COMPLEX ALGEBRAIC SURFACES CLASS 14

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CONTENTS

1. Combinatorial aspects 1
2. del Pezzo surfaces 2
3. Cubic surfaces 3
3.1. Automorphisms of the intersection graph 5
3.2. Every smooth cubic has 27 lines, and is the blow-up of $\mathbb{P}^2$ at 6 points 5

Last day we looked at a lot of rational surfaces. We made use of:

Useful proposition. Consider the blow-up of $\mathbb{P}^2$ at n general points, giving exceptional divisors E_1, ... , E_n. Then the intersection ring on $\mathbb{P}^2$ is given by

$$\mathbb{Z}[H, E_1, \dots, E_n]/H^2 = 1, HE_i = 0, E_i \cdot E_j = 0, E_i^2 = -1.$$

We can understand divisors and sections of divisors in terms of divisors on $\mathbb{P}^2$ with certain multiplicities at the E_j. More precisely: the vector space of sections of $aH - b_1E_1 - \cdots - b_nE_n$ is naturally isomorphic to the vector space of degree a polynomials in $\mathbb{P}^2$ vanishing with multiplicity at least b_i on E_i.

We also proved the following. Suppose S is a surface, and $-K_S$ gives a map to projective space. Then (-1)-curves map isomorphically onto lines. Conversely, if S is a surface, and $-K_S$ gives a map to projective space, then any curve mapping isomorphically onto a line is a (-1)-curve.

Today: Cubic surfaces. But first, some interesting combinatorial remarks, due to Tyler, Diane and others.

1. COMBINATORIAL ASPECTS

Proposition. The automorphism group of the Peterson graph is S_5. (I haven't drawn the Peterson graph in these notes, sorry!)

Proof. Two (equivalent) proofs. Diane's: To each vertex put a size 2 subset of $\{1, \dots, 5\}$. Join them by an edge if the don't intersect.

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Tyler's: Label the edges as follows (3 have label 1, etc.). Observe that as soon as you've labeled one edge with a number, the other two are determined: they are those that are distance 3 from it. Hence if G is the automorphism group of the graph, we have a morphism $G \to S_5$. It is surjective: you can get a 5-cycle (rotate) and a 2-cycle (do it). It is injective: suppose you have an automorphism fixing the colors. We'll show that the three 1-labeled edges are fixed. Look at the two pairs of edges each 1-edged meets; note that this gives a partition of $\{2,3,4,5\}$ into two couples. There are 3 ways to do this, and they correspond to the 3 edges.

Connection between Diane's and Tyler's: in Diane's construction: label an edge with the number missing in its vertex labels. □

If you remember, I described a pattern of automorphism groups of the intersection graphs of the blow-up of $\mathbb{P}^2$ at n up to 9 points. The answer was:

$$\begin{array}{cccccccc} n= & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \dim \mathrm{Aut} & [S_3 \times S_2] & W(A_4)=S_5 & W(D_5) & W(E_6) & W(E_7) & W(E_8) & W(E_9) \end{array}.$$

I expressed disappointment that $S_3 \times S_2$ was not isomorphic to D_{12}, the symmetries of hexagon, because that's what the correct answer was. But in fact:

Proposition (Tyler). $S_3 \times S_2 \cong D_{12}$.

Proof. $x = (12)(12)$ and $y = (12)(123)$ generate $S_3 \times S_2$. They also satisfy the relations defining D_{12}: $x^2 = y^6 = e$, $xyx = y^{-1}$. □

Remark. The map $D_{12} \to S_3$ corresponds to the induced permutation of the 3 diagonals. The map $D_{12} \to S_2$ corresponds to the permutation of the two inscribed equilateral triangles.

2. del Pezzo surfaces

Another remark about these surfaces.

Definition. A surface S is a *del Pezzo surface* if $-K_S$ is ample.

This is yet another ancient idea that remains important. Most recently, they have come up in physics.

Examples from last day: $\mathbb{P}^1 \times \mathbb{P}^1$, and the blow-up of $\mathbb{P}^2$ at up to 8 general points. Let's get rid of that nasty word "general", by making more precise which points you throw out.

Proposition. The blow up of $\mathbb{P}^2$ at up to 8 distinct points, no 3 on a line and no 6 on a conic, is a del Pezzo surface.

Proof. Check using our useful proposition that $-K$ or $-2K$ is very ample. If $n \leq 6$, $-K$ is ample, i.e. cubics vanishing at the n points gives something that separates points and tangent vectors. If $n = 7$ or 8, $-K$ isn't very ample, but $-2K$ is: the linear system corresponding to sextics vanishing at the n points separate points and tangent vectors. $\square$

Proposition. The blow up of $\mathbb{P}^2$ at up to 8 distinct points, with 3 on a line or 6 on a conic, is not a del Pezzo surface.

Proof. First, note that a del Pezzo surface can't have a (-2)-curve, i.e. a genus 0 curve C with self-intersection ≤ -2. Reason: genus formula gives

$$-2 = 2g - 2 = C \cdot (K + C) = C \cdot K + C \cdot C \leq C \cdot K - 2$$

so $0 \leq C \cdot K$. But $-K$ is ample, so $K \cdot C < 0$ for all C.

If at least there are at least 3 points on a line, then there is a genus 0 curve of self-intersection at most -2.

Similarly, if there are at least 6 points on a conic, then there is a genus 0 curve of self-intersection at most -2. $\square$

Remark: Slight extension. You can allow "infinitely near" points: blow up p_1, and blow up a point p_2 on the exceptional divisor E_1; somewhat archaic (but still-used) terminology is that p_2 is an "infinitely near point" to E_2.

But as soon as you blow up a point on an exceptional divisor, you have a genus 0 curve of self-intersection at most -2, and hence it can't be a del Pezzo surface.

Theorem. The only del Pezzo surfaces are the above blow-ups of $\mathbb{P}^2$ (up to 8 points, no 3 on a line, no 6 on a conic), and $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof: later in the course.

3. Cubic surfaces

Blow up $\mathbb{P}^2$ at six points. 27 (-1)-curves. Reminder of where they are: 6 exceptional divisors E_i. $\binom{6}{2}$ lines L_{ij}. 6 conics C_{ijklm}.

Anticanonical map gives embedding in to $\mathbb{P}^3$. (Remember: (-1)-curves $\leftrightarrow$ lines.)

Get smooth cubic surface in $\mathbb{P}^3$ with lines. We'll see: (i) there are no more lines on this surface, (ii) that almost all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points and hence have 27 lines, (iii) all smooth cubic surfaces have 27 lines, and (iv) all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points.

Proposition. We have found all the lines.

Proof. Diophantine equations. Suppose $D = aH - \sum b_i E_i$ is a line, i.e. a (-1)-curve.

First case: some $b_i < 0$. Then $D \cdot E_i = -b_i < 0$. But D is the class of a line, so $D \cdot E_i \geq 0$ *unless* the line and E_i have a component in common. Thus they must be equal, i.e. $D = E_i$. Hence we've found six lines: $E_1, \ldots, E_6$.

Second case: all $b_i \geq 0$. Also note that $a > 0$.

Then degree = 1 means $(3H - \sum E_i) \cdot D = 1$, from which $3a - \sum b_i = 1$.

$D^2 = -1$ implies $a^2 - \sum b_i^2 = -1$.

Recall (or re-prove) the quadratic mean - arithmetic mean inequality:

$$\sqrt{\left(\frac{\sum b_i^2}{6}\right)} \geq \frac{\sum b_i}{6}$$

from which $\sum b_i^2 \geq (\sum b_i)^2/6$. Thus

$$a^2 = \sum b_i^2 - 1 \geq (\sum b_i)^2/6 - 1 = (3a-1)^2/6 - 1$$

from which $6a^2 + 6 \geq 9a^2 - 6a + 1 \Rightarrow 0 \geq 3a^2 - 6a - 5 \Rightarrow 8 \geq 3(a-1)^2$. Thus $a = 1$ or 2.

If $a = 1$, then we are considering the class of lines, then we get $\sum b_i = 2$ and $\sum b_i^2 = 2$, from which two of the b_i's are 1 and the rest are 0.

If $a = 2$, we get $\sum b_i = 5$ and $\sum b_i^2 = 5$. The five b_i's must be equal to 1, and the last equal to 0. □

Proposition. Almost all smooth cubic surfaces are $\mathbb{P}^2$ blown up at six points.

Proof. By dimension count. Dimension of space of cubic surfaces: count cubic equations in 4 variables (20). Subtract 1 to projectivize, to get 19. There is a $\mathbb{P}^{19}$ parametrizing all smooth cubic surfaces. An open set $\mathcal{S}$ corresponds to the smooth ones.

Now how many ways can we get a cubic surface by blowing up six points in the plane? Choose six points in the plane (12 dimensions), except mod out by the automorphisms of the plane $\dim PGL(3) = 8$. Then map to projective space using four linearly independent sections of $3H - \sum E_i$. There is a four-dimensional space of sections, so we have 16 dimensions of choice of 4 sections. But any multiple of such a 4-tuple gives the same embedding, so -1. Get:

$$12 - 8 + 16 - 1 = 19.$$

Thus we have a 19-dimensional family!

It's not an open subset yet; there may be, and in fact are, many ways of representing the surface as a blow-up of six points. But there are only a finite number: any such description corresponds to six lines on the surface, no 2 intersecting. So the image of this 19-dimensional family lies in S, and must be dense in S. □

We'll see soon: Every smooth cubic has 27 lines, and is the blow-up of $\mathbb{P}^2$ at 6 points

3.1. **Automorphisms of the intersection graph.** Again, we can make a graph of the combinatorial structure of the 27 lines. It again is highly symmetric.

Theorem. Its automorphism group is $W(E_6)$, a finite group of order $51840 = 2^7 \times 3^4 \times 5$.

Sketch of proof. Recall the definition of $W(E_6)$. 6 skew lines gives full structure. Get map to $\mathbb{P}^2$. See S_6 in it. Last one: Cremona transformation. $\square$

Interpretation of this group in two ways:

Geometrically: as the monodromy of the 27 lines as you make loops in $\mathcal{S}$.

Algebraically: Let M be the variety parametrizing (surface, line). There is a morphism from M to $\mathcal{S}$, of degree 27. M turns out to be irreducible. Hence we have a field extension of degree 27 $k(\mathcal{S}) \subset k(M)$. If we take the Galois closure of the this field extension, we get a field extension of order 51840, and its Galois group is $W(E_6)$.

Next day we'll show that:

3.2. **Every smooth cubic has 27 lines, and is the blow-up of $\mathbb{P}^2$ at 6 points.** Let S be a smooth cubic surface.

Earlier today (in response to a question of Eric's) I said:

Proposition. S is anticanonically embedded.

Proof: Adjunction formula. S has degree 3, so $K_S = \mathcal{O}_{\mathbb{P}^3}(-4+3)|_S$.

Hence lines correspond to (-1)-curves.

The strategy of proof that every smooth cubic is a blow-up of $\mathbb{P}^2$ at 6 points is as follows.

Proposition. Every cubic surface (even singular ones) contains a line.

Proposition. Given any line l in S, there are exactly 10 other lines in S meeting l (distinct from l). These fall into 5 disjoint pairs of concurrent lines.

Corollary. Any smooth cubic contains two skew lines. Finally:

Theorem. S is the blow-up of $\mathbb{P}^2$ at 6 points.

We'll see instead that S is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points. We'd seen earlier that $\mathbb{P}^2$ blown up at 2 points is $\mathbb{P}^1$ blown up at one point, so we'll be done.

Proof. Let l and l' be disjoint lines. We define rational maps $\phi : l \times l' \dashrightarrow S$ and $\psi : S \dashrightarrow l \times L'$ as follows: if (p, p') is a generic point of $l \times l'$, the line $\langle p, p' \rangle$ meets S in a third point p''. Define $\phi(p, p') = p''$. For $s \in S - l - l'$, set $p = l \cap \langle s, l' \rangle$, $p' = l' \cap \langle s, l \rangle$ and put $\phi(s) = (p, p')$. It is clear that ϕ and ψ are inverses. Moreover, ϕ is a morphism: we can define at a points of l (or l) by replacing the plane $\langle s, l \rangle$ by the tangent plane to S at s

(checking that this gives a morphism, explain). Thus ψ is a birational morphism, and is a composite of blow-ups. Which curves are blown-down? Those lines meeting both l and l'.

We'll see why there are precisely 5 of them, one of each the pairs described above. $\square$