INTRODUCTION TO INFINITE RAMSEY THEORY

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0 Introduction

In combinatorics, Ramsey Theory considers partitions of some mathematical objects and asks the following question: how large must the original object be in order to guarantee that at least one of the parts in the partition exhibits some property? Perhaps the most familiar case is the well-known Pigeonhole Principle: if m pigeonholes house p pigeons where $p > m$, then one of the pigeonholes must contain multiple pigeons. Conversely, the number of pigeons must exceed m in order to guarantee this property.

Ramsey Theory is often discussed in a graph-theoretic context. Ramsey's Theorem, for instance, states that in any coloring of a sufficiently large complete hypergraph, there exists a monochromatic complete subgraph of a desired size. A good deal of research has been generated in this field to bound the size of the smallest such complete hypergraph.

In this exposition, we seek to demonstrate an extension of Ramsey's Theorem onto the domain of infinite sets, as pioneered by Frank P. Ramsey and developed by Waclaw Sierpinski, Paul Erdös, Richard Rado et al. Ramsey Theory will be framed in set-theoretic terms instead in order to enable ourselves to discuss mathematical constructs whose sizes correspond to higher infinities.

We begin by proving the basic analogue of Ramsey's theorem for graphs of size $\aleph_0$, and attempt to generalize it to higher aleph numbers. Then we consider the implication of allowing other parameters to be infinite. Most of the theorems discussed are borrowed from Levy's *Basic Set Theory*, but they are examined in an order that will hopefully assist in grasping a more intuitive picture.

1 Finite Ramsey Theory

Let us begin by visiting finite Ramsey Theory and stating the main results. Instead of relying on the standard definitions in graph theory, we shall treat them as set-theoretic constructs, which admit generalizations more easily.

Definition 1.1. *A* **complete hypergraph** *on a set A, denoted as $[A]^n$, is the set of n-membered subsets of A for some $n \geqslant 1$.*

Definition 1.2. *A* **$\mathfrak{c}$-coloring** *of a complete hypergraph $[A]^n$ is a function $f : [A]^n \to \mathfrak{c}$ where $\mathfrak{c}$ is a cardinal.*

For instance, a $\mathfrak{c}$-coloring of $[A]^1$ can be likened to the task of assigning $|A|$ pigeons to $\mathfrak{c}$ pigeonholes; a $\mathfrak{c}$-coloring of $[A]^2$ can be likened to the task of coloring the edges of the complete graph $K_{|A|}$ with $\mathfrak{c}$ colors. More generally, a $\mathfrak{c}$-coloring of $[A]^n$ is the partition of $[A]^n$ into $\mathfrak{c}$ sets.

Definition 1.3. *A subset $B \subseteq A$ is* **homogeneous** *for $\mathfrak{c}$-coloring $f : [A]^n \to c$ in case $f\,[[B]^n]$ is a singleton set.*

If such subset exists, we say that A "admits" a homogeneous subset B (for f.)

The Pigeonhole Principle and Ramsey's Theorem can now be stated in these definitions.

Proposition 1.4. (The Pigeonhole Principle) *Let $\mathfrak{c}$ be finite, and let A be a finite set with $|A| > \mathfrak{c}$. Then, any $\mathfrak{c}$-coloring of $[A]^1$ admits a homogeneous subset $B \subseteq A$ with $|B| > 1$.*

Theorem 1.5. (Ramsey 1930) *Let $b, n, \mathfrak{c}$ be finite. Then, there exists a finite γ such that any $\mathfrak{c}$-coloring of $[A]^n$ where $|A| > \gamma$ admits a homogeneous subset $B \subseteq A$ of size b.*

Theorem 1.5 tells us that for any finite $\mathfrak{c}$, we can find a homogeneous subset of a desired size for a $\mathfrak{c}$-coloring, as long as the original set A is large enough.

We remark here that the above theorem is actually a simplified version of Ramsey's Theorem. The full statement allows us to impose a different constraint on the size of each part in the partition, but we can forgo its discussion, as the simplified version already suffices to motivate extensions onto infinite sets.

2 Ramsey's Theorem on $\aleph_0$

Theorem 1.5 indicates that large finite sets will admit homogeneous subsets for some coloring. One can naturally extend this idea and ask whether infinite sets also admit homogeneous subsets. The simplest case of this kind arises when the cardinality of these sets is $\aleph_0$, which we explore in this section. We will start with a very weak result, and then progressively strengthen it until we arrive at Theorem 2.8.

Before we begin, it may be useful to introduce a shorthand notation for stating that a set always admits a homogeneous subset for certain coloring, as follows:

Definition 2.1. (Erdös and Rado 1956) *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *be cardinals. Then* $\mathfrak{a} \to (\mathfrak{b})^n_{\mathfrak{c}}$ *shall denote the statement "for all sets* A *and* C *with* $|A| = \mathfrak{a}$ *and* $|C| = \mathfrak{c}$ *and every function* $f : [A]^n \to C$*, there is a homogeneous subset* $B \subseteq A$ *with* $|B| = \mathfrak{b}$*."*

We can immediately restate Theorem 1.5 from the previous section with this shorthand definition:

Theorem 2.2. (Restatement of Theorem 1.5) *Let* $b, n, \mathfrak{c}$ *be finite. Then there exists a finite* γ *such that* $\gamma \to (b)^n_{\mathfrak{c}}$*.*

Note that in Definition 2.1, we are concerned only with the cardinalities of the sets involved, not the sets themselves. This simplification is justified by the fact that any coloring of a set can be translated onto a coloring of an equinumerous set by using the bijective relation between the two sets.

In the same vein, a coloring of a set can be translated onto a coloring of a smaller set by the natural restriction. The next proposition captures this intuition and other related ideas:

Proposition 2.3. (i) *If* $\mathfrak{a} \to (\mathfrak{b})^n_{\mathfrak{c}}$ *holds and* $\mathfrak{a}' \geqslant \mathfrak{a}, \mathfrak{b}' \leqslant \mathfrak{b}, \mathfrak{c}' \leqslant \mathfrak{c}$*, then* $\mathfrak{a}' \to (\mathfrak{b}')^n_{\mathfrak{c}'}$ *holds as well.*
(ii) *For alephs* $\mathfrak{a}$ *and* $\mathfrak{b}$*, if* $m < n$ *and* $\mathfrak{a} \to (\mathfrak{b})^n_{\mathfrak{c}}$*, then also* $\mathfrak{a} \to (\mathfrak{b})^m_{\mathfrak{c}}$*.*

Proof. (i) Take any $\mathfrak{c}'$-coloring of $[A']^n$ where $|A'| = \mathfrak{a}'$. Consider its restriction onto $[A]^n$ where $A \subseteq A'$ has cardinality $\mathfrak{a}$. Since a $\mathfrak{c}'$-coloring is a special case of a $\mathfrak{c}$-coloring, by assumption we have a subset $B \subseteq A \subseteq A'$ of cardinality $\mathfrak{b}$ that is homogeneous for the restriction. Then any subset $B' \subseteq B$ of cardinality $\mathfrak{b}'$ is homogeneous for the original coloring, so $\mathfrak{a}' \to (\mathfrak{b}')^n_{\mathfrak{c}'}$ as desired.

(ii) Let $|A| = \mathfrak{a}$ and let $f : [A]^m \to C$ with $|C| = \mathfrak{c}$. Well-order A and define $f' : [A]^n \to C$ by $f'(u) = f(\text{the set of the smallest } m \text{ members of } u)$. The operation of taking the smallest m members is well-defined via the given well-ordering. Then there exists $B \subseteq A$ of cardinality $\mathfrak{b}$ that is homogeneous for f', and by inspection, B is also homogeneous for f. □

Combining Theorem 2.2 with the proposition we have just proven, we arrive at our first statement on $\aleph_0$:

Corollary 2.4. $\aleph_0 \to (b)^n_{\mathfrak{c}}$ *for any finite* $b, n, \mathfrak{c}$*.*

Proof. By Theorem 2.2, $\gamma \to (b)^n_c$ holds for some finite γ. Applying the proposition with $\aleph_0 > \gamma$ immediately yields $\aleph_0 \to (b)^n_{\mathfrak{c}}$. □

It is not at all surprising that we can find a finite homogeneous subset from a (finite) coloring of an infinite set. In fact, we did not rely on any set-theoretic machinery to obtain this fact. Less obvious is whether there exists an infinite homogeneous subset: for what value of n and $\mathfrak{c}$ does $\aleph_0 \to (\aleph_0)^n_{\mathfrak{c}}$ hold? We will see a surprisingly strong positive result:

Theorem 2.5. (Ramsey 1930) *Let A be a countably infinite set, and suppose we color each pair $\{x, y\} \subseteq A$, where $x \neq y$, either red or black. Then A has an infinite subset B such that all pairs $\{x, y\} \subseteq B$, with $x \neq y$, are of the same color. In other words, $\aleph_0 \to (\aleph_0)^2_2$.*

Proof. For every finite binary sequence s, we shall recursively define a subset $C_s \subseteq A$, and in case $C_s \neq \varnothing$, then also pick a member $x_s \in C_s$. Begin by letting $C_\varnothing = A$.

Now the recursive step: for $s \in \{0,1\}^*$, if $C_s \neq \varnothing$, then we set,

$$\begin{aligned} x_s &= \text{an arbitrary member of } C_s, \\ C_{s|0} &= \{y \in C_s \mid y \neq x_s \wedge \{x_s, y\} \text{ is black}\} \text{ and} \\ C_{s|1} &= \{y \in C_s \mid y \neq x_s \wedge \{x_s, y\} \text{ is red}\}. \end{aligned}$$

If $C_s = \varnothing$, then x_s is not defined, and we set $C_{s|0} = C_{s|1} = \varnothing$.

Now, whenever $C_s \neq \varnothing$, we see that

(1) $\{x_s\}$, $C_{s|0}$ and $C_{s|1}$ form a partition of C_s.

This implies that, for all finite binary sequences s and t where $C_t \neq \varnothing$ and $s \subsetneq t$, the following statements hold:

(2) $C_t \subsetneq C_s$, $x_s \notin C_t$ and $x_s \neq x_t$.

(3) Let b be the bit of t that comes after the portion corresponding to s. So $s|b \sqsubseteq t$, meaning that $x_t \in C_t \subseteq C_{s|b}$. Then, $b = 0$ implies, by definition of $C_{s|0}$, that $\{x_s, x_t\}$ must be black; or else $b = 1$ and $\{x_s, x_t\}$ is red.

Let $T = \{s \mid s \text{ is a finite binary sequence} \wedge C_s \neq \varnothing\}$. By (1) and (2), $\langle T, \subset \rangle$ is a tree, and every initial segment of $s \in T$ is also in T.

Given (1), construction of $\{C_s\}$ can be seen as a recursive partition of A. Thus, by induction, it is easy to see that

(4) $A = \{x_t \mid t \in T \wedge \mathit{length}(t) \leqslant n\} \cup \bigcup_{\mathit{length}(s)=n} C_s$.

For each n, there exists a distinct s such that $\mathit{length}(s) = n$ and C_s is not empty; otherwise, (4) reads that A is the union of a subset of a finite set $\{x_t \mid \mathit{length}(t) \leqslant n\}$ and an empty set, but A is infinite. This particular s belongs to T by definition, so $\mathit{length}(T) = \aleph_0$. Each level T_n of T is finite, since there are only finitely many binary sequences of length n, so König's Lemma [3] applies, and T must have a branch W of length $\aleph_0$.

Let $w = \bigcup W$. Then w is an infinite binary sequence. In w, either 0 or 1 must occur infinitely often; without loss of generality, suppose 0 occurs infinitely often, and let $B = \{x_{w\upharpoonright n} \mid n \in \omega \wedge w(n) = 0\}$, where "$w \upharpoonright n$" denotes the restriction of w onto $[0, n)$, which is then a finite binary string. By our assumption and (2), B is infinite. However, any two different members of B are of the form $x_{w\upharpoonright n}$ and $x_{w\upharpoonright m}$ for some $n > m$ and $w(m) = w(n) = 0$. By (3), we have $\{x_{w\upharpoonright n}, x_{w\upharpoonright m}\}$ colored black. Hence we have shown the existence of an infinite subset B homogeneous for the given 2-coloring. □

Proposition 2.6. *If $\mathfrak{a} \to (\mathfrak{a})^n_2$, then $\mathfrak{a} \to (\mathfrak{a})^n_k$ for all finite $k \geqslant 2$.*

Proof. (By induction) The statement is trivial for $k = 2$. For $k > 3$, let us assume that the statement holds for $2, 3, 4, \ldots, k-1$. Consider $f : [A]^n \to k$ where $|A| = \mathfrak{a}$, and let $g : \{1, \ldots, k\} \to \{1, 2\}$ be defined as

$$g(x) = \begin{cases} 1, & x < k \\ 2, & x = k. \end{cases}$$

Then $g \circ f$ is a 2-coloring of $[A]^n$, so by the inductive hypothesis, there exists a homogeneous subset $B \subset A$ with cardinality $\mathfrak{a}$.

If $g \circ f\,[[B]^n] = 2$, it must be that $f\,[[B]^n] = k$, so B is also homogeneous for f as desired. On the other hand, if $g \circ f\,[[B]^n] = 1$, it implies that $f\,[[B]^n] = \{1, \ldots, k-1\}$. Since $f \upharpoonright_{[B]^n}$ is a $(k-1)$-coloring, by the induction hypothesis, there is a subset $C \subseteq B$ homogeneous for $f \upharpoonright_{[B]^n}$ having cardinality $|B| = \mathfrak{a}$. Then C is homogeneous with respect to f as well, and has the desired cardinality. □

Using the above proposition in conjunction with Theorem 2.5, we can strengthen our result on $\aleph_0$:

Corollary 2.7. $\aleph_0 \to (\aleph_0)^2_c$ *for all finite $c \geqslant 2$.*

The following flagship theorem allows us to substitute an arbitrary finite constant n into the subscript, borrowing the idea explored in Theorem 2.5.

Theorem 2.8. (Ramsey 1930) $\aleph_0 \to (\aleph_0)^n_c$ *for any finite n, c.*

We present two related proofs of this theorem. The first is inspired by Ramsey's original proof [4] and is easier to follow and historically interesting, whereas the second appeals to more formal set theory and formulates the method in a way that can be extended later.

Proof. #1. By Proposition 2.6, it suffices to consider $c = 2$, which we shall prove by induction on n. When $n = 1$, the theorem reads $\aleph_0 \to (\aleph_0)^1_2$, and this is trivially true from the observation that a partition of $\aleph_0$ into two disjoint sets must involve an infinite set.

For $n > 1$, assume that $\aleph_0 \to (\aleph_0)^{n-1}_2$, and let $\Gamma_0 = \aleph_0$. It may be that there exists $x_1 \in \Gamma_0$ and $\Gamma_1 \subseteq \Gamma_0 \backslash \{x_1\}$ so that any n-membered subset of Γ_0 formed by taking x_1 and an $(n-1)$-membered subset of Γ_1 is colored red. Then, it may also be that there exists $x_2 \in \Gamma_1$ and $\Gamma_2 \subseteq \Gamma_1 \backslash \{x_2\}$ so that any n-membered subset of Γ_1 formed by taking x_2 and an $(n-1)$-membered subset of Γ_2 is colored red. Similarly, this process may continue indefinitely, so that we can pick $x_{i+1} \in \Gamma_i$ and $\Gamma_{i+1} \subseteq \Gamma_i \backslash \{x_{i+1}\}$ where any n-membered subset formed by taking x_{i+1} and a $(n-1)$-membered subset of Γ_{i+1} is colored red.

If this is possible, then we have an infinite set $\{x_1, x_2, \ldots\}$ where all the x's are distinct. Let Z be an n-membered subset of $\{x_1, x_2, \ldots\}$. Let $t = \min\{t \mid x_t \in Z\}$. Then Z is a set formed by combining x_t and an $(n-1)$-membered subset of Γ_t, so it must be colored red. This shows that $\{x_1, x_2, \ldots\}$ is homogeneous for the given coloring.

It remains to consider the case in which the process halts at some Γ_i. Let y_1 be any member of Γ_i. Define a 2-coloring g on the $(n-1)$-hypergraph of $\Gamma_i \backslash \{y_1\}$ as follows: $g(X) = f(X \cup \{y_1\})$. By our inductive hypothesis, g admits an infinite homogeneous subset Δ_1. It must be that g does not send $(n-1)$-membered subsets of Δ_1 to red; otherwise Δ_1 qualifies to be $\Gamma_{i+1}$with $x_{i+1} = y_1$.

Now let y_2 be any element of Δ_1. By our preceding argument, there exists an infinite subset Δ_2 such that the union of $\{y_1\}$ with any $(n-1)$-membered subset of Δ_2 is colored the same. This cannot be red; otherwise Δ_2 qualifies to have been Γ_{i+1} with $x_{i+1} = y_2$. We can repeat this argument to generate the pair (y_{j+1}, Δ_{j+1}) from Δ_j indefinitely. In that case, $\{y_1, y_2, \ldots\}$ is homogeneous for the coloring in that any n-membered subset must be colored black.

Hence, in either case we have demonstrated the existence of an infinite homogeneous subset of Γ_0. □

Proof. #2. (By induction on n.) When $n = 1$, the theorem is trivial, as shown in the first proof.

For $n > 1$, assume that $\aleph_0 \to (\aleph_0)^{n-1}_c$. Given $f : [\omega]^n \to c$, we define a tree $\langle T, \supset \rangle$ whose elements are nonempty subsets of ω and whose partial order is given by the inverse of proper inclusion. We shall define recursively the m-th level of T, where $T_0 = \{\omega\}$. Meanwhile, we will simultaneously prove that

(5) T_m is finite,

(6) $\bigcup T_m = \omega \backslash \{\min D \mid D \in \bigcup_{l<m} T_l\}$,

(7) any two different members of T_m are disjoint, and

(8) for every $l < m$ and every $E \in T_m$, there is a unique member of $E_l \in T_l$ such that $E \subset E_l$.

(5)-(8) obviously hold for $m = 0$. Now suppose T_l is already defined for $l \leqslant m$ and that (5)-(8) hold for $l \leqslant m$. We need to define T_{m+1} and prove that (5)-(8) hold as well for $m+1$. Let $E \in T_m$. By (8), for each $l < m$, there is $E_l \in T_l$ that includes E. Set $y_l = \min E_l$ for $l < m$, namely the minimal element of E_l (when taken as a set of ordinals), and $y_m = \min E$. For every $u \in E \backslash \{y_m\}$, let $g_u : [\{y_0, \ldots, y_m\}]^{n-1}$ be the function given by

$$g_u(\{t_1, \ldots, t_{n-1}) = f(\{t_1, \ldots, t_{n-1}, u\})$$

for all $\{t_1, \ldots, t_{n-1}\} \in [\{y_0, \ldots, y_m\}]^{n-1}$. Define an equivalence relation $\approx_E$ on $E\backslash\{y_m\}$ by setting $u \approx_E v$ iff $g_u = g_v$. Let Q_E be the set of all equivalence classes. Since the number of functions mapping $[\{y_0, \ldots, y_m\}]^{n-1}$, which is finite, into c is finite, Q_E is finite as well, and $\bigcup Q_E = E\backslash\{y_m\} = E\backslash\{\min E\}$. We set $T_{m+1} = \bigcup_{E \in T_m} Q_E$. Thus T_{m+1} consists of nonempty subsets of ω, and since T_m is finite by (5), so T_{m+1} is a finite union of finite sets, therefore finite.

$$\begin{aligned}
\bigcup T_{m+1} &= \bigcup_{E \in T_m} \bigcup Q_E = \bigcup_{E \in T_m} (E\backslash\{\min E\}) \\
&= \left[\bigcup T_m\right] \backslash \{\min E \mid E \in T_m\} \text{ by (7)} \\
&= \omega \backslash \left\{ \min D \mid D \in \bigcup_{l<m} T_l \right\} \backslash \{\min E \mid E \in T_m\} \text{ by (6)} \\
&= \omega \backslash \left\{ \min D \mid D \in \bigcup_{l<m+1} T_l \right\}, \text{ so (6) holds for } m+1.
\end{aligned}$$

(7) for $m+1$ follows from (7) for m and the definition of T_{m+1}, since two elements of T_{m+1} are subsets of two elements of T_m, which are disjoint. Finally, the uniqueness in (8) for $m+1$ follows from (7) for m, and the existence for (8) for $m+1$ follows from (8) for m and the definition of T_{m+1}.

T, as constructed above, is clearly a tree of length ω, and by (5), each level is finite. Thus König's Lemma applies again, and T has an infinite branch $\{D_l \mid l < \omega\}$ where $D_l \in T_l$ for $l < \omega$. Denote $\min D_l$ by x_l and let $X = \{x_l \mid l < \omega\}$. For an $(n-1)$-tuple $\{x_{i_1}, \ldots, x_{i_{n-1}}\}$ with $i_1 < i_2 < \cdots < i_{n-1}$, let $m = i_{n-1}$. For all $l > m$, we have $x_l \in D_l \subseteq D_{m+1}$. Since D_{m+1} is an equivalence class of the relation $\approx_{D_m}$, $f(\{x_{i_1}, \ldots, x_{i_{n-1}}, u\}) = g_u(\{x_{i_1}, \ldots, x_{i_{n-1}}\})$ does not depend on u as long as $u \in D_{m+1}$. Hence $f(\{x_{i_1}, \ldots, x_{i_{n-1}}, x_{i_n}\})$ does not depend on i_n, and we can write this as $g(\{x_{i_1}, \ldots, x_{i_{n-1}}\})$. We now have $g : [X]^{n-1} \to c$, but by the induction hypothesis, there is a infinite subset $B \subseteq X$ homogeneous for g. Hence we have $f(\{x_{i_1}, \ldots, x_{i_n}\})$ constant for all $\{x_{i_1}, \ldots, x_{i_n}\} \in [B]^n$, implying that B is homogeneous for f. □

3 Generalizations to Larger Cardinals

In Theorem 2.8, we proved that $\aleph_0$ admits a homogeneous subset of cardinality

$\aleph_0$ for any finite coloring. This is in fact the largest homogeneous subset we could hope to find, since any subset must have a cardinality equal to or less than $\aleph_0$. However, the same result cannot be reproduced in the case of $\aleph_1$ or larger cardinals, as we shall see soon. We assume the Axiom of the Choice in the proofs of this section.

Theorem 3.1. (Sierpinski 1933) $2^{\aleph_0} \nrightarrow (\aleph_1)^2_2$.

Proof. (By counterexample.) Let $<$ be the natural ordering on $\mathbb{R}$, and let $<^*$ be a well-ordering on $\mathbb{R}$. Define a 2-coloring $f : [\mathbb{R}]^2 \to 2$ as follows:

$$f : \{x, y\} \mapsto \begin{cases} 0, & \text{if } x < y \Leftrightarrow x <^* y, \\ 1, & \text{if } \neg(x < y \Leftrightarrow x <^* y). \end{cases}$$

We claim that there is no homogeneous subset $B \subseteq \mathbb{R}$ with cardinality $\aleph_1$. If $f\left[[B]^2\right] = 0$ for some subset $B \subseteq \mathbb{R}$, it must be that $<$ and $<^*$ agree on $x, y \in B$. Thus $<$ well-orders B. Then we can consider a mapping $B \to \mathbb{Q}$ where $b \in B$ is mapped to a rational number between b and $Succ_{<^*}(b)$. (Since $\mathbb{Q}$ is dense in $\mathbb{R}$, a rational number between the two must exist.) Clearly this mapping is injective, so $B \leqslant_c \mathbb{Q} =_c \aleph_0 <_c \aleph_1$. If $f\left[[B]^2\right] = 1$, then the inverse of $<$ well-orders B, which yields the same result. Hence, it must be that no homogeneous subset of $2^{\aleph_0}$ has cardinality $\aleph_1$. □

Corollary 3.2. $\aleph_1 \nrightarrow (\aleph_1)^2_2$.

Proof. Since $2^{\aleph_0} \geqslant \aleph_1$, applying the contrapositive of Proposition 2.3 to Theorem 3.1 yields the corollary directly. □

Theorem 3.3. (Sierpinski 1933) *For every aleph* $\mathfrak{a}$*,* $2^{\mathfrak{a}} \nrightarrow (\mathfrak{a}^+)^2_2$.

Proof. (By counterexample.) This theorem generalizes upon Theorem 3.1; note that Theorem 3.1 is a special case of Theorem 3.3 with $\mathfrak{a} = \aleph_0$.

Lemma 3.4. *For an aleph* κ*, let* $<_A$ *be the induced left-lexicographic ordering of a set* $A \subseteq 2^\kappa$ *whose order type is* κ^+*. In other words, when we treat* $x, y \in A \subseteq 2^\kappa$ *as functions from* κ *to* $\{0, 1\}$*,*

$$x <_A y \Longleftrightarrow_{df} (\exists i \in \kappa)\left[x(i) = 0 \wedge y(i) = 1 \wedge x \upharpoonright i = y \upharpoonright i\right].$$

Let $>_A$ *be the inverse relation. Then neither* $(A, <_A)$ *nor* $(A, >_A)$ *is well-ordered.*

Proof. Suppose $(A, <_A)$ is well-ordered. Since A has order type κ^+, let f be the order-preserving bijection of κ^+ onto A. We claim that, by transfinite induction on μ,

(9) for every ordinal $\mu \leqslant \kappa$, there is an $\alpha_\mu < \kappa^+$ such that for every $\beta \geqslant \alpha_\mu$, $f(\beta) \upharpoonright \mu = f(\alpha_\mu) \upharpoonright \mu$.

For $\mu = 0$, (9) holds with $\alpha_0 = 0$. Suppose now that (9) holds for some μ. If $f(\beta)(\mu)$ (the μ-th bit of $f(\beta) \in A \subseteq 2^\kappa$) is zero for every $\beta \geqslant \alpha_\mu$, then set $\alpha_{\mu+1} = \alpha_\mu$. It is easy to see that whenever $\beta \geqslant \alpha_{\mu+1}$, we have $f(\beta) \upharpoonright \mu = f(\alpha_{\mu+1}) \upharpoonright \mu$ by hypothesis, and also $f(\beta)(\mu) = f(\alpha_{\mu+1})(\mu)$ by assumption. Hence (9) holds for $\mu + 1$. If $f(\beta)(\mu) = 1$ for some $\beta \geqslant \alpha_\mu$, then set $\alpha_{\mu+1}$ to be the least such β in $(A, <_A)$. Because we have assumed that $(A, <_A)$ is well-ordered, we can always find the least such β. Once again, $f(\beta) \upharpoonright \mu = f(\alpha_\mu) \upharpoonright \mu = f(\alpha_{\mu+1}) \upharpoonright \mu$ by hypothesis, and $f(\beta)(\mu) \geqslant_A f(\alpha_{\mu+1})(\mu) = 1$ by lexicographical ordering. Thus $f(\beta)(\mu)$ cannot be 0, and must be 1 as well. So (9) is satisfied for $\mu + 1$.

In case of μ being a limit ordinal, take $\alpha_\mu = \sup_{\lambda<\mu} \alpha_\lambda$. Recall that if the Axiom of Choice holds, all successor cardinals are regular. Hence κ^+ is regular and $\mu \leqslant \kappa$, so we have $\alpha_\mu < \kappa^+$. In addition $f(\alpha_{\lambda+1})(\lambda) = f(\alpha_\mu)(\lambda)$ for all $\lambda < \mu$ by (9) on λ, which gives us what we want.

We have thus shown that (9) holds by transfinite recursion. In particular, when $\mu = \kappa$, we get $f(\beta) \upharpoonright \kappa = f(\alpha_\kappa) \upharpoonright \kappa$ for all $\beta \geqslant \alpha_\kappa$. But since $f(\beta), f(\alpha_\kappa) \in (\kappa \to \{0, 1\})$, the restriction of their domains onto κ trivially preserves them. Hence $f(\beta) = f(\alpha_\kappa)$. But this contradicts our assumption that f is injective.

The proof for $(A, >_A)$ is entirely analogous. □

(Proof of Theorem 3.3 continued.) Again, let $<$ be the lexicographical ordering on $2^{\mathfrak{a}}$, and let $<^*$ be a well-ordering on $\mathfrak{a}^+$. Define $f : [2^{\mathfrak{a}}]^2 \to \{0, 1\}$ as in the proof of Theorem 3.1. If a subset $B \subseteq 2^{\mathfrak{a}}$ of order type κ^+ is homogeneous for f, then either $<_B$ or its inverse must agree with $<^*$ on B, in which case $(B, <_B)$ or $(B, >_B)$ is well-ordered. This is explicitly forbidden by our previous lemma, so it cannot have been that B has order type κ^+. Therefore $2^{\mathfrak{a}} \nrightarrow (\mathfrak{a}^+)^2_2$. □

Corollary 3.5. *For every aleph* $\mathfrak{a}$, $\mathfrak{a}^+ \nrightarrow (\mathfrak{a}^+)^2_2$.

Proof. The corollary follows from the fact that $2^{\mathfrak{a}} \geqslant \mathfrak{a}^+$. □

It is now clear that we cannot hope to achieve $\mathfrak{a} \to (\mathfrak{a})^n_c$ for arbitrary uncountable cardinal $\mathfrak{a}$, but perhaps we will be able to establish some positive results by employing the technique in the proof of Theorem 2.8. In its proof, we showed $\aleph_0 \to (\aleph_0)^n_c$ by obtaining a sequence $\langle x_l \mid l < \omega \rangle$ in $\aleph_0$ such that

(10) for all $i_1 < i_2 < \cdots < i_n$, the color $f(\{x_{i_1}, \ldots, x_{i_n}\})$ depends only on $i_1, \ldots, i_{n-1}$,

which reduced the desired statement to the inductive hypothesis $\aleph_0 \to (\aleph_0)^{n-1}_c$. Similarly, given $\mathfrak{b} \to (\mathfrak{a})^{n-1}_c$, we want to obtain $\mathfrak{d} \to (\mathfrak{a})^n_c$ for as small a $\mathfrak{d}$ as possible by constructing a sequence $\langle x_l \mid l < \mathfrak{b} \rangle$ in $\mathfrak{d}$ that satisfies (10). For this technique to yield what we want, we need $\mathfrak{d} = \mathfrak{b}$, which can be obtained by specifying $\mathfrak{a}$ as a weakly compact cardinal.

Definition 3.6. (i) *An infinite cardinal* λ *is a* **strong limit cardinal** *in case it is regular and for all* $\kappa < \lambda$*, we have* $2^\kappa < \lambda$*.*
(ii) *An uncountable cardinal* Γ *is a* **weakly compact cardinal** *in case it is both a strong limit cardinal and has the tree property: any tree of size* Γ *has a branch of length* Γ *or a level of size* Γ*.*

The simplest example of a strong limit cardinal is $\aleph_0$. On the other hand, the existence of a weakly compact cardinal has not been proven or disproven with the standard axioms of set theory. Hence the following theorem may strike as a vacuous statement, but in fact its converse is also true, which is a result due to Erdös and Tarski in 1961.[2] Therefore, whether there exists any cardinal $\mathfrak{a}$ besides $\aleph_0$ satisfying $\mathfrak{a} \to (\mathfrak{a})^n_{\mathfrak{c}}$ (for all finite n and all $\mathfrak{c} < \mathfrak{a}$) is nicely tied with the existence of weakly compact cardinals.

Theorem 3.7. (Erdös and Tarski 1961) *If an uncountable cardinal* $\mathfrak{a}$ *is also a weakly compact cardinal, then for all n finite and for all* $\mathfrak{c} < \mathfrak{a}$*, we have* $\mathfrak{a} \to (\mathfrak{a})^n_{\mathfrak{c}}$*.*

Proof. As before, we will construct a tree T of non-empty subsets of $\mathfrak{b}$, partially ordered by inverse proper inclusion. One important difference is that now T will have length $\mathfrak{b}$ rather than ω, and each level T_μ is going to have cardinality less than $\mathfrak{b}$ rather than be finite. So construction of T will require transfinite recursion. For all $\mu < \mathfrak{b}$, the following will hold:

(11) $\bigcup T_\mu = \mathfrak{b} \backslash \left\{ \min D \mid D \in \bigcup_{\lambda<\mu} T_\mu \right\}$,

(12) any two different members of T_μ are disjoint, and

(13) for every $\lambda < \mu$ and every $E \in T_\mu$, there is a unique member of $E_\lambda \in T_\lambda$ such that $E \subset E_\lambda$.

Note the similarity to (6)-(8).

Set $T_0 = \{\mathfrak{b}\}$. Then (11)-(13) trivially hold for $\mu = 0$. Now suppose T_μ is defined already and (11)-(13) hold for μ. Let $E \in T_\mu$, and for every $\lambda \leqslant \mu$, let E_λ be the unique member that includes E, as in (13), and let $y_\lambda = \min E_\lambda$. For every $u \in E \backslash \{y_\lambda\}$, let $g_u : [\{y_\lambda \mid \lambda \leqslant \mu\}]^{n-1} \to \mathfrak{c}$ be the function given by $g_u(\{t_1, \ldots, t_{n-1}\}) = f(\{t_1, \ldots, t_{n-1}, u\})$. Proceed to define an equivalent relation $\approx_E$ on $E \backslash \{y_\mu\}$ by setting $u \approx_E v$ iff $g_u = g_v$, and let Q_E be the set of equivalence classes of $\approx_E$.

The number of functions $g : [\{y_\lambda \mid \lambda \leqslant \mu\}]^{n-1} \to \mathfrak{c}$ is $\mathfrak{c}^{|\mu|}$ for μ infinite and at most $\mathfrak{c}^{\aleph_0}$ for μ finite, so we have $|Q_E| \leqslant \mathfrak{c}^{|\mu|+\aleph_0}$. Now set $T_{\mu+1} = \bigcup_{E \in T_\mu} Q_E$, hence

(14) $|T_{\mu+1}| \leqslant |T_\mu| \cdot \mathfrak{c}^{|\mu|+\aleph_0}$.

Meanwhile, (11)-(13) again hold for $\mu+1$. The proof is exactly analogous to how we demonstrated that (6)-(8) held for $m+1$ in Theorem 2.5, with the exception that we have $\mathfrak{b}$ for ω and μ for m.

Note that we are constructing T using transfinite recursion. Hence now we must consider the case in which μ is a limit ordinal. Then, we set $T_\mu = \left\{\bigcap_{\lambda<\mu} h(\lambda) \mid h \text{ is a branch in } T \mid \mu\right\}\backslash\{0\}$. In other words, T_μ is the set of all non-empty intersections along all the branches of $T \mid \mu$. (Since each branch corresponds a sequence of sets that are ordered in inverse proper inclusion, for each branch, we can take the intersection of all these sets.) We can see that (9) holds: let $u \in \mathfrak{b}\backslash\left\{\min D \mid D \in \bigcup_{\lambda<\mu} T_\lambda\right\}$, then, since (9) holds for $\lambda < \mu$ already, we have $u \in \bigcup T_\lambda$, hence for some $E_\lambda \in T_\lambda$ we have $u \in E_\lambda$, and therefore $u \in \bigcup_{\lambda_\mu} E_\lambda \in T_\mu$. Conversely, if $u \in \bigcup T_\mu$, then by definition of T_μ, $u \in \bigcap_{\lambda<\mu} h(\lambda)$ for some branch h of $T|\mu$. Since (9) holds for $\lambda < \mu$, we have $u \notin \left\{\min D \mid D \in \bigcup_{\xi<\lambda} T_\xi\right\}$ for every $\lambda < \mu$.

(10) and (11) for limit ordinal μ follows from the definition of T_μ, and we also have

(15) $\quad |T_\mu| \leqslant \prod_{\lambda<\mu} |T_\lambda|$.

Recall that what we needed is a branch of T of length $\mathfrak{b}$. If $\langle D_\lambda \mid \lambda < \mathfrak{b}\rangle$ is such a branch where $D_\lambda \in T_\lambda$ for $\lambda < \mathfrak{b}$, then set $x_\lambda = \min D_\lambda$ for $\lambda < \mathfrak{b}$. Let $i_1 < i_2 < \cdots < i_n < \mathfrak{b}$ and let $i_{n-1} < j < \mathfrak{b}$. Then we have $i_n, j \geqslant i_{n-1} + 1$, and since $\{D_\lambda \mid \lambda < \mathfrak{b}\}$ is a branch, we have $D_{i_n}, D_j \subseteq D_{i_{n-1}+1}$, hence $x_{i_n}, x_j \in D_{i_{n-1}+1}$. By the definition of $D_{i_{n-1}+1}$, both x_{i_n} and x_j belong to the same equivalence class with respect to $\approx_{D_{i_{n-1}}}$, in which case $f(\{x_{i_1}, \ldots, x_{i_n}\}) = f(\{x_{i_1}, \ldots, x_{i_{n-1}}, x_j\})$ as desired, since f is now independent of its n-th parameter.

It finally remains to show that we can always find a branch of length $\mathfrak{b}$. By induction, $|T_\lambda| \leqslant 2^{|\lambda|+\aleph_0+\mathfrak{c}}$: this is true when $\lambda = 0$, since $|T_0| = |\{\mathfrak{b}\}| = 1$. For $\lambda + 1$, we have, by inductive hypothesis and (14),

$$|T_{\lambda+1}| \leqslant |T_\lambda| \cdot \mathfrak{c}^{|\mu|+\aleph_0} \leqslant 2^{|\lambda|+\aleph_0+\mathfrak{c}} \cdot (2^{\mathfrak{c}})^{|\lambda|+\aleph_0} = (2^{|\lambda|+\aleph_0+\mathfrak{c}})^2 = 2^{|\lambda|+\aleph_0+\mathfrak{c}}.$$

(Recall that if the Axiom of Choice holds, the product of two infinite cardinals is the maximum of the two. Cardinal arithmetic of the exponent above is thus simplified.)

For a limit ordinal λ we have, by (15),

$$|T_\lambda| \leqslant \prod_{\mu<\lambda} |T_\mu| \leqslant \prod_{\mu<\lambda} 2^{|\mu|+\aleph_0+\mathfrak{c}} = 2^{[\sum_{\mu<\lambda} |\mu|]+|\lambda|\cdot\aleph_0+|\lambda|\cdot\mathfrak{c}} = 2^{|\lambda|+\aleph_0+\mathfrak{c}}.$$

Since $\mathfrak{b}$ is weakly compact, it is also a strong limit cardinal, meaning $2^{|\lambda|+\aleph_0+\mathfrak{c}} = 2^\lambda < \mathfrak{b}$ for $\lambda < \mathfrak{b}$. Hence $|T_\lambda| < \mathfrak{b}$ for $\lambda < \mathfrak{b}$. At this point we can invoke the tree property of a weakly compact cardinal, and conclude that T must have a branch of length $\mathfrak{b}$, since it has no level of size $\mathfrak{b}$. □

Theorem 3.8. (Erdös and Tarski 1961) *If an uncountable cardinal $\mathfrak{a}$ satisfies $\mathfrak{a} \to (\mathfrak{a})^n_{\mathfrak{c}}$ for all finite n and all $\mathfrak{c} < \mathfrak{a}$, then $\mathfrak{a}$ is weakly compact.*

Proof. First, regularity of $\mathfrak{a}$ will be shown in Proposition 4.1. Next, suppose the contrary that $\mathfrak{a}$ is not a strong limit cardinal. Then there must exist κ such that $\kappa < \mathfrak{a} \leqslant 2^\kappa$. By Theorem 3.3, we have $2^\kappa \nrightarrow (\kappa^+)^2_2$, which implies $\mathfrak{a} \nrightarrow (\mathfrak{a})^2_2$ via Proposition 2.3, yielding a contradiction. Finally, to show the tree property, take $\langle T, <_T \rangle$ to be a $\mathfrak{a}$-tree. We can assume that $T = \mathfrak{a}$ for convenience. For $x < \mathfrak{a}$ at level m or higher, define $\pi_m(x)$ to be the predecessor of x at the m-th level. Now, we can specify a linear ordering on T as follows:

$$x <^{\#} y \Longleftrightarrow_{df} \begin{array}{l} \pi_m(x) < \pi_m(y) \text{ where } m \text{ is the least} \\ \text{level at which the predecessors differ.} \end{array}$$

Now construct $f : [\mathfrak{a}]^2 \to \{0, 1\}$ by letting $f(x, y) = 0$ if $x <^{\#} y$ and $f(x, y) = 1$ otherwise. Since $\mathfrak{a} \to (\mathfrak{a})^2_2$ holds, there is a homogeneous subset W of cardinality $\mathfrak{a}$ for f.

If T has a level of size $\mathfrak{a}$, we are trivially done, so suppose the contrary. Then for each $m < \mathfrak{a}$, there is a $\sigma_m < \mathfrak{a}$ such that if $x \in W$ satisfies $\sigma_m <^{\#} x$, then x must be at level m or higher. By definition of the linear ordering above, if $x <^{\#} y$ are both at level m or higher, then $\pi_m(x) = \pi_m(y)$ or $\pi_m(x) <^{\#} \pi_m(y)$. Therefore, if $f\left[[W]^2\right] = 0$, then $\{\pi_m(x) \mid \sigma_m < x \wedge x \in W\}$ is non-decreasing in the linear ordering with respect to x. Then there exists $\tau_m < \mathfrak{a}$ and a b_m such that $\tau_m < x \wedge x \in W$ implies $\pi_m(x) = b_m$. Then $\{b_m \mid m < \mathfrak{a}\}$ is a branch of length $\mathfrak{a}$. Hence $\mathfrak{a}$ has the tree property. If $f\left[[W]^2\right] = 1$, the same follows analogously.

We have shown that $\mathfrak{a}$ is a regular strongly limit cardinal with the tree property. By definition, $\mathfrak{a}$ is then a weakly compact cardinal. □

We close this section by remarking that in the proof of Theorem 3.7, even if $\mathfrak{a}$ is not a weakly compact cardinal, it is possible to obtain $\mathfrak{d} \to (\mathfrak{a})^n_c$ from $\mathfrak{b} \to (\mathfrak{a})^{n-1}_c$ where $\mathfrak{d}$ is the set of binary sequences of length less than $\mathfrak{b}$, which is a result due to Erdös and Rado in 1956. [2]

4 Generalizations in Other Parameters

So far we have focused on relations of the form $\mathfrak{a} \to (\mathfrak{b})^n_c$ where n and c are finite. In this section we briefly relax these constraints and extend our investigation onto cases in which n and c are infinite.

(We have already derived some applicable results in the previous sections: Proposition 2.6 also holds when n is infinite, and so does Theorem 3.7 when $\mathfrak{c} < \mathfrak{a}$ is infinite.)

Proposition 4.1. *For all alephs* $\mathfrak{a}$*,* $\mathfrak{a} \to (\mathfrak{a})^1_{\mathfrak{c}}$ *iff* $\mathfrak{c} < cf(\mathfrak{a})$ *where* $cf(\mathfrak{a})$ *is the cofinality of* $\mathfrak{a}$*.*

Proof. By definition, in order to express $\mathfrak{a}$ as a union of sets of smaller cardinality, we require $cf(\mathfrak{a})$-many sets. Hence, if $\mathfrak{a}$ has been partitioned by a $\mathfrak{c}$-coloring with $\mathfrak{c} < cf(\mathfrak{a})$, it must be that one of the parts has cardinality of $\mathfrak{a}$. On the other hand, if $\mathfrak{c} \geqslant cf(\mathfrak{a})$, then there exists a partition into $\mathfrak{c}$-many sets with each part having smaller cardinality than $\mathfrak{a}$, so no homogeneous subset of $\mathfrak{a}$ would exist for this partition. □

Now we ask when the relation $\mathfrak{a} \to (\mathfrak{b})^n_c$ holds with $n = \aleph_0$; namely, what happens if we color each countably infinite subset of $\mathfrak{a}$? Note that in order to have a subset of cardinality $\aleph_0$, both $\mathfrak{a}$ and $\mathfrak{b}$ must be infinite. Hence, the weakest form of the relation would read:

$$\mathfrak{a} \to (\aleph_0)^{\aleph_0}_2,$$

according to Proposition 2.3. Nonetheless, it turns out that whenever $\mathfrak{a}$ is an aleph, the relation fails to hold. This is the strongest result we could have hoped to obtain, as we demonstrate in the next theorem.

Theorem 4.2. *For all alephs* κ*,* $\kappa \nrightarrow (\aleph_0)^{\aleph_0}_2$*.*

Proof. Let $\{A_\alpha \colon \alpha < \kappa^{\aleph_0}\}$ be a well-ordering of the set $\kappa^{\aleph_0}$. Define a two-coloring of $\kappa^{\aleph_0}$ by the following rule: $A_\alpha \in \kappa^{\aleph_0}$ is colored red iff $A_\beta \not\subset A_\alpha$ for every $\beta < \alpha$. Now let X be an infinite subset of κ. We show that $X^{\aleph_0}$ contains both red and non-red elements.

First, let $X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \cdots$ be an arbitrary sequence of infinite subsets of X. This can be constructed by choosing X_1 and removing one element at a time. Then let $Y = \{\alpha \mid (\exists r)[A_\alpha = X_r]\}$, and take its least element in the given well-ordering. Denote this element by $A_\alpha = X_r$ for some α, r; also let β be such that $A_\beta = X_{r+1}$. Then we see that $A_\beta \subset A_\alpha$ while $\alpha < \beta$ by choice of α, implying that $A_\beta = X_{r+1} \in X^{\aleph_0}$ is not colored red.

Second, let $\alpha = \min\{\beta \mid A_\beta \subset X\}$. Then $A_\alpha \in X^{\aleph_0}$, and also A_α is colored red by definition. Hence, X contains both red and non-red elements, meaning that it must be that $\mathfrak{a} \nrightarrow (\aleph_0)^{\aleph_0}_2$. □

References

[1] Hajnal, András; Hamburger, Peter, *Set Theory,* (New York: Cambridge University Press, 1999)

[2] Levy, Azriel, *Basic Set Theory,* (Mineola, NY: Dover, 2002)

[3] Moschovakis, Yiannis, *Notes on Set Theory,* (New York: Springer, 2006)

[4] Ramsey, Frank P., *On a Problem of Formal Logic,* Proc. London Math. Soc. series 2, vol. 30 (1930), pp.264-286.