FOUNDATIONS OF ALGEBRAIC GEOMETRY CLASS 18

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CONTENTS

1. Proper morphisms 1

1. PROPER MORPHISMS

I'll now tell you about a new property of morphisms, the notion of *properness*. You can think about this in several ways.

Recall that a map of topological spaces (also known as a continuous map!) is said to be proper if the preimage of compact sets is compact. Properness of morphisms is an analogous property. For example, proper varieties over $\mathbb{C}$ will be the same as compact in the "usual" topology. Alternatively, we will see that projective A-schemes are proper over A — this is the hardest thing we will prove — so you can see this as nice property satisfied by projective schemes, and quite convenient to work with.

A (continuous) map of topological spaces $f : X \to Y$ is **closed** if for each closed subset $S \subset X$, $f(S)$ is also closed. A morphism of schemes is closed if the underlying continuous map is closed. We say that a morphism of schemes $f : X \to Y$ is **universally closed** if for every morphism $g : Z \to Y$, the induced morphism $Z \times_Y X \to Z$ is closed. In other words, a morphism is universally closed if it remains closed under any base change. (A note on terminology: if P is some property of schemes, then a morphism of schemes is said to be "universally P" if it remains P under any base change.)

A morphism $f : X \to Y$ *is* **proper** *if it is separated, finite type, and universally closed.* A scheme X is often said to be proper if some implicit structure morphism is proper. For example, a k-scheme X is often described as proper if $X \to \operatorname{Spec} k$ is proper. (A k-scheme is often said to be *complete* if it is proper. We will not use this terminology.)

Let's try this idea out in practice. We expect that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, because the complex manifold corresponding to $\mathbb{A}^1_{\mathbb{C}}$ is not compact. However, note that this map is separated (it is a map of affine schemes), finite type, and closed. So the "universally" is what matters here.

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1.A. EXERCISE. Show that $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$ is not proper, by finding a base change that turns this into a non-closed map. (Hint: Consider $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.)

1.1. As a first example: closed immersions are proper. They are clearly separated, as affine morphisms are separated. They are finite type. After base change, they remain closed immersions, and closed immersions are always closed.

1.2. *Proposition. —*

(a) *The notion of "proper morphism" is stable under base change.*
(b) *The notion of "proper morphism" is local on the target (i.e.* $f : X \to Y$ *is proper if and only if for any affine open cover* $U_i \to Y$*,* $f^{-1}(U_i) \to U_i$ *is proper). Note that the "only if" direction follows from (a) — consider base change by* $U_i \hookrightarrow Y$*.*
(c) *The notion of "proper morphism" is closed under composition.*
(d) *The product of two proper morphisms is proper (i.e. if* $f : X \to Y$ *and* $g : X' \to Y'$ *are proper, where all morphisms are morphisms of* Z*-schemes) then* $f \times g : X \times_Z X' \to Y \times_Z Y'$ *is proper.*
(e) *Suppose*

(1)
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Z & \end{array}$$

is a commutative diagram, and g *is proper, and* h *is separated. Then* f *is proper.*

A sample application of (e): A morphism (over $\operatorname{Spec} k$) from a proper k-scheme to a separated k-scheme is always proper.

Proof. (a) We have already shown that the notions of separatedness and finite type are local on the target. The notion of closedness is local on the target, and hence so is the notion of universal closedness.

(b) The notions of separatedness, finite type, and universal closedness are all preserved by fibered product. (Notice that this is why universal closedness is better than closedness — it is automatically preserved by base change!)

(c) The notions of separatedness, finite type, and universal closedness are all preserved by composition.

(d) By (a) and (c), this follows from an earlier exercise showing that a property of morphisms preserved by composition and base change is also preserved by products.

(e) Closed immersions are proper, so we invoke the Cancellation Theorem for properties of morphisms. □

We now come to the most important example of proper morphisms.

1.3. *Theorem. — Projective* A*-schemes are proper over* A.

It is not easy to come up with an example of an A-scheme that is proper but not projective! We will see a simple example of a proper but not projective surface, . Once we discuss blow-ups, I'll give Hironaka's example of a proper but not projective nonsingular ("smooth") threefold over $\mathbb{C}$.

Proof. The structure morphism of a projective A-scheme $X \to \operatorname{Spec} A$ factors as a closed immersion followed by $\mathbb{P}^n_A$. Closed immersions are proper, and compositions of proper morphisms are proper, so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper. We have already seen that this morphism is finite type (an earlier easy exercise) and separated (shown last week by hand), so it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is universally closed. As $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} A$, it suffices to show that $\mathbb{P}^n_X := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} X \to X$ is closed for any scheme X. But the property of being closed is local on the target on X, so by covering X with affine open subsets, it suffices to show that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed. This is important enough to merit being stated as a Theorem.

1.4. *Theorem. —* $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ *is a closed morphism.*

This is sometimes called the fundamental theorem of elimination theory. Here are some examples to show you that this is a bit subtle.

First, let $A = k[a, b, c, \dots, i]$, and consider the closed subscheme of $\mathbb{P}^2_A$ (taken with coordinates x, y, z) corresponding to $ax + by + cz = 0$, $dx + ey + fz = 0$, $gx + hy + iz = 0$. Then we are looking for the locus in $\operatorname{Spec} A$ where these equations have a non-trivial solution. This indeed corresponds to a Zariski-closed set — where

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

As a second example, let $A = k[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$. Now consider the closed subscheme of $\mathbb{P}^1_A$ (taken with coordinates x and y) corresponding to $a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m = 0$ and $b_0 x^n + b_1 x^{m-1} y + \cdots + b_n y^n = 0$. Then we are looking at the locus in $\operatorname{Spec} A$ where these two polynomials have a common root — this is known as the *resultant*.

More generally, this question boils down to the following question. Given a number of homogeneous equations in $n+1$ variables with indeterminate coefficients, Proposition 1.4 implies that one can write down equations in the coefficients that will precisely determine when the equations have a nontrivial solution.

Proof of Theorem 1.4. Suppose $Z \hookrightarrow \mathbb{P}^n_A$ is a closed *subset*. We wish to show that $\pi(Z)$ is closed.

Suppose $y \notin \pi(Z)$ is a *closed* point of $\operatorname{Spec} A$. We'll check that there is a distinguished open neighborhood $D(f)$ of y in $\operatorname{Spec} A$ such that $D(f)$ doesn't meet $\pi(Z)$. (If we could show this for *all* points of $\pi(Z)$, we would be done. But I prefer to concentrate on closed points for now.) Suppose y corresponds to the maximal ideal $\mathfrak{m}$ of A. We seek $f \in A - \mathfrak{m}$ such that $\pi^* f$ vanishes on Z.

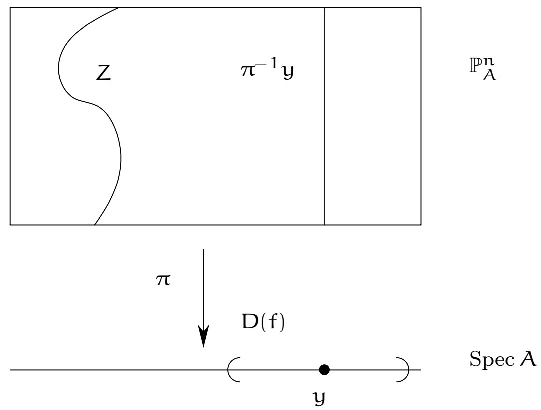

FIGURE 1

Let $U_0, \ldots, U_n$ be the usual affine open cover of $\mathbb{P}^n_A$. The closed subsets $\pi^{-1}y$ and Z do not intersect (see Figure 1). On the affine open set U_i, we have two closed subsets $Z \cap U_i$ and $\pi^{-1}y \cap U_i$ that do not intersect, which means that the ideals corresponding to the two closed sets generate the unit ideal, so in the ring of functions $A[x_{0/i}, x_{1/i}, \ldots, x_{n/i}]$. on U_i, we can write

$$1 = a_i + \sum m_{ij} g_{ij}$$

where $m_{ij} \in \mathfrak{m}$, and a_i vanishes on Z. Note that $a_i, g_{ij} \in A[x_{0/i}, \ldots, x_{n/i}]$, so by multiplying by a sufficiently high power x_i^n of x_i, we have an equality

$$x_i^N = a_i' + \sum m_{ij} g_{ij}'$$

on U_i, where both sides are expressions in $S_\bullet = A[x_0, \ldots, x_n]$. We may take N large enough so that it works for all i. Thus for N' sufficiently large, we can write any monomial in $x_1, \ldots, x_n$ of degree N' as something vanishing on Z plus a linear combination of elements of $\mathfrak{m}$ times other polynomials. Hence

$$S_{N'} = I(Z)_{N'} + \mathfrak{m} S_{N'}$$

where $I(Z)_*$ is the graded ideal of functions vanishing on Z. We now need Nakayama's lemma. If you haven't seen this result before, we will prove it next week. We will use the following form of it: if M is a finitely generated module over A such that $M = \mathfrak{m}M$ for some maximal ideal $\mathfrak{m}$, then there is some $f \notin \mathfrak{m}$ such that $fM = 0$. Applying this in the case where $M = S_{N'}/I(Z)_{N'}$, we see that there exists $f \in A - \mathfrak{m}$ such that

$$f S_{N'} \subset I(Z)_{N'}.$$

Thus we have found our desired f.

We now tackle Theorem 1.4 in general. Suppose $y = [\mathfrak{p}]$ not in the image of Z. Applying the above argument in $\operatorname{Spec} A_\mathfrak{p}$, we find $S_{N'} \otimes A_\mathfrak{p} = I(Z)_{N'} \otimes A_\mathfrak{p} + \mathfrak{m} S_{N'} \otimes A_\mathfrak{p}$, from which $g(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$ for some $g \in A_\mathfrak{p} - \mathfrak{p}A_\mathfrak{p}$, from which $(S_{N'}/I(Z)_{N'}) \otimes A_\mathfrak{p} = 0$. As $S_{N'}$ is a finitely generated A-module, there is some $f \in A - \mathfrak{p}$ with $fS_N \subset I(Z)$ (if the module-generators of $S_{N'}$, and $f_1, \ldots, f_a$ are annihilate the generators respectively, then take $f = \prod f_i$), so once again we have found $D(f)$ containing $\mathfrak{p}$, with (the pullback of) f vanishing on Z. $\square$

Notice that projectivity was essential to the proof: we used graded rings in an essential way.

This also concludes the proof of Theorem 1.3.

1.5. *Corollary. — Finite morphisms are proper.*

Proof. Suppose $f : X \to Y$ is a finite morphism. As properness is local on the base, to check properness of f, we may assume Y is affine. But finite morphisms to $\operatorname{Spec} A$ are projective , and projective morphisms are proper. $\square$

In particular, as promised in our initial discussion of finiteness:

1.6. *Corollary. — Finite morphisms are closed.*

1.7. Unproved facts that may help you correctly think about finiteness.

We conclude with some interesting facts that we will prove later. They may shed some light on the notion of finiteness.

A morphism is finite if and only if it is proper and affine, if and only if it is proper and quasifinite. We have verify the "only if" parts of this statement; the "if" parts are harder.

As an application: quasifinite morphisms from proper schemes to separated schemes are finite. Here is why: suppose $f : X \to Y$ is a quasifinite morphism over Z, where X is proper over Z. Then by the Cancellation Theorem for properties of morphisms, $X \to Y$ is proper. Hence as f is quasifinite and proper, f is finite.

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