# FOUNDATIONS OF ALGEBRAIC GEOMETRY PROBLEM SET 10

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**This set covers classes 19 and 20.**

Please *read all of the problems*, and ask me about any statements that you are unsure of, even of the many problems you won't try. Hand in nine solutions, where each "-" problem is worth half a solution, each "+" problem is worth one-and-a-half, and each "++" problem is worth two. *You are allowed to hand in up to three problems from previous sets that you have not done.* If you are ambitious (and have the time), go for more. Try to solve problems on a range of topics. You are encouraged to talk to each other, and to me, about the problems. Some of these problems require hints, and I'm happy to give them!

**1.** Show that if $Y$ is an irreducible subset of a scheme $X$ with generic point $y$, show that the codimension of $Y$ is the dimension of the local ring $\mathcal{O}_{X,y}$.

**2-.** Show that

$$\operatorname{codim}_X Y + \dim Y \leq \dim X. \tag{1}$$

**3++.** Show that if $f : B \to A$ is a ring homomorphism, and $(b_1, \dots, b_n) = 1$ in $B$, and $B_{b_i} \to A_{f(b_i)}$ is integral, then $f$ is integral. Thus we can define the notion of **integral morphism** of schemes.

**4+.** Show that the notion of integral homomorphism is well behaved with respect to localization and quotient of $B$, and quotient of $A$, but not localization of $A$. Show that the notion of integral extension is well behaved with respect to localization and quotient of $B$, but not quotient of $A$. If possible, draw pictures of your examples.

**5.** Show that if $B$ is an integral extension of $A$, and $C$ is an integral extension of $B$, then $C$ is an integral extension of $A$.

**6-.** *(finite = integral + finite type)* Show that a morphism is finite if and only if it is integral and finite type.

**7-.** *(reality check)* The morphism $k[t] \to k[t]_{(t)}$ is not integral, as $1/t$ satisfies no monic polynomial with coefficients in $k[t]$. Show that the conclusion of the Going-up theorem fails.

**8.** Show that the special case of the Going-Up Theorem where $A$ is a field translates to: if $B \subset A$ is a subring with $A$ integral over $B$, then $B$ is a field. Prove this. (Hint: all you need to do is show that all nonzero elements in $B$ have inverses in $B$. Here is the start: If $b \in B$, then $1/b \in A$, and this satisfies some integral equation over $B$.)

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**9+.** *(important but straightforward exercise, sometimes also called the going-up theorem)* Show that if $\mathfrak{q}_1 \subset \mathfrak{q}_2 \subset \cdots \subset \mathfrak{q}_n$ is a chain of prime ideals of $B$, and $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of $A$ such that $\mathfrak{p}_i$ "lies over" $\mathfrak{q}_i$ (and $m < n$), then the second chain can be extended to $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ so that this remains true.

**10++.** Show that if $f : \operatorname{Spec} A \to \operatorname{Spec} B$ corresponds to an integral *extension of rings*, then $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$. (Hint: show that a chain of prime ideals downstairs gives a chain upstairs, by the previous exercise, of the same length. Conversely, a chain upstairs gives a chain downstairs. We need to check that no two elements of the chain upstairs goes to the same element $[\mathfrak{q}] \in \operatorname{Spec} B$ of the chain downstairs. As integral extensions are well-behaved by localization and quotients of $\operatorname{Spec} B$ (Exercise ), we can replace $B$ by $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ (and $A$ by $A \otimes_B (B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}})$). Thus we can assume $B$ is a field. Hence we must show that if $\phi : k \to A$ is an integral extension, then $\dim A = 0$. Outline of proof: Suppose $\mathfrak{p} \subset \mathfrak{m}$ are two prime ideals of $\mathfrak{p}$. Mod out by $\mathfrak{p}$, so we can assume that $A$ is a domain. I claim that any non-zero element is invertible: Say $x \in A$, and $x \neq 0$. Then the minimal monic polynomial for $x$ has non-zero constant term. But then $x$ is invertible — recall the coefficients are in a field.)

**11.** *(Nakayama's lemma version 3)* Suppose $A$ is a ring, and $I$ is an ideal of $A$ contained in all maximal ideals. Suppose $M$ is a *finitely generated* $A$-module, and $N \subset M$ is a submodule. If $N/IN \to M/IM$ an isomorphism, then $M = N$. (This can be useful, although it won't come up again for us.)

**12+.** *(Nakayama's lemma version 4)* Suppose $(A, \mathfrak{m})$ is a local ring. Suppose $M$ is a finitely-generated $A$-module, and $f_1, \dots, f_n \in M$, with (the images of) $f_1, \dots, f_n$ generating $M/\mathfrak{m}M$. Then $f_1, \dots, f_n$ generate $M$. (In particular, taking $M = \mathfrak{m}$, if we have generators of $\mathfrak{m}/\mathfrak{m}^2$, they also generate $\mathfrak{m}$.)

**13.** *(Nakayama's lemma version 5)* Prove Nakayama version 1 without the hypothesis that $M$ is finitely generated, but with the hypothesis that $I^n = 0$ for some $n$. (This argument does *not* use the trick.) This result is quite useful, although we won't use it.

**14+.** *(used in the proof of Algebraic Hartogs' Lemma)* Suppose $S$ is a subring of a ring $A$, and $r \in A$. Suppose there is a faithful $S[r]$-module $M$ that is finitely generated as an $S$-module. Show that $r$ is integral over $S$. (Hint: look carefully at the proof of Nakayama's Lemma version 1, and change a few words.)

**15+.** *(Nullstellensatz from dimension theory)*
(a) Suppose $A = k[x_1, \dots, x_n]/I$, where $k$ is an algebraically closed field and $I$ is some ideal. Then the maximal ideals are precisely those of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$. This version (the "weak Nullstellensatz") was stated earlier.
(b) Suppose $A = k[x_1, \dots, x_n]/I$ where $k$ is not necessarily algebraically closed. Show that every maximal ideal of $A$ has a residue field that is a finite extension of $k$. This version was stated in earlier. (Hint for both parts: the maximal ideals correspond to dimension $0$ points, which correspond to transcendence degree $0$ extensions of $k$, i.e. finite extensions of $k$. If $k = \overline{k}$, the maximal ideals correspond to surjections $f : k[x_1, \dots, x_n] \to k$. Fix one such surjection. Let $a_i = f(x_i)$, and show that the corresponding maximal ideal is $(x_1 - a_1, \dots, x_n - a_n)$.)

**16.** *(important)* Suppose $X$ is an irreducible variety. Show that $\dim X$ is the transcendence degree of the function field (the stalk at the generic point) $\mathcal{O}_{X,\eta}$ over $k$. Thus (as the generic point lies in all non-empty open sets) the dimension can be computed in any open set of $X$. (This is not true in general, see the pathology in the notes.)

**17.** Suppose $f(x,y)$ and $g(x,y)$ are two complex polynomials ($f, g \in \mathbb{C}[x,y]$). Suppose $f$ and $g$ have no common factors. Show that the system of equations $f(x,y) = g(x,y) = 0$ has a finite number of solutions. (This isn't essential for what follows. But it is a basic fact, and very believable.)

**18.** Suppose $X \subset Y$ is an inclusion of irreducible $k$-varieties, and $\eta$ is the generic point of $X$. Show that $\dim X + \dim \mathcal{O}_{Y,\eta} = \dim Y$. Hence show that $\dim X + \operatorname{codim}_Y X = \dim Y$. Thus for varieties, the inequality $\dim X + \operatorname{codim}_Y X \leq \dim Y$ is always an equality.

**19.** Show that $\operatorname{Spec} k[w,x,y,z]/(wz - xy, wy - x^2, xz - y^2)$ is an integral *surface*. You might expect it to be a curve, because it is cut out by three equations in 4-space. (You may recognize this as the affine cone over the twisted cubic.) It turns out that you actually need three equations to cut out this surface. The first equation cuts out a threefold in four-space (by Krull's theorem, see later). The second equation cuts out a surface: our surface, along with another surface. The third equation cuts out our surface, and removes the "extraneous component". One last aside: notice once again that the cone over the quadric surface $k[w,x,y,z]/(wz - xy)$ makes an appearance.)

**20++.** Reduce the proof of Chevalley's theorem to the following statement: suppose $f : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ is a dominant morphism, where $A$ and $B$ are domains, and $f$ corresponds to $\phi : B \to B[x_1, \dots, x_n]/I \cong A$. Then the image of $f$ contains a dense open subset of $\operatorname{Spec} B$. (Hint: Make a series of reductions. The notion of constructable is local, so reduce to the case where $Y$ is affine. Then $X$ can be expressed as a finite union of affines; reduce to the case where $X$ is affine. $X$ can be expressed as the finite union of irreducible components; reduce to the case where $X$ is irreducible. Reduce to the case where $X$ is reduced. By considering the closure of the image of the generic point of $X$, reduce to the case where $Y$ also is integral (irreducible and reduced), and $X \to Y$ is dominant. Use Noetherian induction in some way on $Y$.)

**21.** What is the dimension of $\operatorname{Spec} k[w,x,y,z]/(wz - xy, y^{17} + z^{17})$? (Be careful to check they hypotheses before invoking Krull!)

**22.** *(important for later)* (a) *(Hypersurfaces meet everything of dimension at least 1 in projective space — unlike in affine space.)* Suppose $X$ is a closed subset of $\mathbb{P}^n_k$ of dimension at least 1, and $H$ a nonempty hypersurface in $\mathbb{P}^n_k$. Show that $H$ meets $X$. (Hint: consider the affine cone, and note that the cone over $H$ contains the origin. Use Krull's Principal Ideal Theorem.)
(b) (Definition: Subsets in $\mathbb{P}^n$ cut out by linear equations are called **linear subspaces**. Dimension 1, 2 linear subspaces are called **lines** and **planes** respectively.) Suppose $X \hookrightarrow \mathbb{P}^n_k$ is a closed subset of dimension $r$. Show that any codimension $r$ linear space meets $X$. Hint: Refine your argument in (a). (In fact any two things in projective space that you might expect to meet for dimensional reasons do in fact meet. We won't prove that here.)
(c) Show further that there is an intersection of $r + 1$ hypersurfaces missing $X$. (The key

step: show that there is a hypersurface of sufficiently high degree that doesn't contain every generic point of $X$. Show this by induction on the number of generic points. To get from $n$ to $n+1$: take a hypersurface not vanishing on $p_1, \dots, p_n$. If it doesn't vanish on $p_{n+1}$, we're done. Otherwise, call this hypersurface $f_{n+1}$. Do something similar with $n+1$ replaced by $i$ ($1 \leq i \leq n$). Then consider $\sum_i f_1 \cdots \widehat{f_i} \cdots f_{n+1}$.)

**23-.** Show that it is false that if $X$ is an integral scheme, and $U$ is a non-empty open set, then $\dim U = \dim X$.

**24.** Suppose $f$ is an element of a normal domain $A$, and $f$ is contained in no codimension 1 primes. Show that $f$ is a unit.

**25.** Suppose $f$ and $g$ are two global sections of a Noetherian normal scheme, not vanishing at any associated point, with the same poles and zeros. Show that each is a unit times the other.

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