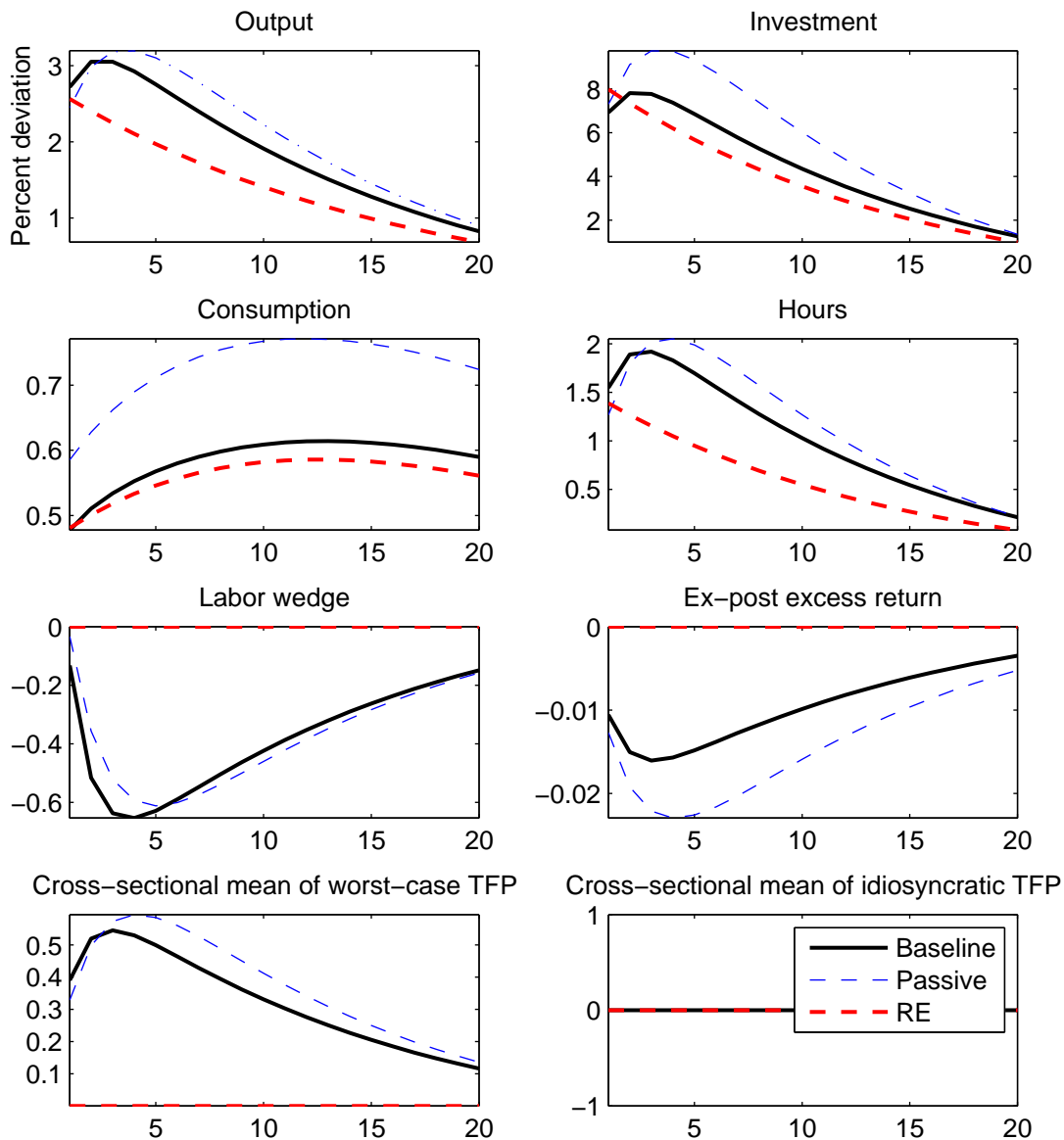


Figure 2: Impulse response for a 1% increase in aggregate productivity. Thick black solid line is our baseline model with active learning, thin blue dashed line is the model with passive learning, and thick red dashed line is the frictionless RBC model.

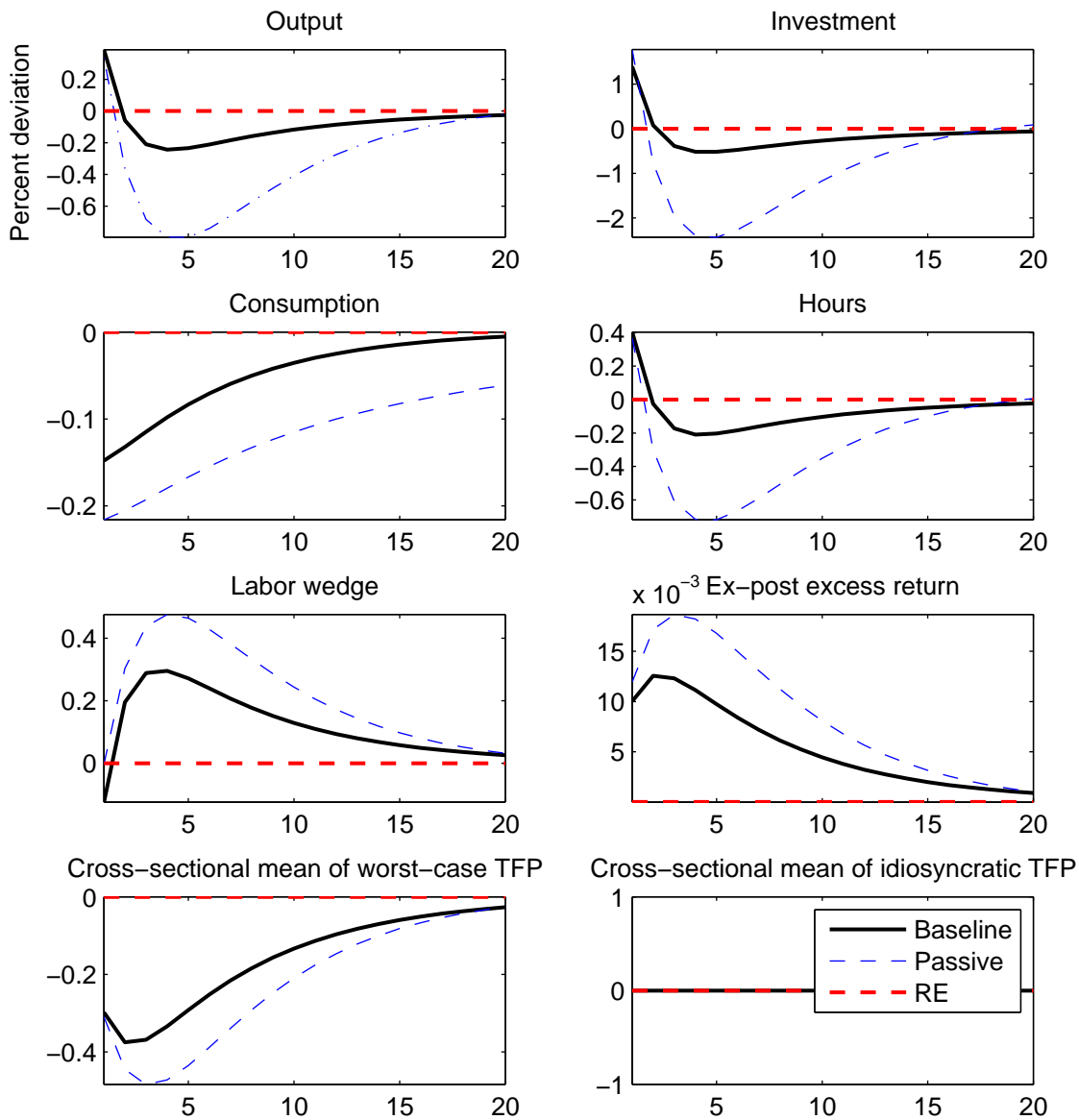


Figure 3: Impulse response for a 1% increase in the cross-sectional standard deviation of idiosyncratic productivity. Thick black solid line is our baseline model with active learning, thin blue dashed line is the model with passive learning, and thick red dashed line is the frictionless RBC model.

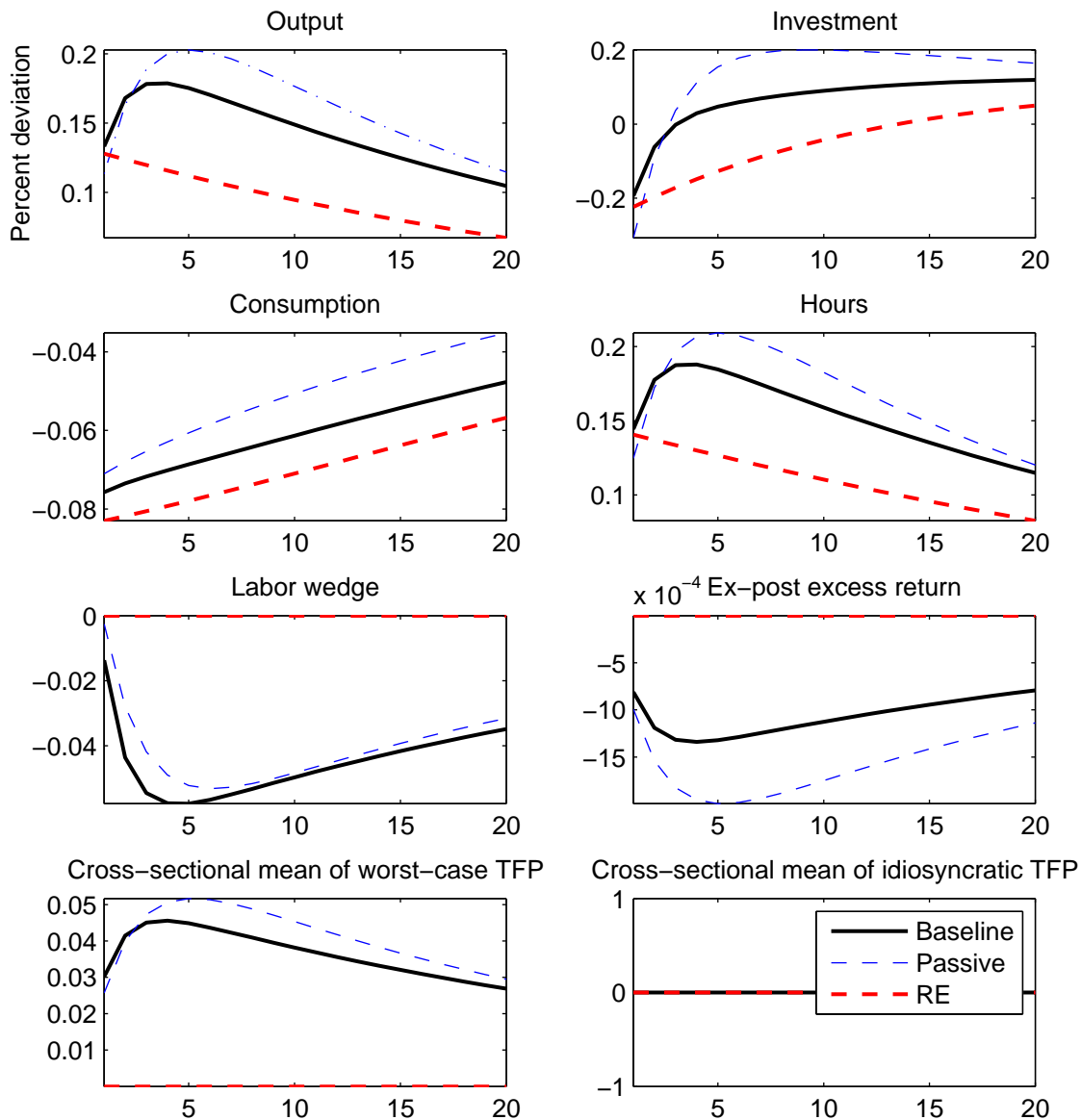


Figure 4: Impulse response for a 1% increase in government spending share to output. Thick black solid line is our baseline model with active learning, thin blue dashed line is the model with passive learning, and thick red dashed line is the frictionless RBC model.

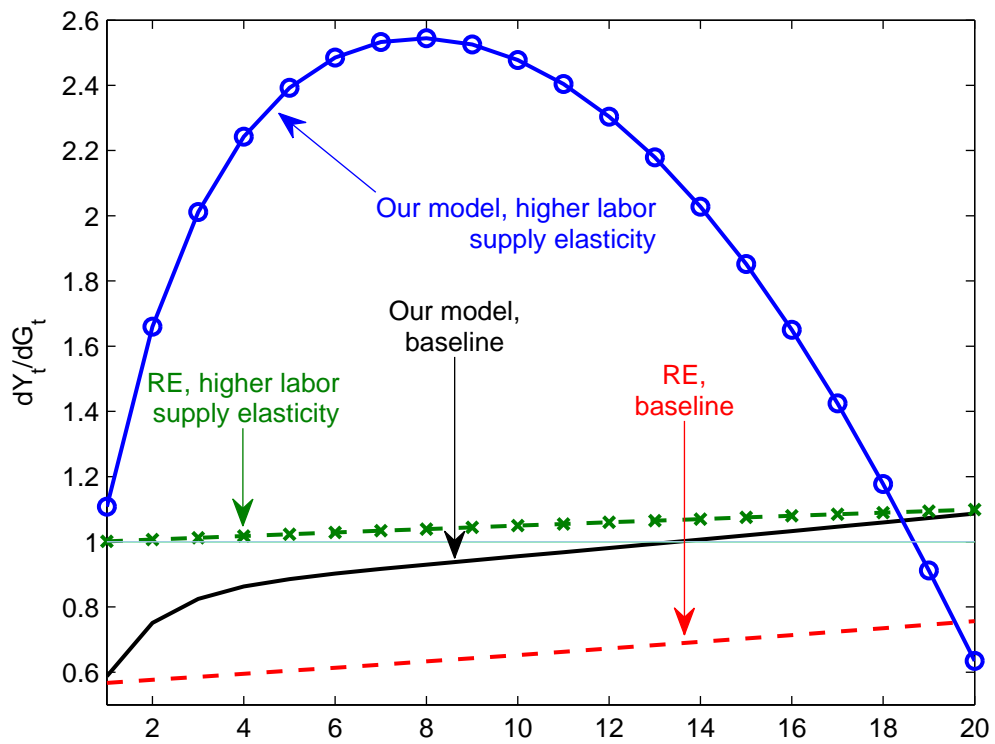


Figure 5: Government spending multiplier.

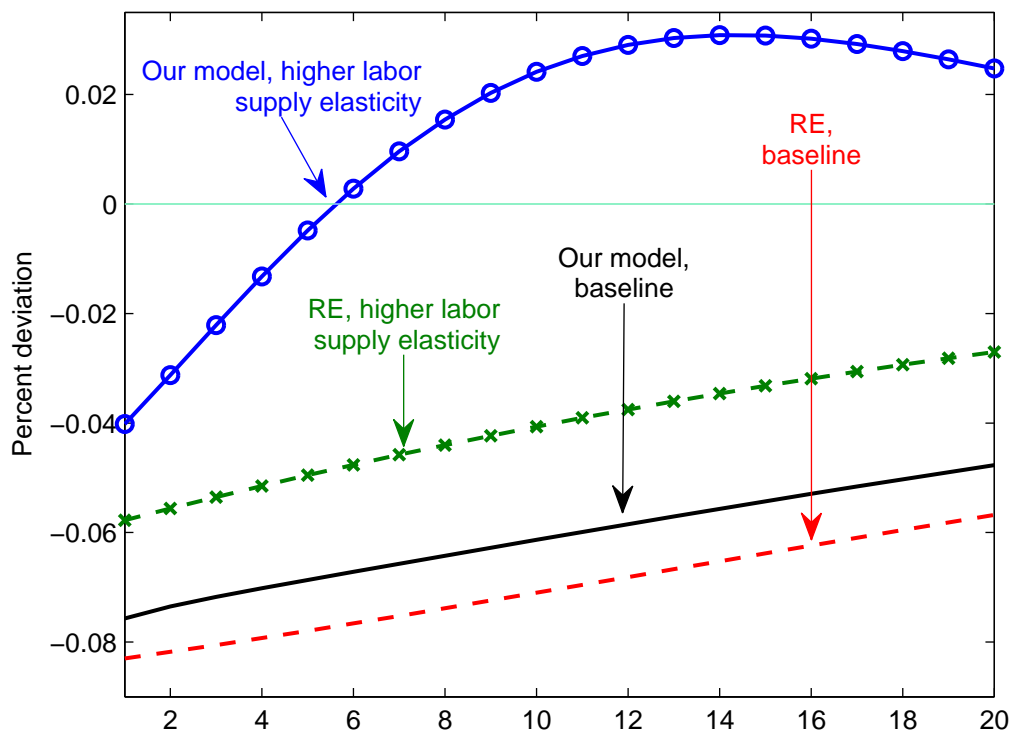


Figure 6: Consumption response to a 1% increase in government spending share to output.

Table 1: Parameters

	Description	Value
<i>Technology and preference</i>		
$\gamma$	Labor augmenting technology growth	1.004
$\alpha$	Capital share	0.3
$\beta$	Discount factor	0.99
$\eta$	Inverse Frisch elasticity	0.5
$\delta_0$	SS depreciation	0.025
$\delta_2/\delta_1$	Convexity of depreciation	0.3
$\eta_a$	Size of entropy constraint	1
$\bar{\Sigma}$	SS posterior variance	0.12
$\bar{g}$	SS share of government spending to output	0.2
<i>Shocks</i>		
$\rho_z$	Idiosyncratic technology	0.7
$\sigma_z$	Idiosyncratic technology	0.4
$\rho_A$	Aggregate technology	0.95
$\rho_g$	Government spending	0.95
$\rho_\sigma$	Firm-level dispersion	0.85

Table 2: HP-filtered moments

	Data	Our model	RBC
<i>Standard deviations</i>			
$\sigma(y)$	1.11	1.11	0.88
$\sigma(c)/\sigma(y)$	0.72	0.16	0.20
$\sigma(i)/\sigma(y)$	3.57	2.60	3.12
$\sigma(h)/\sigma(y)$	1.64	0.64	0.55
<i>Correlations with output</i>			
$\sigma(c, y)$	0.86	0.95	0.94
$\sigma(i, y)$	0.92	0.99	0.99
$\sigma(h, y)$	0.88	0.99	0.99
<i>Correlations with labor wedge</i>			
$\sigma(y, \pi)$	-0.83	-0.86	0
$\sigma(c, \pi)$	-0.84	-0.83	0
$\sigma(i, \pi)$	-0.83	-0.85	0
$\sigma(h, \pi)$	-0.97	-0.88	0
<i>Autocorrelations (levels)</i>			
$\sigma(y_t, y_{t-1})$	0.89	0.78	0.69
$\sigma(c_t, c_{t-1})$	0.85	0.75	0.73
$\sigma(i_t, i_{t-1})$	0.93	0.79	0.69
$\sigma(h_t, h_{t-1})$	0.95	0.82	0.69
<i>Autocorrelations (growth rates)</i>			
$\sigma(\Delta y_t, \Delta y_{t-1})$	0.39	0.14	-0.05
$\sigma(\Delta c_t, \Delta c_{t-1})$	0.50	0.07	0.04
$\sigma(\Delta i_t, \Delta i_{t-1})$	0.51	0.16	-0.05
$\sigma(\Delta h_t, \Delta h_{t-1})$	0.71	0.26	-0.05

*Notes:* Both data and model moments are in logs, HP-filtered ( $\lambda = 1600$ ) if the variables are in levels, and multiplied by 100 to express them in percentage terms. The model moments are the median values from 200 replications of simulations of 120 periods (after throwing away the initial 50 periods). The sample period for the data is 1985Q1–2014Q4. We choose the standard deviation of the aggregate technology shock so that the output standard deviation in the baseline model matches the data. Other aggregate shocks are set to zero.