

Holography: Map AdS/CFT to 3+1 LF Theory

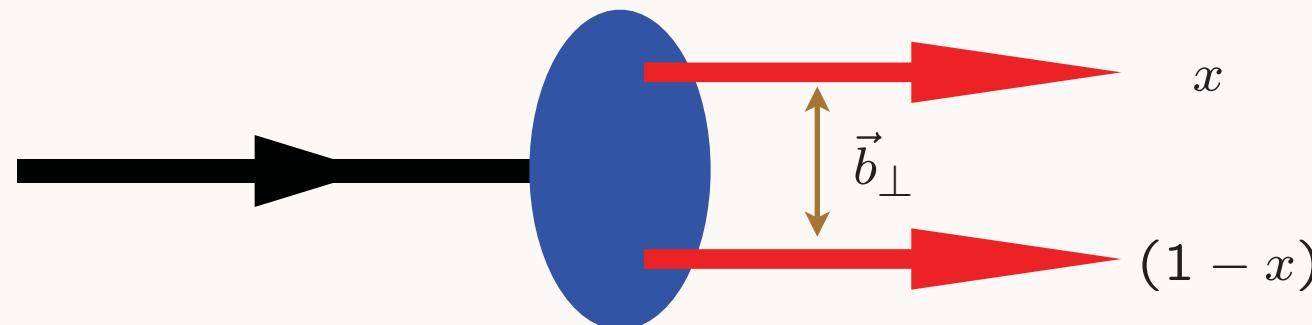
Relativistic LF radial equation

Frame Independent

$$\left[-\frac{d^2}{d\zeta^2} + V(\zeta) \right] \phi(\zeta) = \mathcal{M}^2 \phi(\zeta)$$

$$\zeta^2 = x(1-x)b_\perp^2.$$

G. de Teramond, sjb



Effective conformal potential:

$$V(\zeta) = -\frac{1-4L^2}{4\zeta^2} + \kappa^4 \zeta^2$$

confining potential:

Consider the AdS_5 metric:

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2).$$

ds^2 invariant if $x^\mu \rightarrow \lambda x^\mu$, $z \rightarrow \lambda z$,

Maps scale transformations to scale changes of the the holographic coordinate z .

We define light-front coordinates $x^\pm = x^0 \pm x^3$.

$$\text{Then } \eta^{\mu\nu}dx_\mu dx_\nu = dx_0^2 - dx_3^2 - dx_\perp^2 = dx^+ dx^- - dx_\perp^2$$

and

$$ds^2 = -\frac{R^2}{z^2}(dx_\perp^2 + dz^2) \text{ for } x^+ = 0.$$

Light-Front AdS_5 Duality

- ds^2 is invariant if $dx_\perp^2 \rightarrow \lambda^2 dx_\perp^2$, and $z \rightarrow \lambda z$, at equal LF time.
- Maps scale transformations in transverse LF space to scale changes of the holographic coordinate z .
- Holographic connection of AdS_5 to the light-front.
- The effective wave equation in the two-dim transverse LF plane has the Casimir representation L^2 corresponding to the $SO(2)$ rotation group [The Casimir for $SO(N) \sim S^{N-1}$ is $L(L+N-2)$].

- Light-front Hamiltonian equation

$$H_{LF}|\phi\rangle = \mathcal{M}^2|\phi\rangle,$$

leads to effective LF Schrödinger wave equation (KKSS)

$$\left[-\frac{d^2}{d\zeta^2} - \frac{1 - 4L^2}{4\zeta^2} + \kappa^4\zeta^2 + 2\kappa^2(L-1) \right] \phi(\zeta) = \mathcal{M}^2\phi(\zeta)$$

with eigenvalues $\mathcal{M}^2 = 4\kappa^2(n + L)$ and eigenfunctions

$$\phi_L(\zeta) = \kappa^{1+L} \sqrt{\frac{2n!}{(n+L)!}} \zeta^{1/2+L} e^{-\kappa^2\zeta^2/2} L_n^L(\kappa^2\zeta^2).$$

- Transverse oscillator in the LF plane with $SO(2)$ rotation subgroup has Casimir L^2 representing rotations for the transverse coordinates \mathbf{b}_\perp in the LF.
- SW model is a remarkable example of integrability to a non-conformal extension of AdS/CFT [Chim and Zamolodchikov (1992) - Potts Model.]

Example: Pion LFWF

- Two parton LFWF bound state:

$$\tilde{\psi}_{\bar{q}q/\pi}^{HW}(x, \mathbf{b}_\perp) = \frac{\Lambda_{\text{QCD}} \sqrt{x(1-x)}}{\sqrt{\pi} J_{1+L}(\beta_{L,k})} J_L\left(\sqrt{x(1-x)} |\mathbf{b}_\perp| \beta_{L,k} \Lambda_{\text{QCD}}\right) \theta\left(\mathbf{b}_\perp^2 \leq \frac{\Lambda_{\text{QCD}}^{-2}}{x(1-x)}\right),$$

$$\tilde{\psi}_{\bar{q}q/\pi}^{SW}(x, \mathbf{b}_\perp) = \kappa^{L+1} \sqrt{\frac{2n!}{(n+L)!}} [x(1-x)]^{\frac{1}{2}+L} |\mathbf{b}_\perp|^L e^{-\frac{1}{2}\kappa^2 x(1-x)\mathbf{b}_\perp^2} L_n^L(\kappa^2 x(1-x)\mathbf{b}_\perp^2).$$

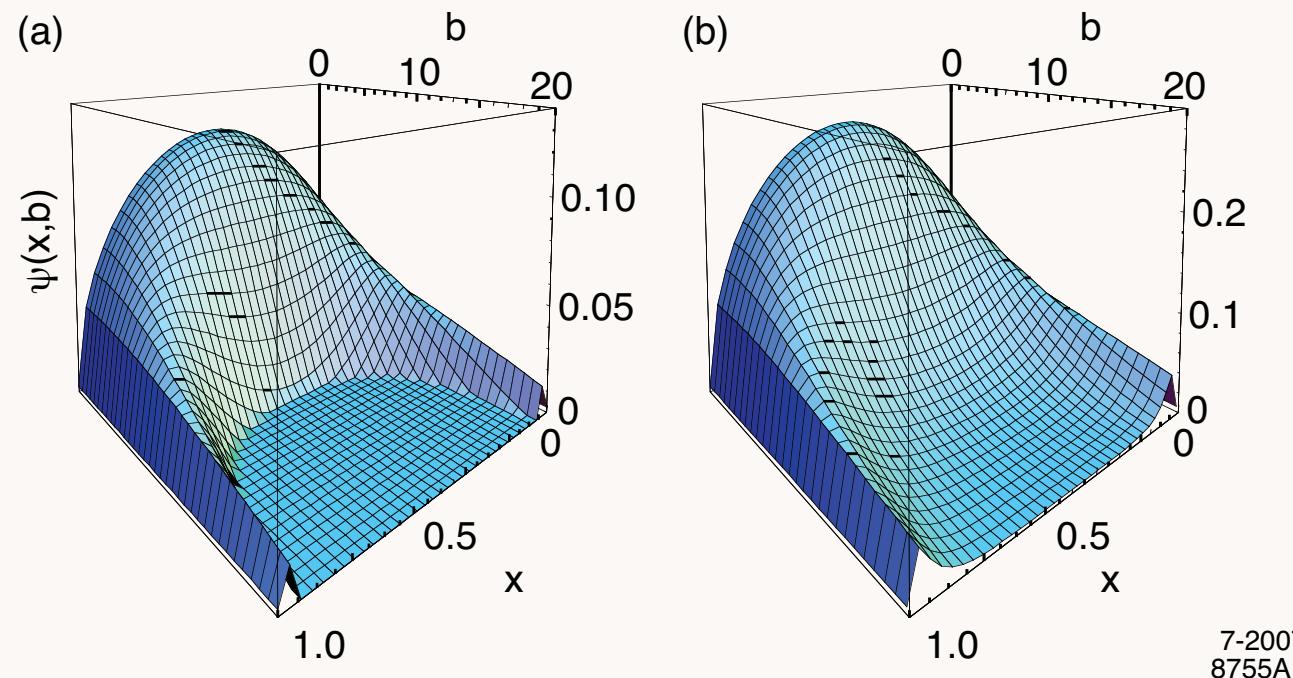
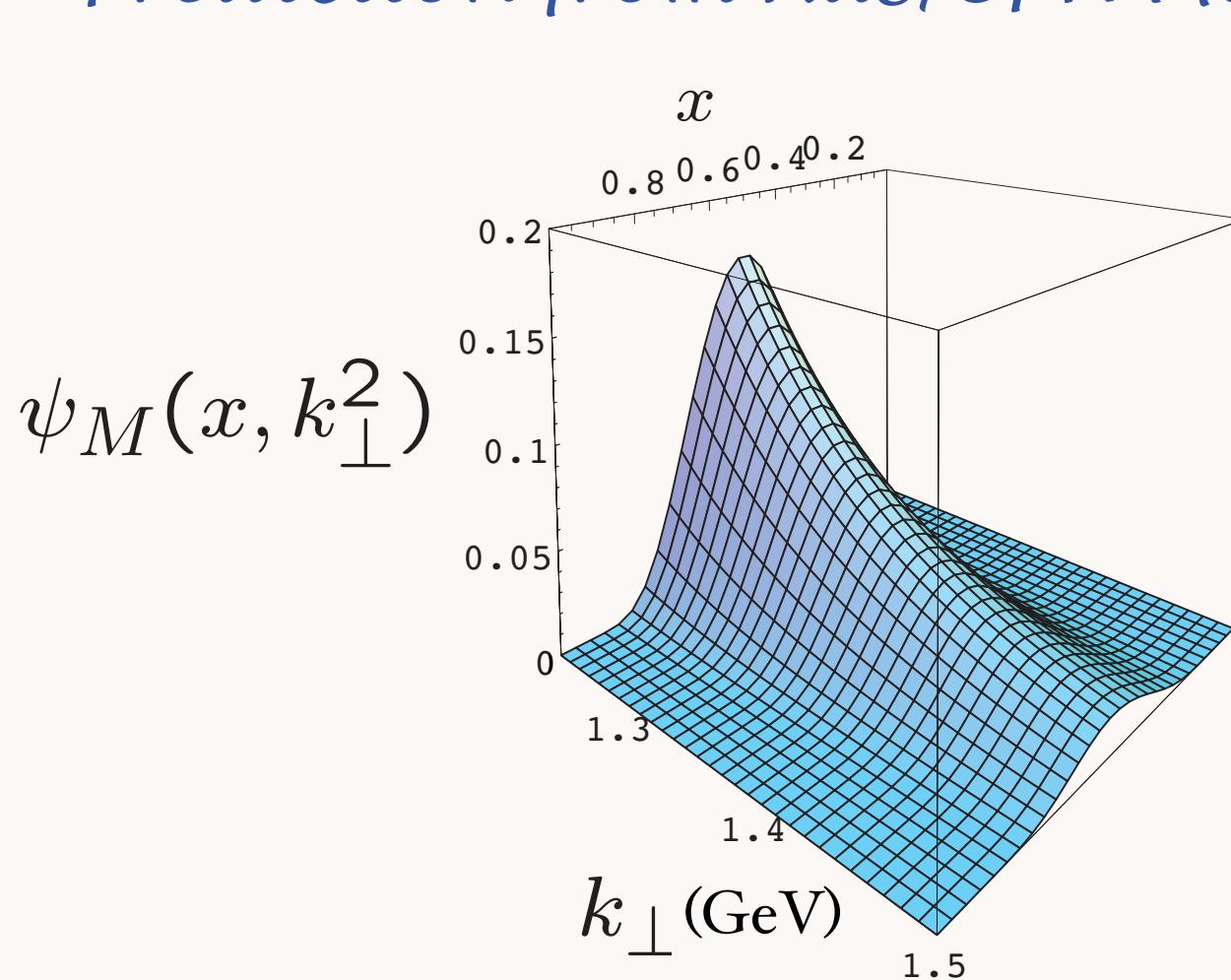


Fig: Ground state pion LFWF in impact space. (a) HW model $\Lambda_{\text{QCD}} = 0.32 \text{ GeV}$, (b) SW model $\kappa = 0.375 \text{ GeV}$.

Prediction from AdS/CFT: Meson LFWF



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**“Soft Wall”
model**

$\kappa = 0.375 \text{ GeV}$

massless quarks

$$\psi_M(x, k_\perp) = \frac{4\pi}{\kappa \sqrt{x(1-x)}} e^{-\frac{k_\perp^2}{2\kappa^2 x(1-x)}}$$

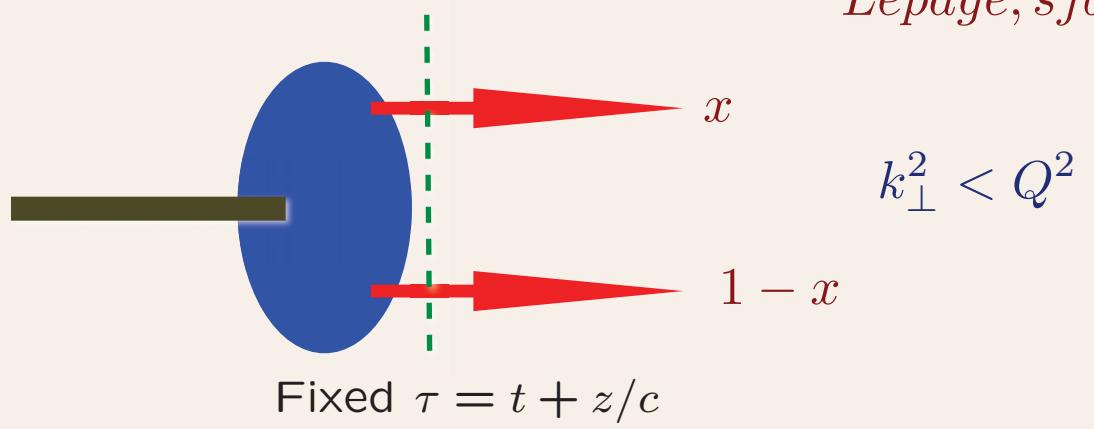
$$\phi_M(x, Q_0) \propto \sqrt{x(1-x)}$$

Hadron Distribution Amplitudes

Lepage, sjb

$$\phi_H(x_i, Q)$$

$$\sum_i x_i = 1$$



- Fundamental gauge invariant non-perturbative input to hard exclusive processes, heavy hadron decays. Defined for mesons, baryons

Lepage, sjb

- Evolution Equations from PQCD,
OPE, Conformal Invariance

Frishman, Lepage, Sachrajda, sjb

Peskin Braun

Efremov, Radyushkin Chernyak et al

- Compute from valence light-front wavefunction in
light-cone gauge

$$\phi_M(x, Q) = \int_Q d^2 \vec{k} \psi_{q\bar{q}}(x, \vec{k}_\perp)$$

Second Moment of Pion Distribution Amplitude

$$\langle \xi^2 \rangle = \int_{-1}^1 d\xi \xi^2 \phi(\xi)$$

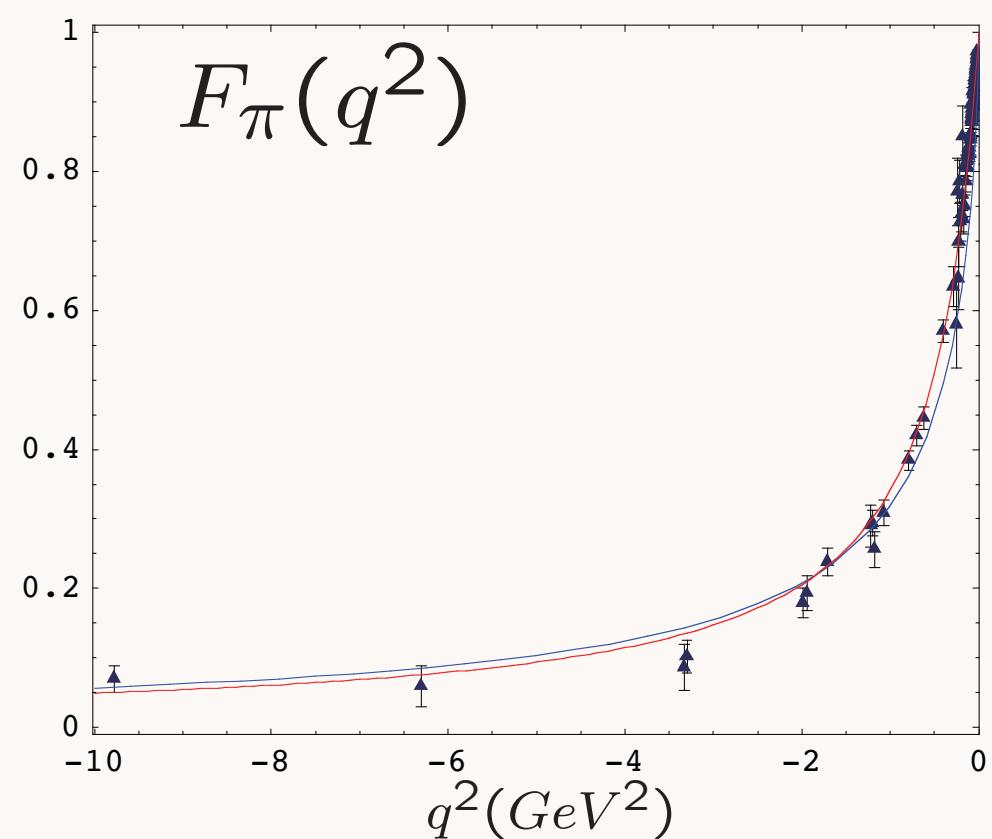
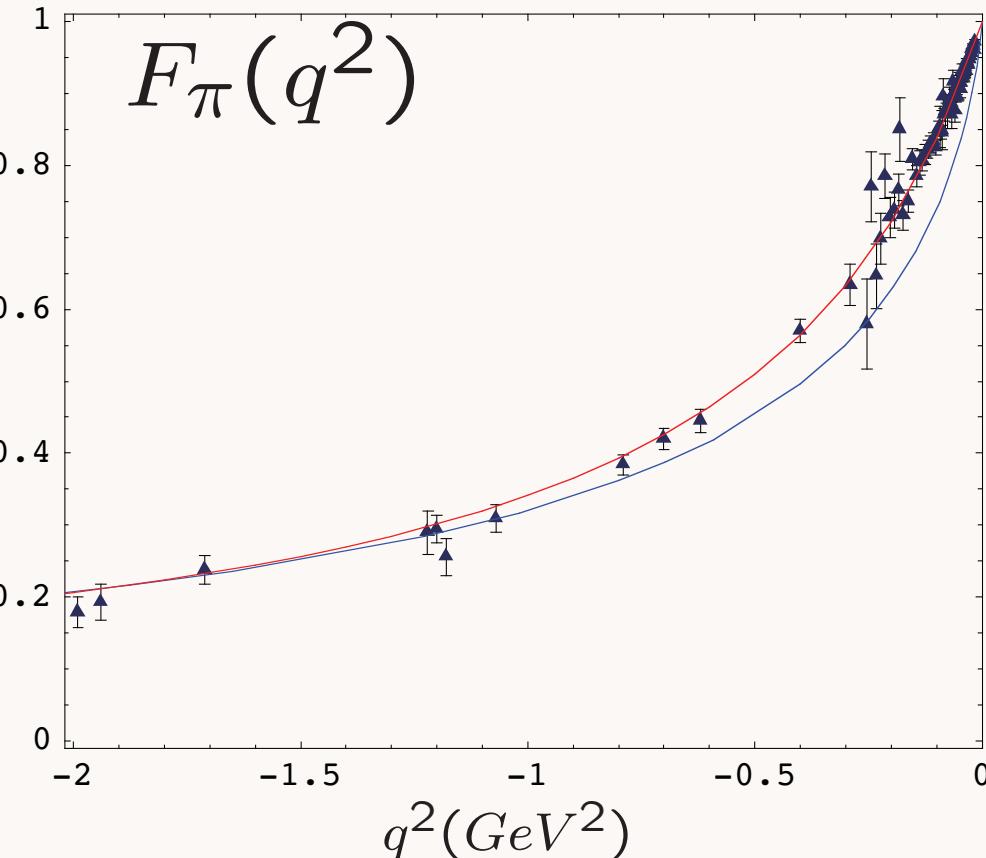
$$\xi = 1 - 2x$$

$$\langle \xi^2 \rangle = 1/5 \quad \phi_{asympt} \propto x(1-x)$$

$$\langle \xi^2 \rangle = 1/4 \quad \phi_{AdS/QCD} \propto \sqrt{x(1-x)}$$

Sachrajda Lattice: $\langle \xi^2 \rangle = 0.28 \pm 0.02$

Spacelike pion form factor from AdS/CFT



Data Compilation from Baldini, Kloe and Volmer



SW: Harmonic Oscillator Confinement



HW: Truncated Space Confinement

One parameter - set by pion decay constant

de Teramond, sjb

Note: Contributions to Mesons Form Factors at Large Q in AdS/QCD

- Write form factor in terms of an effective partonic transverse density in impact space \mathbf{b}_\perp

$$F_\pi(q^2) = \int_0^1 dx \int db^2 \tilde{\rho}(x, b, Q),$$

with $\tilde{\rho}(x, b, Q) = \pi J_0 [b Q(1 - x)] |\tilde{\psi}(x, b)|^2$ and $b = |\mathbf{b}_\perp|$.

- Contribution from $\rho(x, b, Q)$ is shifted towards small $|\mathbf{b}_\perp|$ and large $x \rightarrow 1$ as Q increases.

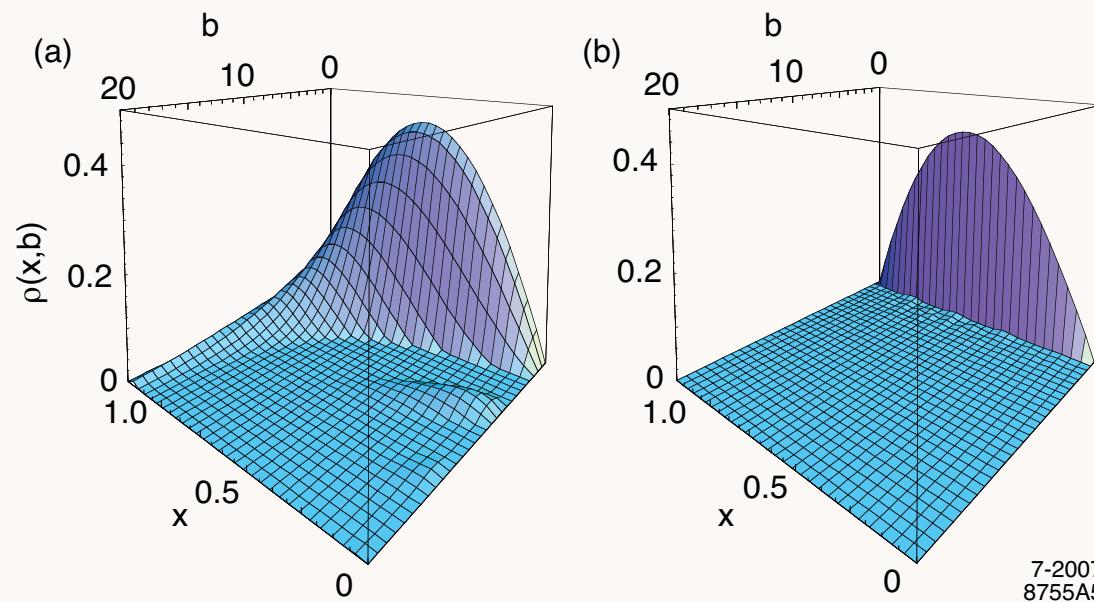


Fig: LF partonic density $\rho(x, b, Q)$: (a) $Q = 1$ GeV/c, (b) very large Q .

Example: Evaluation of QCD Matrix Elements

- Pion decay constant f_π defined by the matrix element of EW current J_W^+ :

$$\langle 0 | \bar{\psi}_u \gamma^+ \frac{1}{2} (1 - \gamma_5) \psi_d | \pi^- \rangle = i \frac{P^+ f_\pi}{\sqrt{2}}$$

with

$$|\pi^-\rangle = |d\bar{u}\rangle = \frac{1}{\sqrt{N_C}} \frac{1}{\sqrt{2}} \sum_{c=1}^{N_C} \left(b_c^\dagger d_{c\downarrow}^\dagger d_{c\uparrow}^\dagger - b_c^\dagger d_{c\uparrow}^\dagger d_{c\downarrow}^\dagger \right) |0\rangle.$$

- Find light-front expression (Lepage and Brodsky '80):

$$f_\pi = 2\sqrt{N_C} \int_0^1 dx \int \frac{d^2 \vec{k}_\perp}{16\pi^3} \psi_{\bar{q}q/\pi}(x, k_\perp).$$

- Using relation between AdS modes and QCD LFWF in the $\zeta \rightarrow 0$ limit

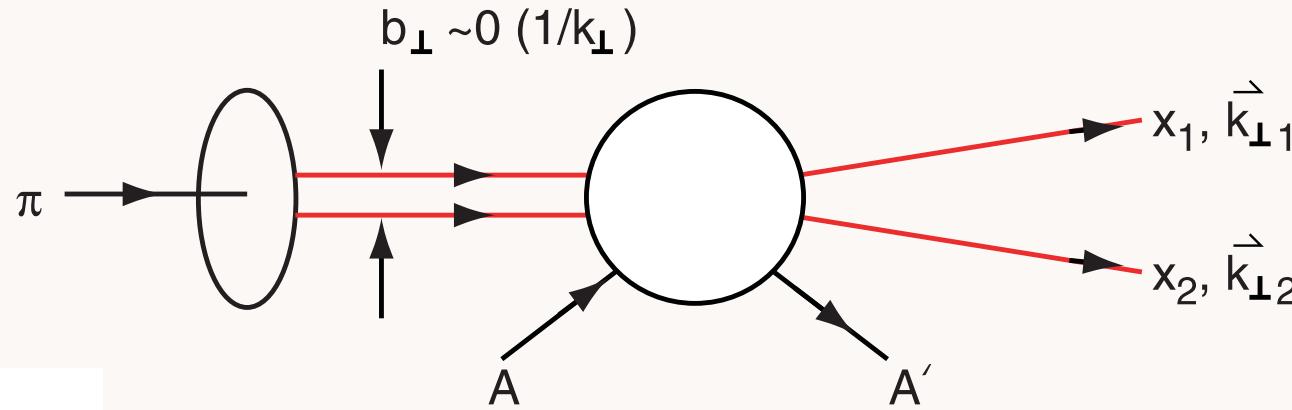
$$f_\pi = \frac{1}{8} \sqrt{\frac{3}{2}} R^{3/2} \lim_{\zeta \rightarrow 0} \frac{\Phi(\zeta)}{\zeta^2}.$$

- Holographic result ($\Lambda_{\text{QCD}} = 0.22 \text{ GeV}$ and $\kappa = 0.375 \text{ GeV}$ from pion FF data): Exp: $f_\pi = 92.4 \text{ MeV}$

$$f_\pi^{HW} = \frac{\sqrt{3}}{8J_1(\beta_{0,k})} \Lambda_{\text{QCD}} = 91.7 \text{ MeV}, \quad f_\pi^{SW} = \frac{\sqrt{3}}{8} \kappa = 81.2 \text{ MeV},$$

Diffractive Dissociation of Pion into Quark Jets

E791 Ashery et al.



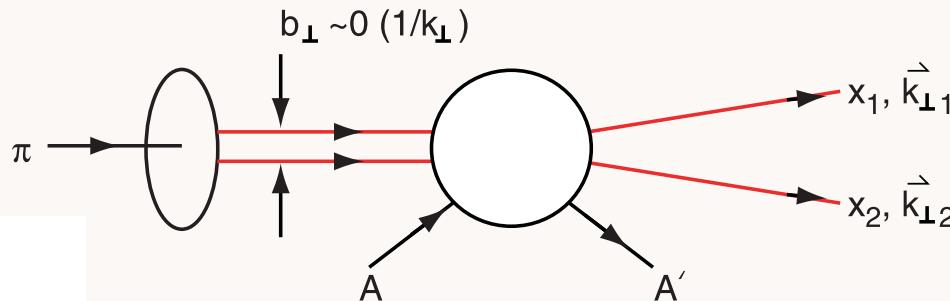
$$M \propto \frac{\partial^2}{\partial^2 k_\perp} \psi_\pi(x, k_\perp)$$

Measure Light-Front Wavefunction of Pion

Minimal momentum transfer to nucleus

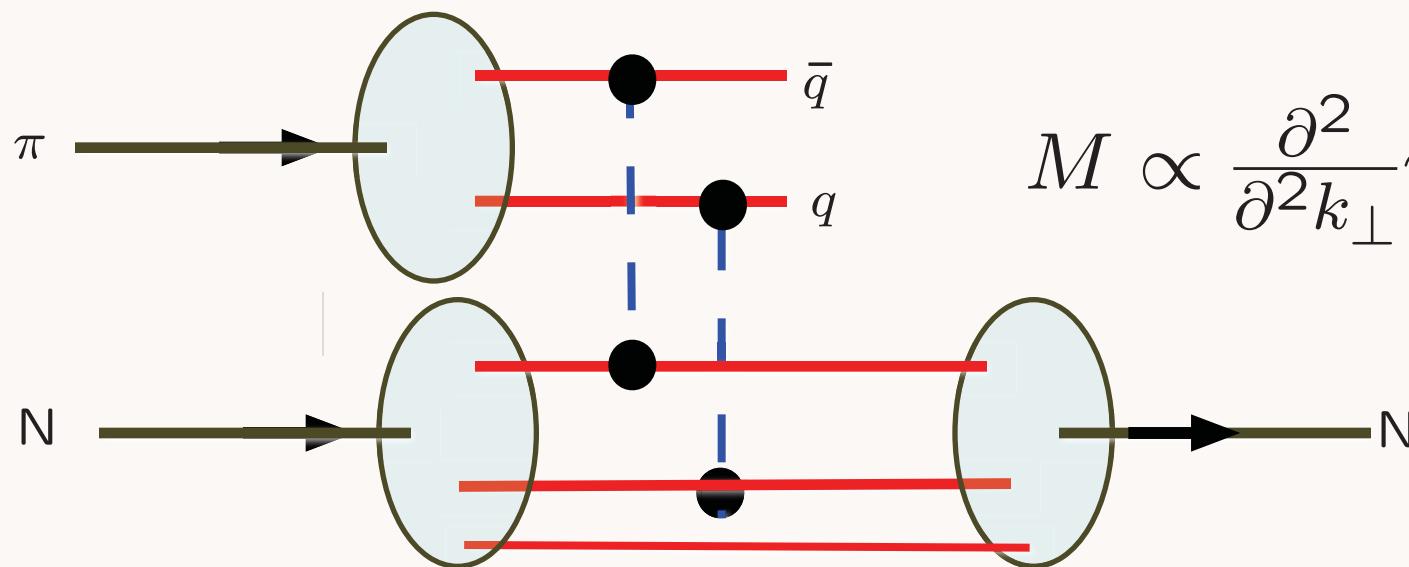
Nucleus left Intact!

E791 FNAL Diffractive DiJet

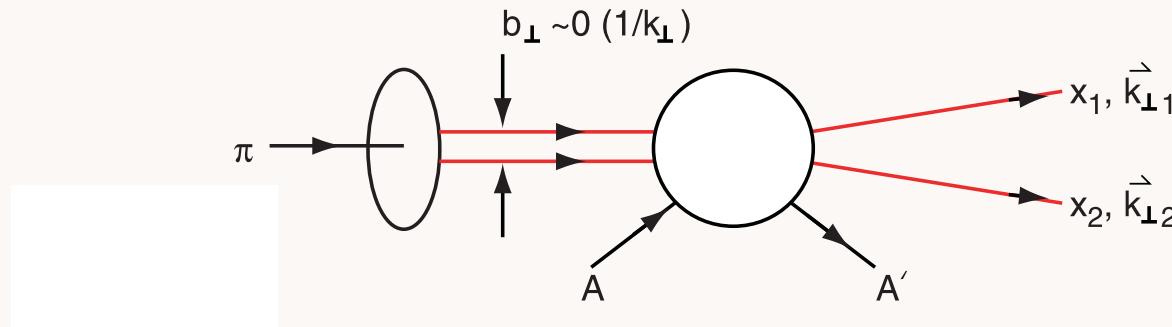


Gunion, Frankfurt, Mueller, Strikman, sjb
 Frankfurt, Miller, Strikman

Two-gluon exchange measures the second derivative of the pion light-front wavefunction



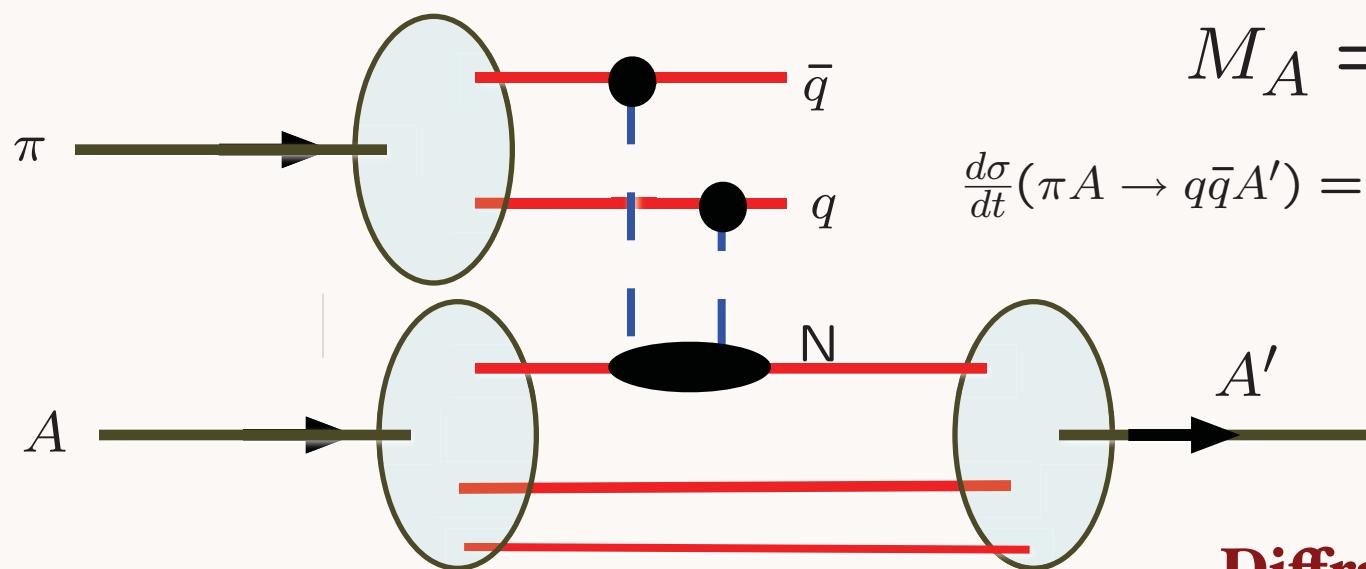
Key Ingredients in E791 Experiment



Brodsky Mueller
Frankfurt Miller Strikman

*Small color-dipole moment pion not absorbed;
interacts with each nucleon coherently*

QCD COLOR Transparency



$$M_A = A M_N$$

$$\frac{d\sigma}{dt}(\pi A \rightarrow q\bar{q}A') = A^2 \frac{d\sigma}{dt}(\pi N \rightarrow q\bar{q}N') F_A^2(t)$$

Target left intact

Diffraction, Rapidity gap

Color Transparency

Bertsch, Gunion, Goldhaber, sjb
A. H. Mueller, sjb

- Fundamental test of gauge theory in hadron physics
- Small color dipole moments interact weakly in nuclei
- Complete coherence at high energies
- Clear Demonstration of CT from Diffractive Di-Jets

Measure pion LFWF in diffractive dijet production Confirmation of color transparency

A-Dependence results: $\sigma \propto A^\alpha$

<u>k_t range (GeV/c)</u>	<u>α</u>	<u>α (CT)</u>	
$1.25 < k_t < 1.5$	$1.64 +0.06 -0.12$	1.25	
$1.5 < k_t < 2.0$	1.52 ± 0.12	1.45	
$2.0 < k_t < 2.5$	1.55 ± 0.16	1.60	
<hr/>			Ashery E791
<hr/> α (Incoh.) = 0.70 ± 0.1			

Conventional Glauber Theory Ruled
Out !

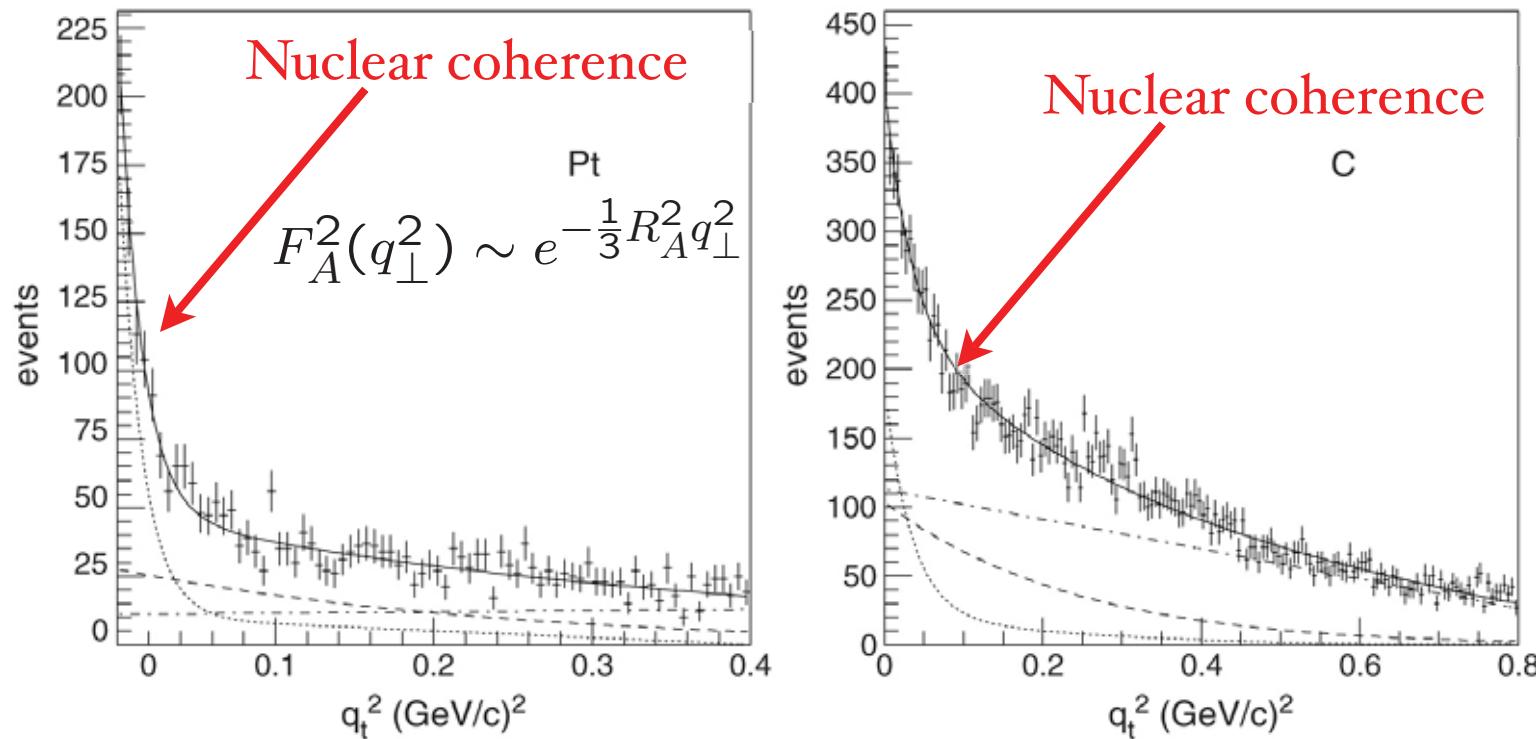
Factor of 7

- Fully coherent interactions between pion and nucleons.
- Emerging Di-Jets do not interact with nucleus.

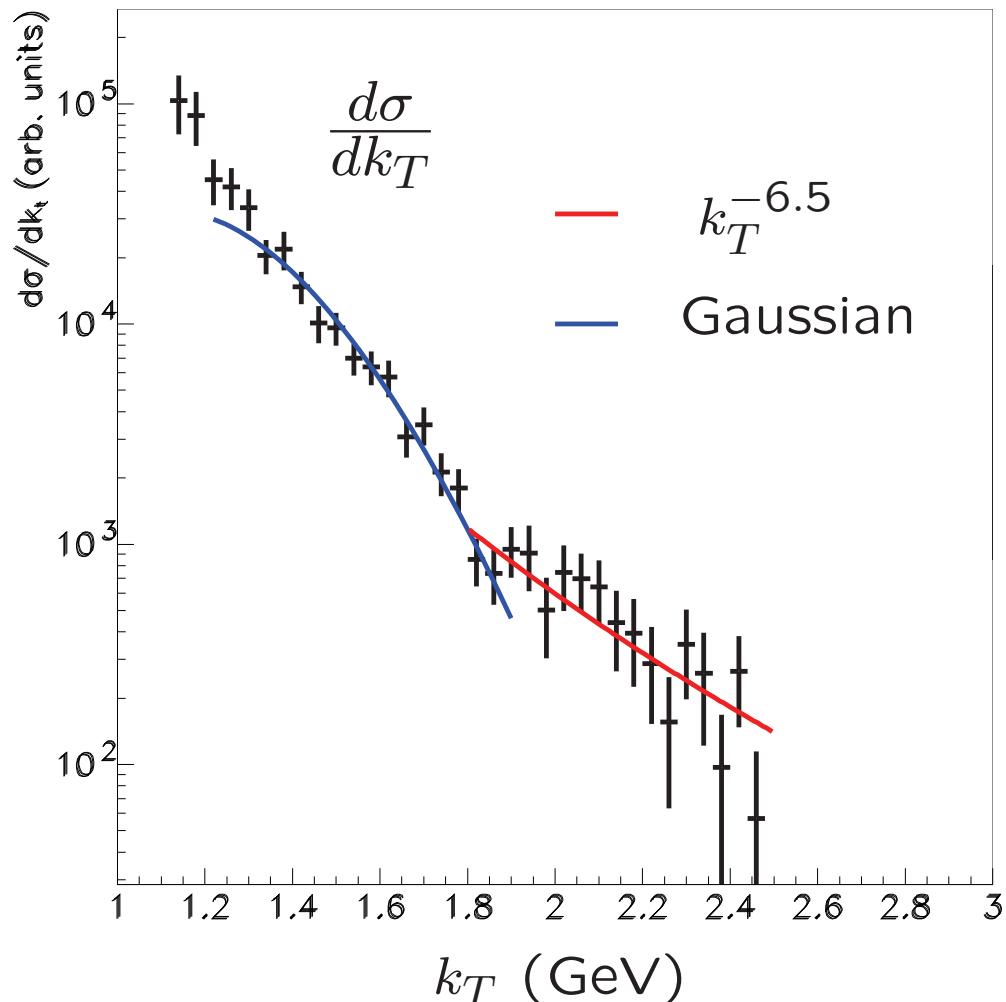
$$\mathcal{M}(\mathcal{A}) = \mathcal{A} \cdot \mathcal{M}(\mathcal{N})$$

$$\frac{d\sigma}{dq_t^2} \propto A^2 \quad q_t^2 \sim 0$$

$$\sigma \propto A^{4/3}$$



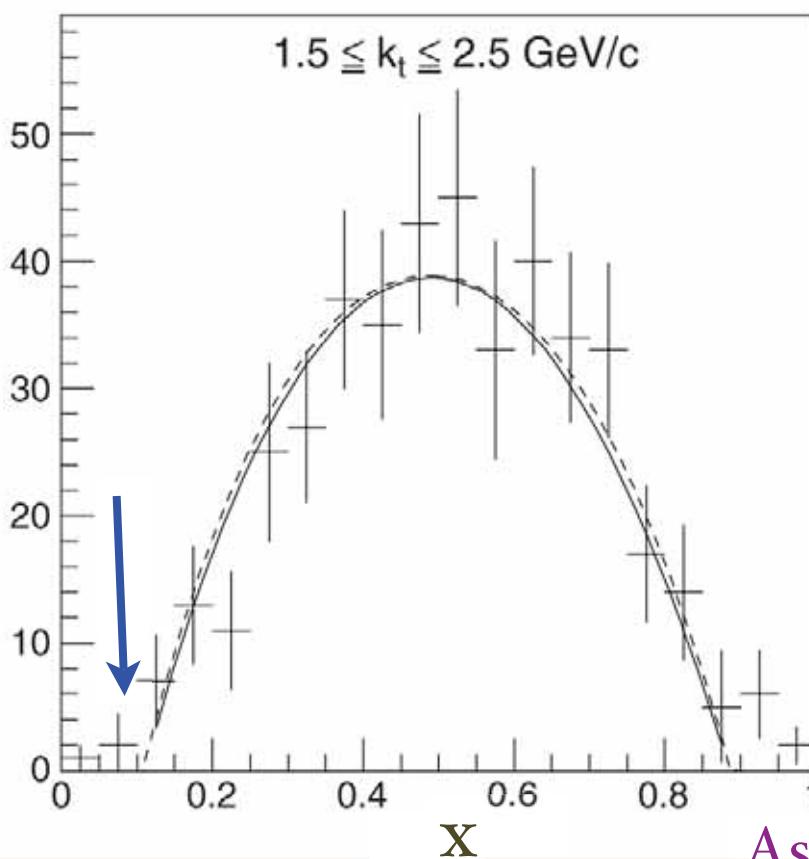
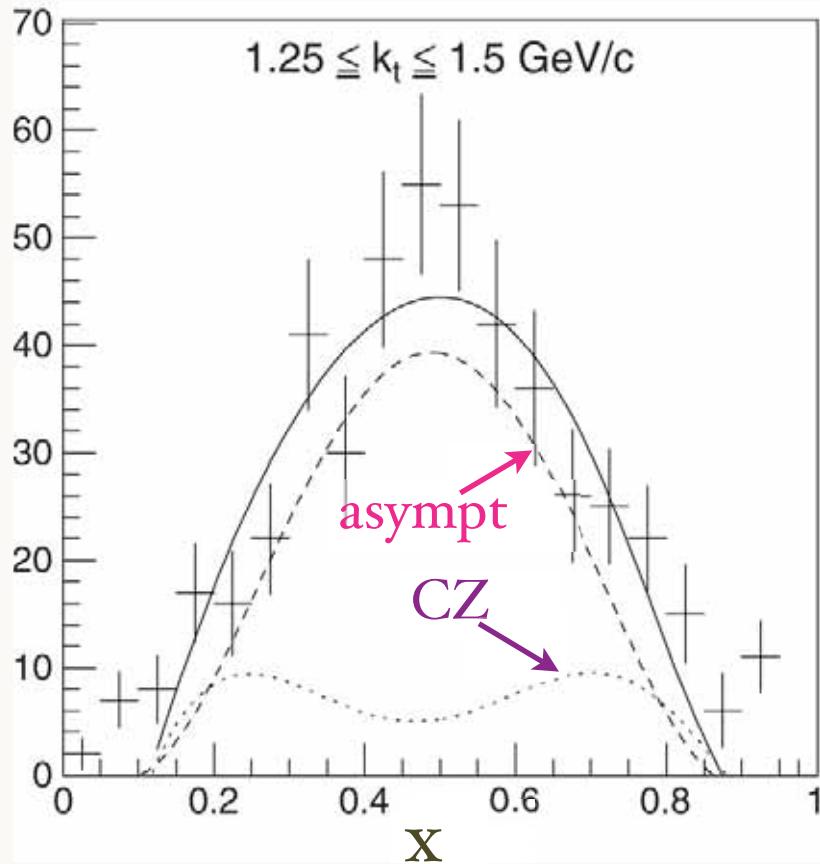
E791 Diffractive Di-Jet transverse momentum distribution



Two Components

High Transverse momentum dependence $k_T^{-6.5}$
consistent with PQCD,
ERBL Evolution

Gaussian component similar
to AdS/CFT HO LFWF



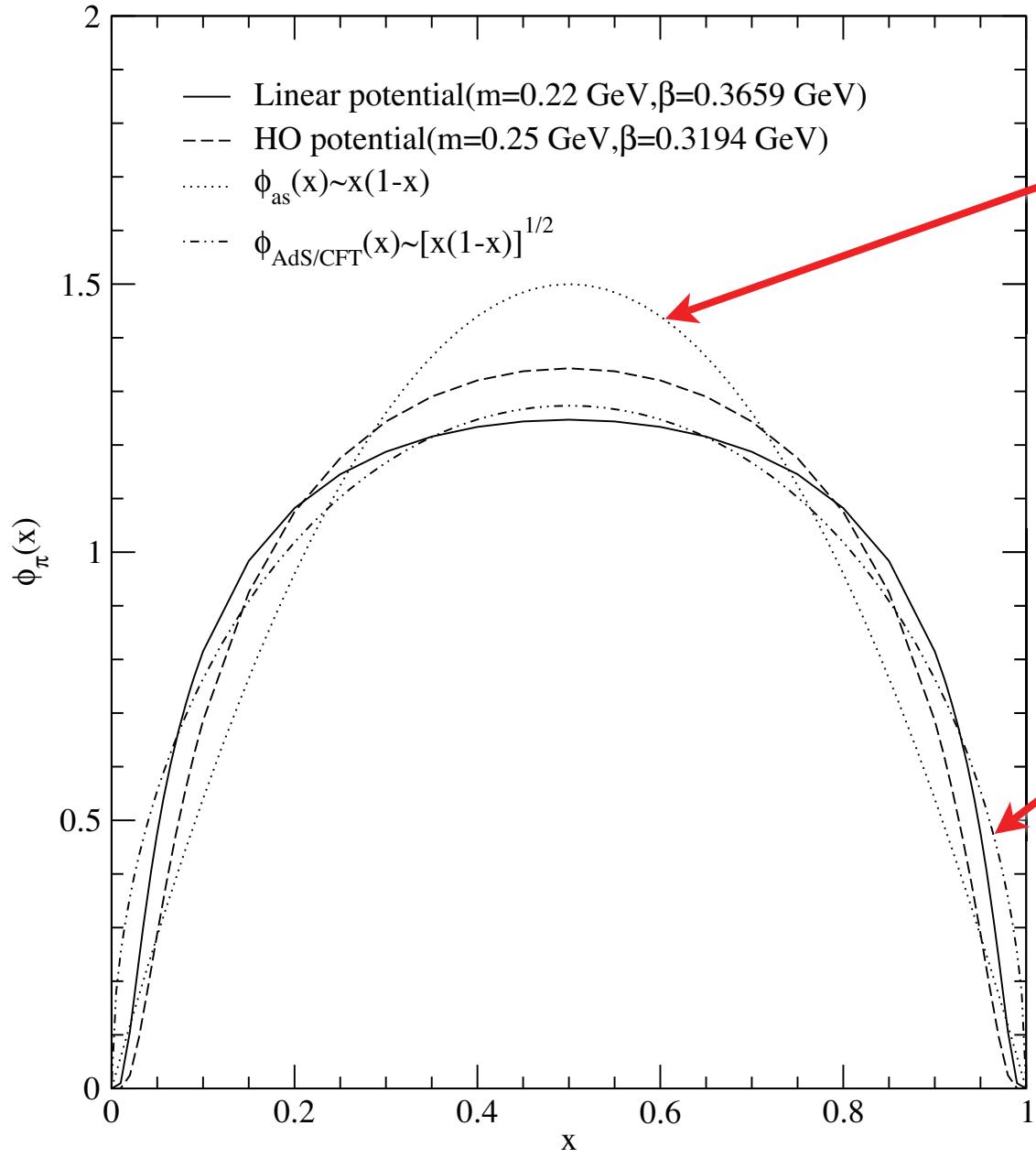
Ashery E791

Narrowing of x distribution at higher jet transverse momentum

x distribution of diffractive dijets from the platinum target for $1.25 \leq k_t \leq 1.5 \text{ GeV}/c$ (left) and for $1.5 \leq k_t \leq 2.5 \text{ GeV}/c$ (right). The solid line is a fit to a combination of the asymptotic and CZ distribution amplitudes. The dashed line shows the contribution from the asymptotic function and the dotted line that of the CZ function.

Possibly two components:
Nonperturbative (AdS/CFT) and
Perturbative (ERBL)
Evolution to asymptotic distribution

$$\phi(x) \propto \sqrt{x(1-x)}$$



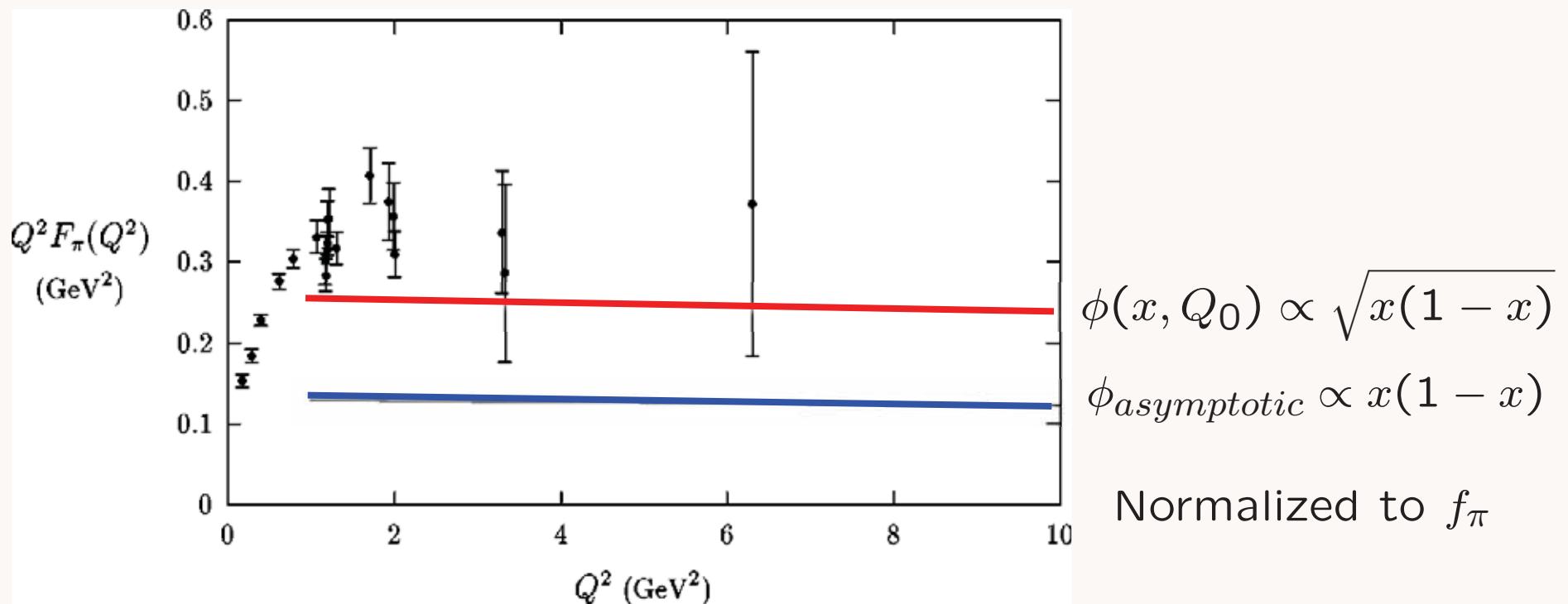
$$\phi_{asympt} \sim x(1 - x)$$

AdS/CFT:

$$\phi(x, Q_0) \propto \sqrt{x(1 - x)}$$

Increases PQCD leading twist prediction
 $F_\pi(Q^2)$ by factor $16/9$

$$F_\pi(Q^2) = \int_0^1 dx \phi_\pi(x) \int_0^1 dy \phi_\pi(y) \frac{16\pi C_F \alpha_V(Q_V)}{(1-x)(1-y)Q^2}$$

**AdS/CFT:**

Increases PQCD leading twist prediction for $F_\pi(Q^2)$ by factor 16/9

Higher Spin Bosonic Modes HW

- Each hadronic state of integer spin $S \leq 2$ is dual to a normalizable string mode

$$\Phi(x, z)_{\mu_1 \mu_2 \dots \mu_S} = \epsilon_{\mu_1 \mu_2 \dots \mu_S} e^{-iP \cdot x} \Phi_S(z).$$

with four-momentum P_μ and spin polarization indices along the 3+1 physical coordinates.

- Wave equation for spin S -mode [W. S. I'Yi, Phys. Lett. B 448, 218 \(1999\)](#)

$$[z^2 \partial_z^2 - (d+1-2S)z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2] \Phi_S(z) = 0,$$

- Solution

$$\tilde{\Phi}(z)_S = \left(\frac{z}{R}\right)^S \Phi(z)_S = C e^{-iP \cdot x} z^{\frac{d}{2}} J_{\Delta - \frac{d}{2}}(z \mathcal{M}) \epsilon(P)_{\mu_1 \mu_2 \dots \mu_S},$$

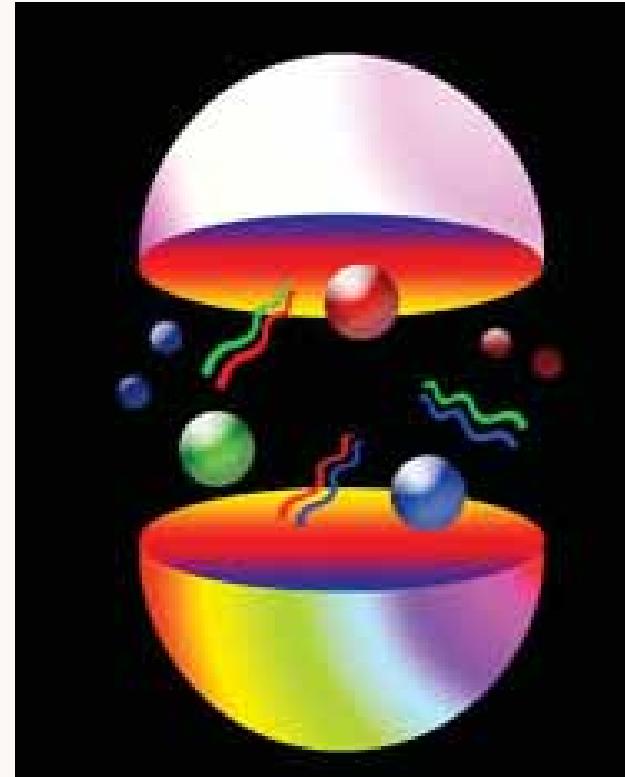
- We can identify the conformal dimension:

$$\Delta = \frac{1}{2} (d + \sqrt{(d-2S)^2 + 4\mu^2 R^2}).$$

- Normalization:

$$R^{d-2S-1} \int_0^{\Lambda_{\text{QCD}}^{-1}} \frac{dz}{z^{d-2S-1}} \Phi_S^2(z) = 1.$$

Baryons in AdS/CFT

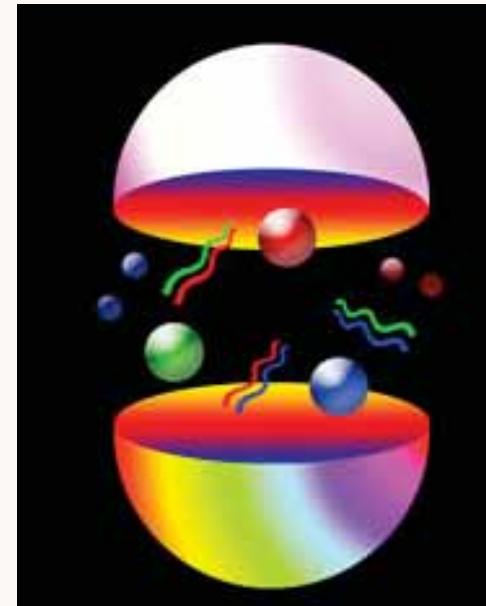


**Baryons in “bottom-up” approach to holographic QCD:
{ GdT and SJB (2004) }.**

**“Top-down” Sakai-Sugimoto model:
{ Hong, Rho, Yee and Yi (2007);
Hata, Sakai, Sugimoto, Yamato (2007) }.**

- Baryons Spectrum in "bottom-up" holographic QCD
GdT and Brodsky: hep-th/0409074, hep-th/0501022.

Baryons in AdS/CFT



- Action for massive fermionic modes on AdS_{d+1} :

$$S[\bar{\Psi}, \Psi] = \int d^{d+1}x \sqrt{g} \bar{\Psi}(x, z) \left(i\Gamma^\ell D_\ell - \mu \right) \Psi(x, z).$$

- Equation of motion: $(i\Gamma^\ell D_\ell - \mu) \Psi(x, z) = 0$

$$\left[i \left(z\eta^{\ell m} \Gamma_\ell \partial_m + \frac{d}{2} \Gamma_z \right) + \mu R \right] \Psi(x^\ell) = 0.$$

Baryons

Holographic Light-Front Integrable Form and Spectrum

- In the conformal limit fermionic spin- $\frac{1}{2}$ modes $\psi(\zeta)$ and spin- $\frac{3}{2}$ modes $\psi_\mu(\zeta)$ are **two-component spinor** solutions of the Dirac light-front equation

$$\alpha\Pi(\zeta)\psi(\zeta) = \mathcal{M}\psi(\zeta),$$

where $H_{LF} = \alpha\Pi$ and the operator

$$\Pi_L(\zeta) = -i \left(\frac{d}{d\zeta} - \frac{L + \frac{1}{2}}{\zeta} \gamma_5 \right),$$

and its adjoint $\Pi_L^\dagger(\zeta)$ satisfy the commutation relations

$$[\Pi_L(\zeta), \Pi_L^\dagger(\zeta)] = \frac{2L+1}{\zeta^2} \gamma_5.$$

- Supersymmetric QM between bosonic and fermionic modes in AdS?

- Note: in the Weyl representation ($i\alpha = \gamma_5\beta$)

$$i\alpha = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

- Baryon: twist-dimension $3 + L$ ($\nu = L + 1$)

$$\mathcal{O}_{3+L} = \psi D_{\{\ell_1 \dots D_{\ell_q} \psi D_{\ell_{q+1}} \dots D_{\ell_m}\}} \psi, \quad L = \sum_{i=1}^m \ell_i.$$

- Solution to Dirac eigenvalue equation with UV matching boundary conditions

$$\psi(\zeta) = C\sqrt{\zeta} [J_{L+1}(\zeta\mathcal{M})u_+ + J_{L+2}(\zeta\mathcal{M})u_-].$$

Baryonic modes propagating in AdS space have two components: orbital L and $L + 1$.

- Hadronic mass spectrum determined from IR boundary conditions

$$\psi_{\pm}(\zeta = 1/\Lambda_{\text{QCD}}) = 0,$$

given by

$$\mathcal{M}_{\nu,k}^+ = \beta_{\nu,k}\Lambda_{\text{QCD}}, \quad \mathcal{M}_{\nu,k}^- = \beta_{\nu+1,k}\Lambda_{\text{QCD}},$$

with a scale independent mass ratio.

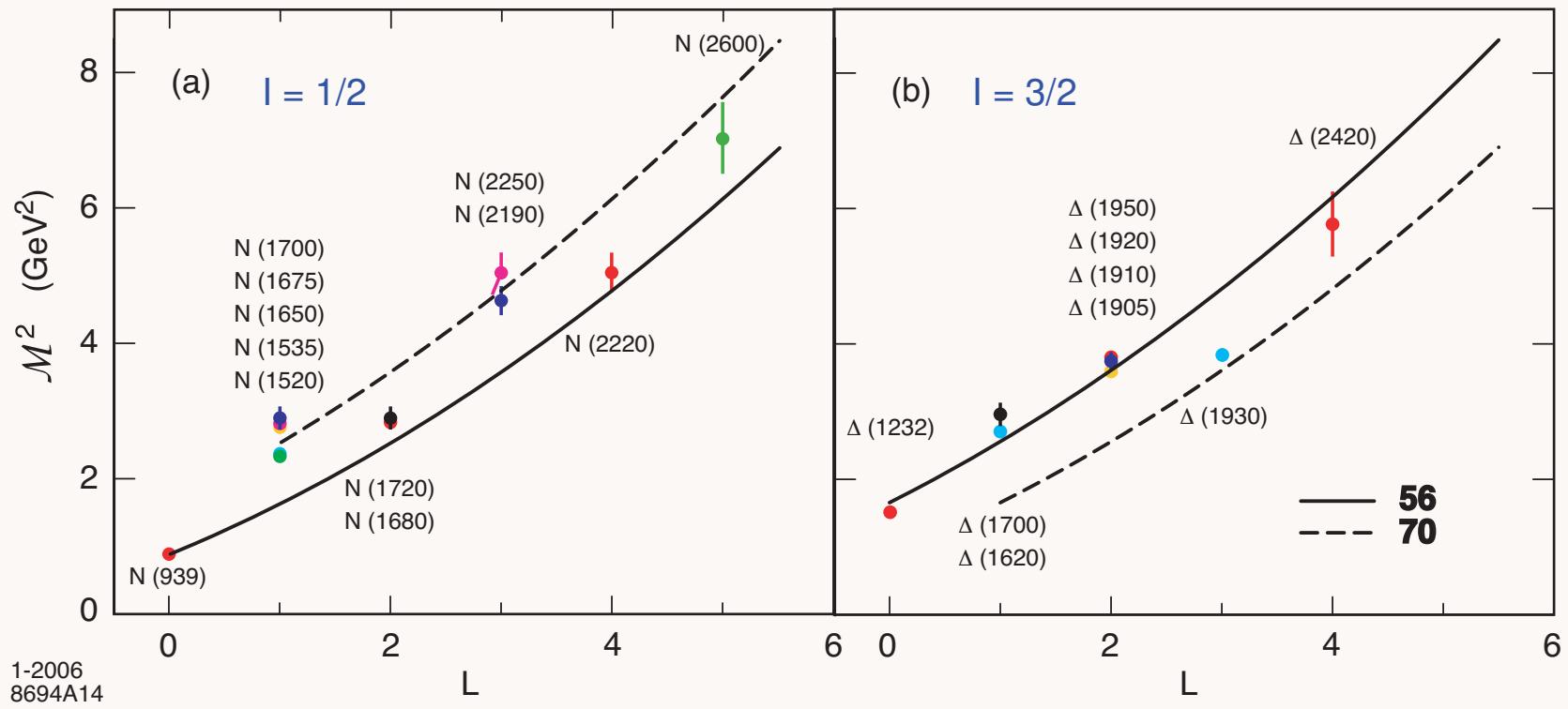


Fig: Light baryon orbital spectrum for $\Lambda_{QCD} = 0.25$ GeV in the HW model. The **56** trajectory corresponds to L even $P = +$ states, and the **70** to L odd $P = -$ states.

$SU(6)$	S	L	Baryon State			
56	$\frac{1}{2}$	0				$N \frac{1}{2}^+(939)$
	$\frac{3}{2}$	0				$\Delta \frac{3}{2}^+(1232)$
70	$\frac{1}{2}$	1			$N \frac{1}{2}^-(1535)$	$N \frac{3}{2}^-(1520)$
	$\frac{3}{2}$	1			$N \frac{1}{2}^-(1650)$	$N \frac{3}{2}^-(1700)$
	$\frac{1}{2}$	1			$\Delta \frac{1}{2}^-(1620)$	$\Delta \frac{3}{2}^-(1700)$
56	$\frac{1}{2}$	2			$N \frac{3}{2}^+(1720)$	$N \frac{5}{2}^+(1680)$
	$\frac{3}{2}$	2			$\Delta \frac{1}{2}^+(1910)$	$\Delta \frac{3}{2}^+(1920)$
70	$\frac{1}{2}$	3			$N \frac{5}{2}^-$	$N \frac{7}{2}^-$
	$\frac{3}{2}$	3			$N \frac{5}{2}^-$	$N \frac{7}{2}^-(2190)$
	$\frac{1}{2}$	3			$\Delta \frac{5}{2}^-(1930)$	$\Delta \frac{7}{2}^-$
56	$\frac{1}{2}$	4			$N \frac{7}{2}^+$	$N \frac{9}{2}^+(2220)$
	$\frac{3}{2}$	4			$\Delta \frac{5}{2}^+$	$\Delta \frac{7}{2}^+$
70	$\frac{1}{2}$	5			$\Delta \frac{9}{2}^+$	$\Delta \frac{11}{2}^+(2420)$
	$\frac{3}{2}$	5			$N \frac{9}{2}^-$	$N \frac{11}{2}^-(2600)$
					$N \frac{7}{2}^-$	$N \frac{9}{2}^-$
					$N \frac{11}{2}^-$	$N \frac{13}{2}^-$

Non-Conformal Extension of Algebraic Structure (Soft Wall Model)

- We write the Dirac equation

$$(\alpha \Pi(\zeta) - \mathcal{M}) \psi(\zeta) = 0,$$

in terms of the matrix-valued operator Π

$$\Pi_\nu(\zeta) = -i \left(\frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} \gamma_5 - \kappa^2 \zeta \gamma_5 \right),$$

and its adjoint Π^\dagger , with commutation relations

$$[\Pi_\nu(\zeta), \Pi_\nu^\dagger(\zeta)] = \left(\frac{2\nu + 1}{\zeta^2} - 2\kappa^2 \right) \gamma_5.$$

- Solutions to the Dirac equation

$$\begin{aligned} \psi_+(\zeta) &\sim z^{\frac{1}{2}+\nu} e^{-\kappa^2 \zeta^2/2} L_n^\nu(\kappa^2 \zeta^2), \\ \psi_-(\zeta) &\sim z^{\frac{3}{2}+\nu} e^{-\kappa^2 \zeta^2/2} L_n^{\nu+1}(\kappa^2 \zeta^2). \end{aligned}$$

- Eigenvalues

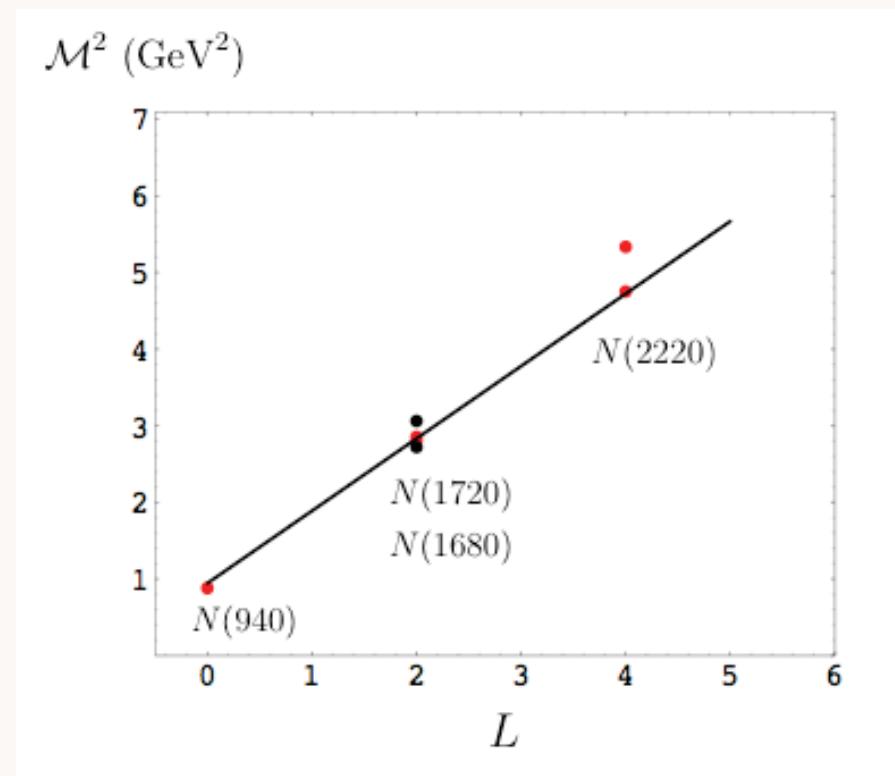
$$\mathcal{M}^2 = 4\kappa^2(n + \nu + 1).$$

- Baryon: twist-dimension $3 + L$ ($\nu = L + 1$)

$$\mathcal{O}_{3+L} = \psi D_{\{\ell_1 \dots D_{\ell_q}} \psi D_{\ell_{q+1} \dots D_{\ell_m}\}} \psi, \quad L = \sum_{i=1}^m \ell_i.$$

- Define the zero point energy (identical as in the meson case) $\mathcal{M}^2 \rightarrow \mathcal{M}^2 - 4\kappa^2$:

$$\mathcal{M}^2 = 4\kappa^2(n + L + 1).$$



Proton Regge Trajectory $\kappa = 0.49\text{GeV}$

Space-Like Dirac Proton Form Factor

- Consider the spin non-flip form factors

$$F_+(Q^2) = g_+ \int d\zeta J(Q, \zeta) |\psi_+(\zeta)|^2,$$

$$F_-(Q^2) = g_- \int d\zeta J(Q, \zeta) |\psi_-(\zeta)|^2,$$

where the effective charges g_+ and g_- are determined from the spin-flavor structure of the theory.

- Choose the struck quark to have $S^z = +1/2$. The two AdS solutions $\psi_+(\zeta)$ and $\psi_-(\zeta)$ correspond to nucleons with $J^z = +1/2$ and $-1/2$.
- For $SU(6)$ spin-flavor symmetry

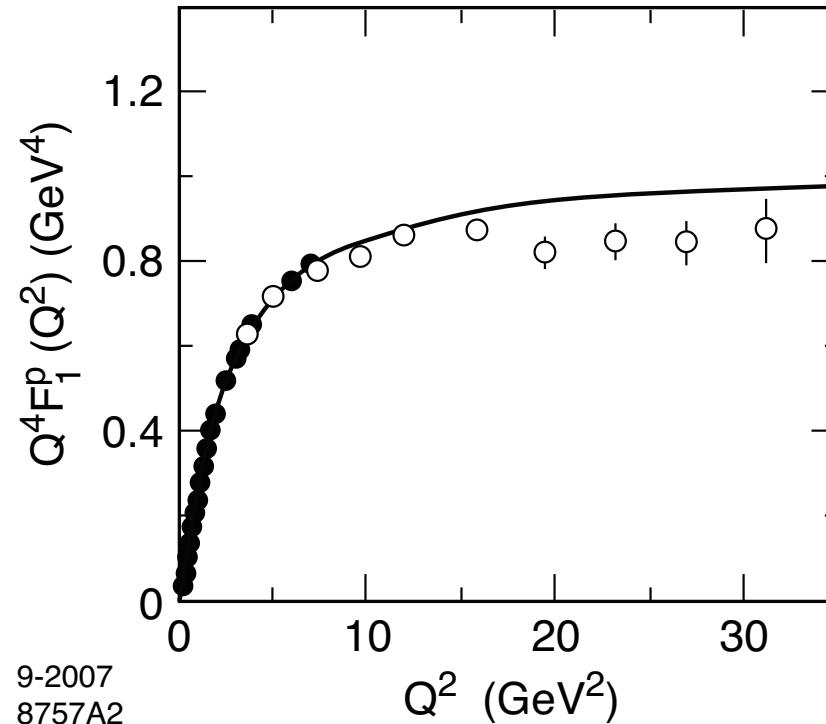
$$F_1^p(Q^2) = \int d\zeta J(Q, \zeta) |\psi_+(\zeta)|^2,$$

$$F_1^n(Q^2) = -\frac{1}{3} \int d\zeta J(Q, \zeta) [|\psi_+(\zeta)|^2 - |\psi_-(\zeta)|^2],$$

where $F_1^p(0) = 1$, $F_1^n(0) = 0$.

- Scaling behavior for large Q^2 : $Q^4 F_1^p(Q^2) \rightarrow \text{constant}$

Proton $\tau = 3$

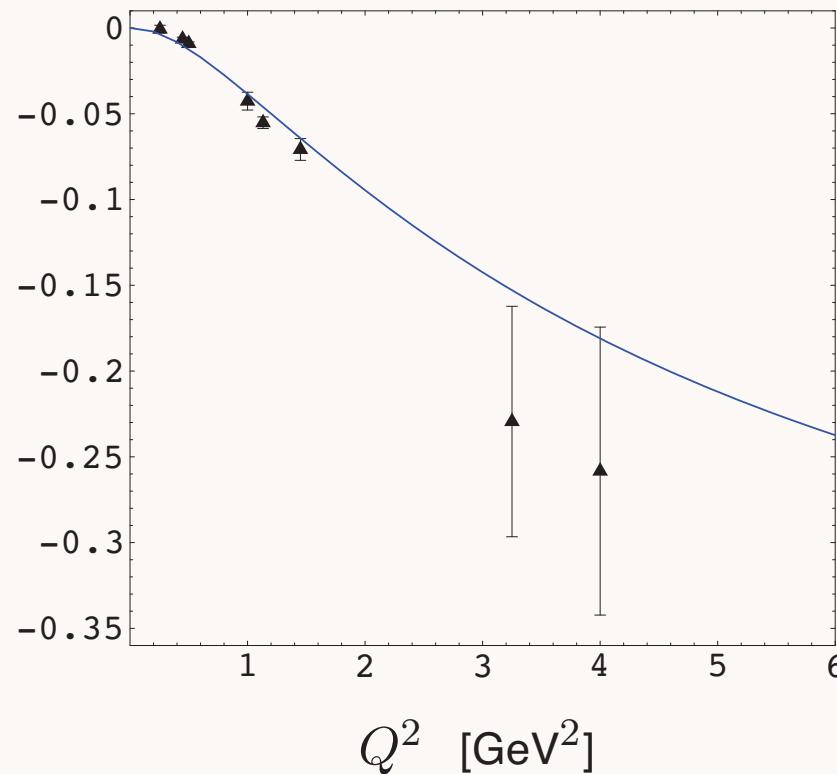


SW model predictions for $\kappa = 0.424 \text{ GeV}$. Data analysis from: M. Diehl *et al.* Eur. Phys. J. C **39**, 1 (2005)

Dirac Neutron Form Factor (Valence Approximation)

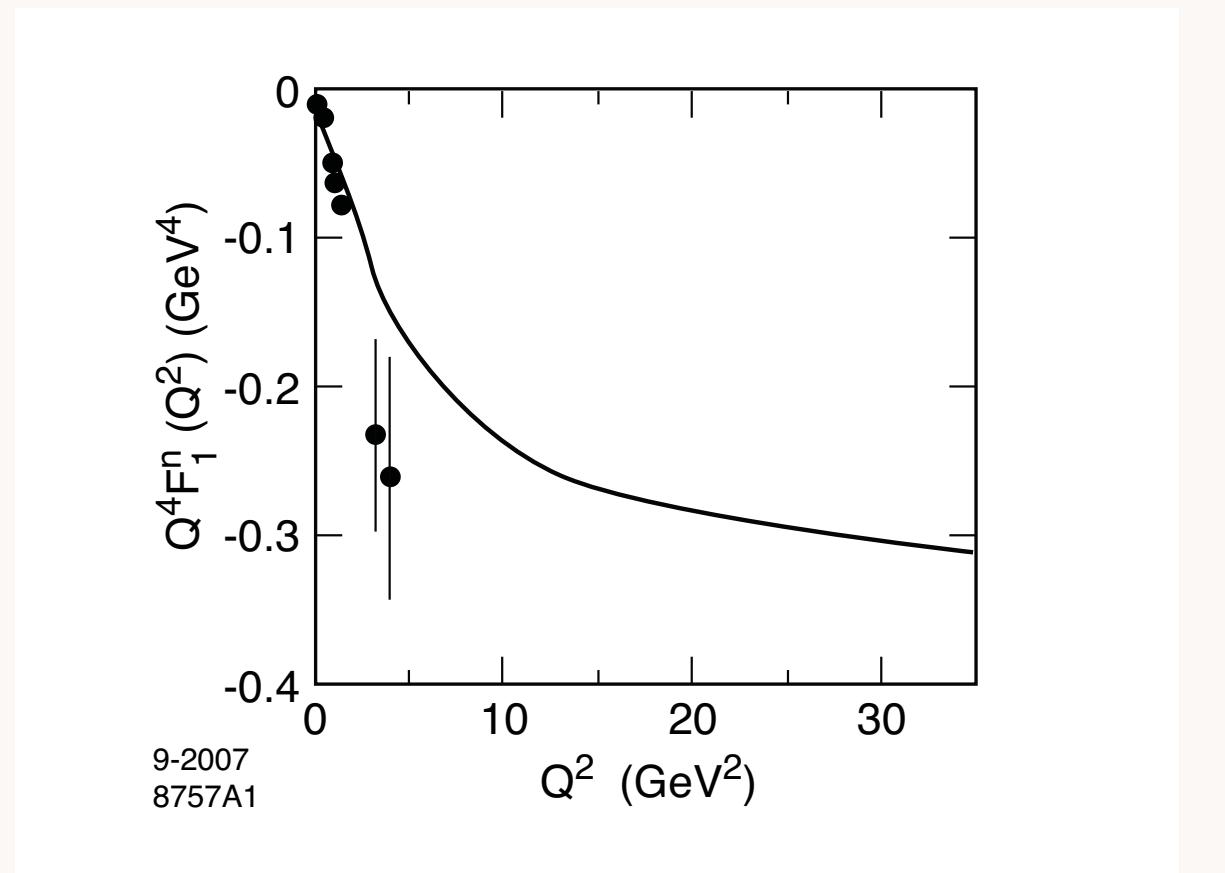
Truncated Space Confinement

$$Q^4 F_1^n(Q^2) \text{ [GeV}^4]$$



Prediction for $Q^4 F_1^n(Q^2)$ for $\Lambda_{\text{QCD}} = 0.21$ GeV in the hard wall approximation. Data analysis from Diehl (2005).

- Scaling behavior for large Q^2 : $Q^4 F_1^n(Q^2) \rightarrow \text{constant}$ Neutron $\tau = 3$

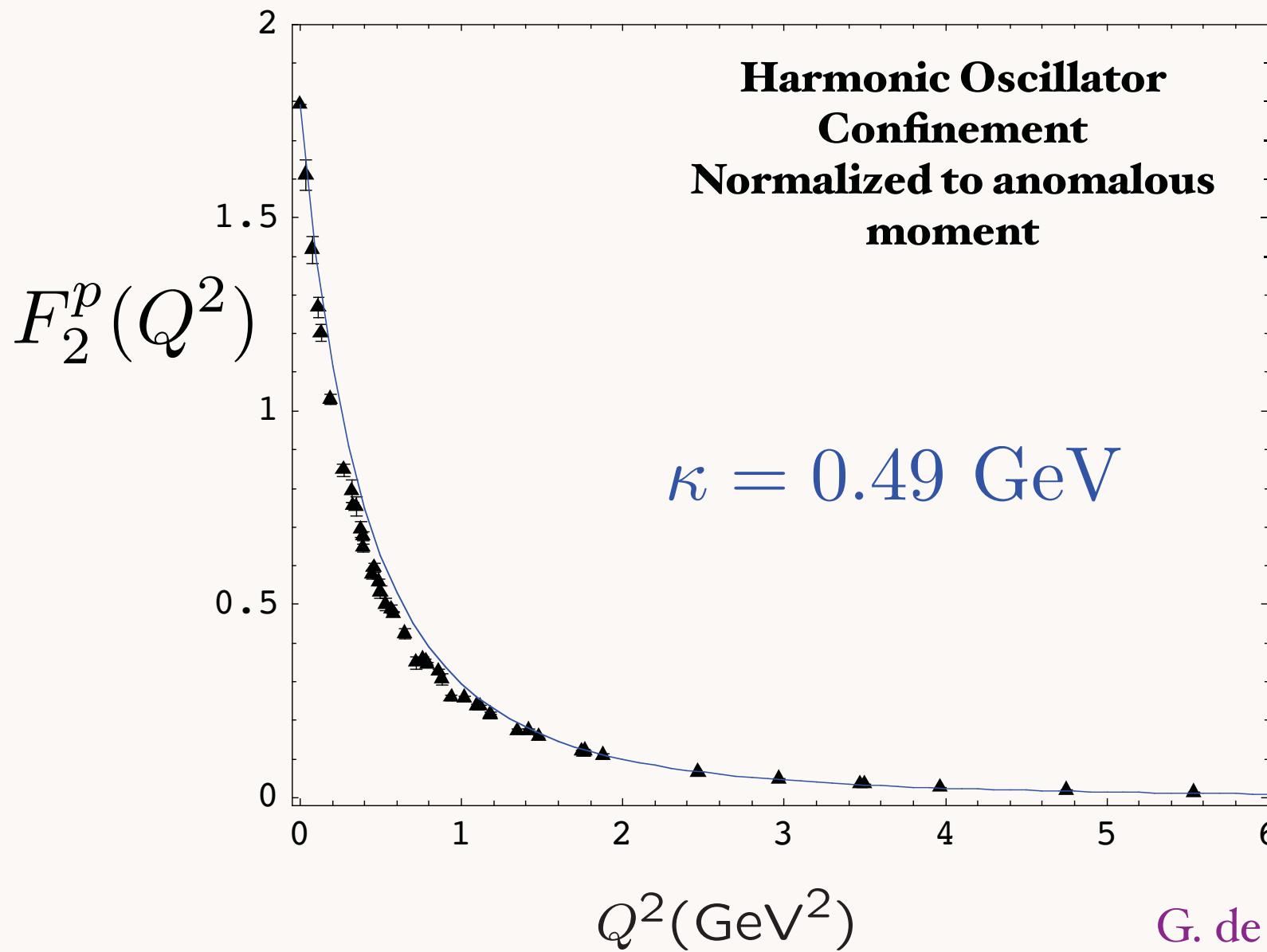


SW model predictions for $\kappa = 0.424$ GeV. Data analysis from M. Diehl *et al.* Eur. Phys. J. C **39**, 1 (2005).

Spacelike Pauli Form Factor

Preliminary

From overlap of $L = 1$ and $L = 0$ LFWFs



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November 20, 2007

AdS/QCD
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Holographic Connection between LF and AdS/CFT

- Predictions for hadronic spectra, light-front wavefunctions, interactions
- Deduce meson and baryon wavefunctions, distribution amplitude, structure function from holographic constraint
- Identification of Orbital Angular Momentum Casimir for $\text{SO}(2)$: LF Rotations
- Extension to massive quarks