COMPLEX ALGEBRAIC SURFACES CLASS 3

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CONTENTS

Recap of last time.

The times have changed to Wednesdays and Fridays 2:10–3:25. We will likely meet on a couple of Mondays as well for 50 minutes to catch up if need be; if that happens, I'll warn you in advance.

We've been discussing line bundles, a.k.a. invertible sheaves; they form a group called $\mathrm{Pic}(X)$, a.k.a. $H^1(X, \mathcal{O}_X^*)$.

We related invertible sheaves and divisors. In particular, for a smooth variety, we showed that $\mathrm{Pic}(X) \cong Div(X)$ modulo linear equivalence. The divisors linearly equivalent to 0 were the divisors of rational functions. This involved the construction of the invertible sheaf $\mathcal{O}(D)$, where D is a divisor.

If X is a (proper, nonsingular) curve, then there is a degree map $\mathrm{Pic}(X) \to \mathbb{Z}$.

As examples, we showed that $\mathrm{Pic}(\mathbb{A}^1) \cong \{1\}$, by showing directly that any point was rationally equivalent to 0. Secondly, we showed that $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$ by showing that any two points were rationally equivalent, so $\mathrm{Pic}(\mathbb{P}^1)$ was generated by $\mathcal{O}(pt)$. Using the degree map, for example, we showed that $\mathcal{O}(n(pt)) \neq \mathcal{O}$.

1. ANOTHER EXTENDED EXAMPLE: $\mathbb{P}^n$

This actually generalizes.

Date: Wednesday, October 9.

First, $\operatorname{Pic}(\mathbb{A}^n) = \{1\}$. Reason: Let $\mathbb{A}^n = \operatorname{Spec}[x_1, \ldots, x_n]$. Then the *irreducible divisors* correspond to irreducible polynomials (up to scalars). This requires the *fact* that all codimension one loci are the zero sets of polynomials in the n variables. (This isn't hard, but involves the algebraic theory of dimension, so we won't go into it here.) Thus any divisor is the divisor of a rational function, e.g. $(y - x^2) - 3x - 2y = div((y - x^2)/(x^3 y^2))$.

Next: on to $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$. (Should I say something about Proj notation? The x_i are projective coordinates.) The irreducible divisors correspond to irreducible *homogeneous* polynomials in the x_i of positive degree. Reason: For future use, let U_0, ... , U_n be the basic affine open sets; U_i is where $x_i \neq 0$. Note that $\mathbb{P}^n - U_0 = (x_0 = 0) = (x_0)$, so the only irreducible divisor missing U_0 is the hyperplane $x_0 = 0$. Given an irreducible divisor meeting U_0, it is irreducible in U_0, and hence corresponds to an irreducible polynomial $p(u_1, \ldots, u_n)$ in the coordinates of U_0 (where $u_i = x_i/x_0$), the full divisor is the closure of its restriction to U_0. In confusing math-ese: $D = \overline{D|_{U_0}}$. Then check that D is the vanishing set of the polynomial $x_0^{\deg p} p(u_1, \ldots, u_n)$, and that this is an irreducible homogeneous in the x_i.

Finally, **Theorem.** $\operatorname{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Remark. The generator is the class of the hyperplane H. As with $\mathbb{P}^1$, we use the notation $\mathcal{O}_{\mathbb{P}^n}(d)$ for $\mathcal{O}_{\mathbb{P}^n}(dH)$. (Hence we have a degree map here too.)

Proof. We need to show (i) that any divisor is linearly equivalent to a multiple of H, and (ii) that $\mathcal{O}(dH) \neq 0$.

(i) Given an irreducible divisor $p(\vec{x}) = 0$, note that $p/x_0^{d:=\deg p}$ is a rational function, and its divisor is $(p) - d(x_0)$, so $(p) = dH$ in the Picard group.

(ii) Interpret H as $\mathcal{O}_{\mathbb{P}^n}((x_0))$. Consider the immersion of a line in projective space $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$, $[x; y] \mapsto [x; y; 0; ...; 0]$. Pullback gives a map $\operatorname{Pic}(\mathbb{P}^n) \to \operatorname{Pic}(\mathbb{P}^1)$. Take a hyperplane meeting the line at a point, e.g. (x_0). Check that $\mathcal{O}_{\mathbb{P}^n}(H)$ pulls back to $\mathcal{O}_{\mathbb{P}^1}(pt)$, which has infinite order. $\square$

I next want to give you a way of interpreting sections of $\mathcal{O}_{\mathbb{P}^n}$ over any open set.

1.1. **Useful interpretation of rational sections of $\mathcal{O}_{\mathbb{P}^n}(d)$.** Rational sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ correspond to rational homogeneous functions $P(x_0, \ldots, x_n)/Q(x_0, \ldots, x_n)$ with degree d. This behaves well with respect to multiplication, i.e. if you have a rational sections s resp. t of $\mathcal{O}_{\mathbb{P}^n}(d)$ resp. $\mathcal{O}_{\mathbb{P}^n}(e)$, then st is a rational section of $\mathcal{O}_{\mathbb{P}^n}(d+e)$, and if $t \neq 0$ then s/t is a rational section of $\mathcal{O}_{\mathbb{P}^n}(d-e)$, and this interpretation respects multiplication and division.

If you want to see actual sections over an open set U, you allow poles away from U. For example, global sections correspond to the vector space of degree d in the $n+1$ projective co-ordinates x_0, ... , x_n. (As a corollary, $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d}$.) For example, $\mathcal{O}_{\mathbb{P}^2}(2)$ is a 6-dimensional vector space, with generators x_0^2, ... , $x_1 x_2$. What's the divisor corresponding to global section $x_0 x_1 + x_0 x_2$? Ans: $(x_0) + (x_1 + x_2)$.

Proof: **Exercise.**

I'll do it for global sections of $\mathbb{P}^2$, so you can see the bijection.

Let me pick coordinates on the three open sets:

- $U_0 = \{x_0 \neq 0\}$. Coordinates (u_1, u_2) where $(x_0; x_1; x_2) = (1; u_1; u_2)$, so $u_1 = x_1/x_0$.
- $U_1 = \{x_1 \neq 0\}$. Coordinates (v_0, v_2) where $(x_0; x_1; x_2) = (v_0; 1; v_2)$.
- $U_2 = \{x_2 \neq 0\}$. Coordinates (w_0, w_1) where $(x_0; x_1; x_2) = (w_0; w_1; 1)$.

Then on the overlap, you can check things like $w_0 = v_0 w_1$ etc.

Let's consider $\mathcal{O}_{\mathbb{P}^2}(2)$ in its guise of $\mathcal{O}_{\mathbb{P}^2}((x_0)^2)$.

- Restrict first to U_0. We're allowed to have polynomials in u_1 and u_2.
- Restrict to U_1. We're allowed to have polynomials in v_0 and v_2, and poles of order up to 2 in v_0.
- Restrict to U_2. We're allowed to have polynomials in w_0 and w_1, and poles of order up to 2 in w_0.

Then we have gluing data. Now you do the algebra, and when the dust settles, what this corresponds to are polynomials in u_1 and u_2 of degree up to 2:

$$?+?u_1+?u_2+?u_1^2+?u_1u_2+?u_2^2 = \frac{1}{x_0^2}(x_0^2+?x_0x_1+?x_0x_2+?x_1^2+?x_1x_2+?x_2^2).$$

So we made a choice of manifestation of $\mathcal{O}_{\mathbb{P}^2}(2)$ by picking the divisor x_0^2, and we got homogeneous polynomials with denominator x_0^2. Now you check that if you picked a different manifestation $\mathcal{O}_{\mathbb{P}^2}(2) \cong \mathcal{O}_{\mathbb{P}^2}((p))$ where p is degree 2, then the last line would have been

$$\frac{1}{p}(x_0^2+?x_0x_1+?x_0x_2+?x_1^2+?x_1x_2+?x_2^2),$$

and in fact the isomorphism between the two would preserve the degree 2 polynomial. Hence this correspondence between the vector space of global sections and the vector space of degree 2 polynomials is well-defined.

This argument extends to (i) rational sections and (ii) $\mathbb{P}^n$ without change.

Let me make this very explicit. Suppose I have a section of $\mathcal{O}_{\mathbb{P}^2}(2)$ that I'm calling $x_0^2 - x_1x_2$. You would like to see it as an element of $\mathcal{O}_{\mathbb{P}^2}(D)$, where D has degree 2, for example $\mathcal{O}_{\mathbb{P}^2}(x_0x_2)$. Then the corresponding element of $\mathcal{O}_{\mathbb{P}^2}(D)$ is $(x_0^2 - x_1x_2)/(x_0x_2)$. The content of this "theorem" is that this is a well-defined bijection, independent of your choice of D.

1.2. **The canonical sheaf of $\mathbb{P}^n$.** Now let's find the canonical sheaf of $\mathbb{P}^n$. My goal here is (i) to show you that the canonical sheaf isn't scary, and (ii) to actually get the number.

Theorem. $\mathcal{K}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. *Remember this!*

Proof. I'll just do the case $n = 2$ for notational convenience. The sections of $\mathcal{K}$ over U_0 are $p(u_1, u_2)du_1 \wedge du_2$. Let's take the section $du_1 du_2$ over this open set, and see what its poles and zeroes are over all of $\mathbb{P}^2$. There's only one divisor away from U_0, $x_0 = 0$, so we need only check this divisor.

Let's look over U_1, with coordinates v_0 and v_2. Remember that $(1; u_1; u_2) = (v_0; 1; v_2)$, so $u_1 = 1/v_0$ and $u_2 = v_2/v_0$.

Thus in terms of the coordinates of U_1 the section transforms into:

$$du_1 \wedge du_2 = \left(-\frac{1}{v_0^2}dv_0\right) \wedge \left(\frac{v_0 dv_2 - v_2 dv_0}{v_0^2}\right) = -\frac{1}{v_0^3}dv_0 \wedge dv_2.$$

So we see a pole along $v_0 = 0$ of order 3, as desired. □

1.3. **Sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ and maps to projective space; more generally, invertible sheaves and maps to projective space.** One of the central facts about invertible sheaves on proper schemes X is loosely that global sections give maps to projective space. I want to show this to you first in the case of $\mathcal{O}_{\mathbb{P}^2}(d)$ to show you that this isn't hard, and then I'll return to a general discussion of varieties.

First, let me take 4 sections of $\mathcal{O}_{\mathbb{P}^2}(2)$. For convenience, let me take the projective coordinates to be x, y, z, rather than x_0, x_1, x_2. I'll choose the 4 sections: x^2, y^2, z^2, xy. Then for any point of $\mathbb{P}^2$, the point $[x^2, y^2, z^2, xy]$ is a well-defined point of $\mathbb{P}^3$. (Explain: (i) even though x etc. isn't well-defined, this is a well-defined point of projective space, and (ii) these are never all zero.)

More generally: **Definition.** For an arbitrary scheme X with invertible sheaf $\mathcal{L}$, a vector space of global sections with basis $s_0, \dots, s_n$ is said to be *base-point free* if they have no common zeros on X. Then a basepoint free vector space V of $n+1$ global sections gives a map to $\mathbb{P}^n$. If you unwind the definition carefully, you'll see that this gives $X \to \mathbb{P}V^*$.

I should also then define base points: **Definition.** given a vector space of global sections, their locus of common zeros is called the *base locus*, or *base points*. (Normally you take the scheme-theoretic intersection.)

Important fact: there is a converse to this construction. If X is proper (not necessarily nonsingular): there is a bijection between $\pi : X \to \mathbb{P}^n$ and $(X, \mathcal{L}, (s_0, \dots, s_n)/k^*)$ where $s_i \in \Gamma(X, \mathcal{L})$. The bijection from right to left was described before. In the other direction: $\mathcal{L} = \pi^*\mathcal{O}(1)$, and $s_i = \pi^* x_i$.

To see if you understand this fact, here's an immediate consequence: **Exercise.** The only morphisms from $\mathbb{P}^n$ to $\mathbb{P}^m$ if $m < n$ are the constant maps.

Back to the example of $\mathbb{P}^2$ and $\mathcal{O}_{\mathbb{P}^2}(2)$: The sheaf $\mathcal{O}_{\mathbb{P}^2}(2)$. Six sections.

$$[x; y; z] \to [x^2; y^2; z^2; xy; yz; zx].$$

(Draw a picture.) Hyperplane sections correspond to conics. The degree of a subvariety X of $\mathbb{P}^n$ can be defined as the number of intersection points of X with a general linear space of complementary dimension. Hence the degree of this embedded $\mathbb{P}^2$ is 4. (Common terminology that won't come up in this course: This is an example of the famous *Veronese* embedding of $\mathbb{P}^2$. In general, use $\mathbb{P}^n$ and its space of global sections $\mathcal{O}_{\mathbb{P}^n}(d)$ to get a map to a big projective space $\mathbb{P}^N$, where $N = \binom{d+n+1}{d} - 1$ (I think!). **Exercise.** The degree of the Veronese-embedded space is d^n by a similar argument.

Definition. An invertible sheaf $\mathcal{L}$ is *very ample* if the global sections of $\mathcal{L}$ gives a closed immersion into projective space.

Fact. equivalent to: "separates points and tangent vectors". (Explain why, loosely. Separating points means there is a hyperplane passing through one point but not the other; that means that they don't map to the same point in projective space. Separating tangent vectors loosely means that by the implicit function theorem, you have a local isomorphism. Complex geometers might buy this.)

Again, hyperplane sections correspond to $H^0(X, \mathcal{L})$.

Definition. The corresponding map to projective space is called a *linear system*. (I'm not sure if I'll use this terminology, but I want to play it safe.)

$$|\mathcal{L}| : X \to \mathbb{P}^n = \mathbb{P}H^0(X, \mathcal{L})^*.$$

Definition. An invertible sheaf is *ample* if some power of it is very ample.

Note: A very ample sheaf on a curve has positive degree. Hence an ample sheaf on a curve has positive degree. We'll soon see that this is an "if and only if".

Fact (Serre vanishing). Suppose $\mathcal{M}$ is any coherent sheaf e.g. an invertible sheaf, or more generally a locally free sheaf (essentially, a vector bundle), and $\mathcal{L}$ is *ample*. Then for $n >> 0$, $H^i(X, \mathcal{M} \otimes \mathcal{L}^n) = 0$ for $i > 0$.

Next day: Serre duality; Riemann-Roch; lots of applications to curves. Then on to surfaces!