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Controllability of Mobile Robots with Kinematic Constraints

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Abstract: *We address the controllability problem for robot systems subject to kinematic constraints on the velocity and its application to path planning. We show that the well-known Controllability Rank Condition Theorem is applicable to these systems when there are inequality constraints on the velocity in addition to equality constraints, and/or when the constraints are non-linear instead of linear. This allows us to infer a whole set of new results on the controllability of robotic systems subject to non-integrable kinematic constraints (called nonholonomic systems). A car with limited steering angle is one example of such a system. For example, we show that:*

1) An n-body car system, which consists of a car towing $n-1$ trailers, is controllable for $n < 4$ even if the steering angle is limited.

2) An n-body car ($n < 4$) that can only turn left is still maneuverable on the right.

3) If there is a path for an n-body car system ($n < 4$) with limited steering angle in a given environment, then there is another path that uses only the extremal values of the steering angle. We conjecture that these results are true for all n. However, we have only been able to prove them for $n < 4$.

The same kind of results as above can also be obtained for any nonholonomic system additionally subject to inequality constraints on the velocity.

We present experiments with simulated nonholonomic systems that illustrate these results. These experiments were conducted by using a general path planner previously described in [Barraquand and Latombe 89b].

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1 Introduction

In this paper we address the controllability problem for robot systems in the presence of kinematic constraints on the velocity and its application to path planning. Kinematic constraints on the velocity can be divided into two main classes:

-Equality constraints, which are generally caused by one or several ***rolling contacts*** between rigid bodies and express the fact that the relative velocity of two points in contact is zero. When these constraints are non-integrable, they make the dimension of the space of achievable velocities smaller than the dimension of the robot's configuration space. They are called nonholonomic equality constraints. We call ***nonholonomic robot*** a robot whose motions are constrained by nonholonomic constraints.

-Inequality constraints, which are generally caused by mechanical stops of the robot kinematics. Unlike equality constraints, they do not reduce the dimension of the space of achievable velocities. Instead, they restrict locally the authorized directions for the velocity to a subset of the tangent space.

A car is a typical example of a nonholonomic mechanical system. In the absence of obstacles, it can attain any position in the plane with any orientation. Hence, the configuration space is three-dimensional. However, assuming no slipping of the wheels on the ground, the velocity of the midpoint between the two rear wheels of the car is always tangent to the car orientation. The space of achievable velocities at any configuration is thus two-dimensional. This corresponds to an equality constraint on the velocity. If in addition the steering angle of the front wheels is limited by a mechanical stop, the space of achievable velocities at any configuration is further restricted to a two-dimensional cone around the neutral position. This corresponds to an inequality constraint on the velocity.

Collision-free path planning consists of constructing a path in the free subset of the configuration space – i.e. the set of configurations where the robot has no contact or intersection with the obstacles – between two input configurations. Kinematic constraints require that the tangent to the path at any configuration be within the subspace of velocities selected by the constraints. A collision-free path for a nonholonomic robot typically has to include "maneuvers", i.e. backing-up configurations where the robot stops and changes the sign of the velocity (think, for example, of the parallel parking of a car along a sidewalk). Finding a feasible path between two configurations is one difficult problem. Another one is to minimize the number of maneuvers, or at least to keep it reasonable, whenever possible.

The first part of this paper is a generalization of the mathematical analysis of nonholonomic constraints developed in [Barraquand and Latombe 89b]. Using standard results in nonlinear control theory (namely, the Controllability Rank Condition Theorem for Non-Linear Systems), we state a general result applicable to robot systems subject to equality and/or inequality constraints on the velocity. We then infer new results on the controllability of robotic systems.

The second part of the paper applies these results to a special type of robotic system, which we name the n-body car system. A n-body car robot is kinematically similar to an automobile car towing $n-1$ trailers, each hooked up to the next. We give an explicit analytical expression of the differential equations of motion for any multi-body system. Using the general result of the first part of the paper, we show that a single-body, two-body, or three-body car system is controllable whenever there are at least two different admissible positions of the steering wheel. The following three statements are direct consequences of this result:

1) An n-body car ($n < 4$) is controllable even if the steering angle is limited.

2) An n-body car ($n < 4$) that can only turn left is still maneuverable on the right.

3) If there exists a path for an n-body car ($n < 4$) with limited steering angle in a given environment, then there exists another path that uses only the extremal values of the steering angle.

We conjecture that these results are true for all n. However, we have only been able to prove them for $n < 4$.

Finally, in the third part of the paper, we present experiments with simulated nonholonomic systems that illustrate the above results. These experiments were conducted by using a path planner previously described in [Barraquand and Latombe 89b]. The experiments reported in this paper were carried out with a version of the planner specifically designed to optimize the number of maneuvers. In this version, the planner applies a brute force method that consists of performing a dynamic programming search in the discretized configuration space with the number of maneuvers as the cost function. Despite its conceptual simplicity, the planner is relatively fast with reasonable, but non trivial examples. To our knowledge, this is the first implemented planner capable of finding collision-free path with quasi-minimal number of maneuvers (at the resolution of the configuration space representation) for nonholonomic robots.

Possible applications of the planner include navigation of autonomous robots, automated parking of personal cars and trucks, autonomous navigation of luggage carriers in airport facilities, automatic planning of the movements of machines in a construction site, and computer-aided design of access ports for trucks in industrial and commercial facilities.

2 Relation to Other Work

Research on collision-free path planning has been very active during the past ten years (e.g. see [Latombe 90]). Today, the mathematical and computational structures of path planning for holonomic robots are reasonably well-understood. Practical planners have also been implemented in more or less specific cases (e.g., [Brooks and Lozano-Perez 83] [Gouzenes 84] [Laugier and Germain 85] [Faverjon 86] [Faverjon and Tournassoud 87] [Lozano-Perez 87] [Barraquand and Latombe 89a]). However, path planning with nonholonomic constraints has attracted much less interest so far.

The problem was first introduced by Laumond [Laumond 86], who proved that the single-body car, i.e. a car without a trailer, is controllable. Laumond's constructive proof can be used to design an actual two-phase planner. In the first phase, the planner generates a free path by ignoring the nonholonomic constraints. In the second phase, it transforms this path into a topologically equivalent free path that satisfies the nonholonomic constraints. The transformation is done by introducing as many backing-up maneuvers as necessary. However, the number of maneuvers generated by this planner is unbounded (see [Laumond et al. 88]), even when there exist collision-free paths with no or few maneuvers that satisfy the nonholonomic constraint. Building on his previous work, Laumond sketched an algorithm for planning smooth – i.e. maneuver-free – collision-free paths of a nonholonomic circular robot whose turning radius is lower-bounded [Laumond 87]. But this algorithm fails whenever all free paths require one or more maneuvers.

Another planning approach has been specifically developed and implemented for one-body car robots. It consists of planning the motion of the robot in a network of "corridors" [Tournassoud and Jehl 88] or "lanes" [Wilfong 88] that are extracted from the workspace in a first phase of processing. Local planning techniques are used to generate turns for transferring the robot from a corridor (or lane) to another at a "crossroad" of the network. The difficulty of the approach is that for most workspaces one cannot define an intrinsic set of corridors (or lanes). In addition, the various turns that have to be generated along a path usually interact, and local planning techniques may result in quite inefficient paths.

The motion of a point along paths having lower-bounded curvature radius has been investigated in [Fortune and Wilfong 88] and [Jacobs and Canny 89]. Fortune and Wilfong proposed an exact algorithm for deciding whether there exists a feasible path between two configurations among polygonal obstacles. Jacobs and Canny adapted one of the results presented in [Fortune and Wilfong 88] and described an implemented planner. This planner consists of discretizing the boundary of the obstacles and connecting the points resulting from this discretization by paths of standard shapes, called "jumps", which are made of circular arcs and straight segments.

[Li and Canny 89] and [Laumond and Simeon 89] pointed out that results in nonlinear control theory where transposable in order to characterize the controllability of robots subject to linear equality constraints on the velocity. [Laumond and Simeon 89] and [Barraquand and Latombe 89b] used these results to prove the controllability of the two-body car system. However, both proofs consider only the linear equality constraint on the velocity imposed by the rolling of the wheels of the ground. They are not applicable when there is an additional inequality constraint imposed on the steering angle by the mechanical stops.

Our contributions to robot motion planning with nonholonomic constraints reported in this paper are the following:

1) We show that a careful instantiation of the Controllability Rank Condition Theorem subsumes and generalizes most previous results on the controllability of nonholonomic robots.

2) We apply this result to the one-body, two-body, and three-body car systems and we derive new formal results related to the controllability of these systems.

3) We describe a planner that has been able to solve more complex planning problems with nonholonomic robots than any previous implemented planner.

3 Non-linear Control Systems

Throughout this paper, we regard a nonholonomic robot as a nonlinear control system. In this section we recall the definition of some basic concepts in controllability theory and we review important results related to non-linear control ([Hermann and Krener 77]).

3.1 Definition of Controllability

Let us consider a control system of the form:

$$\dot{q} = f(q, u) \tag{1}$$

where $u \in \Omega$, a measurable subset of R^m, $q \in \mathcal{C}$, a connected manifold of dimension n. f is smooth as a function of q. Ω represents the *control space* of the system, i.e. the set of admissible control values. $\mathcal{C}$ represents the *state space*, or *configuration space*, i.e. the set of distinguishable states that the system may take at any given time.

Given a subset $U \subset \mathcal{C}$, the configuration $q_1 \in U$ is said to be *U-accessible* from $q_0 \in U$ if there exists a bounded measurable control $u(t), t \in [t_0, t_1]$ such that the solution

$q(t)$ of the system (1) satisfies: $q(t_0) = q_0, q(t_1) = q_1$ and $q(t) \in U, \forall t \in [t_0, t_1]$. We write $q_1 A_U q_0$. The set of points U-accessible from q_0 will be denoted by $A_U(q_0)$.

The system (1) is *controllable at* q_0 iff $A_{\mathcal{C}}(q_0) = \mathcal{C}$. If this is true for any point q_0, then the system is *controllable.* This simply means that any point is $\mathcal{C}$-accessible from any other point. This global notion of controllability is not well-suited to mathematical characterizations. A much more useful concept is the local notion of controllability defined below.

The system (1) is *locally controllable at* q_0 iff for every neighborhood U of q_0, $A_U(q_0)$ is also a neighborhood of q_0. It is *locally controllable* iff this is true for every q_0.

Accessibility is a reflexive and transitive relation, but not necessarily symmetric. The symmetric closure of this relation is called weak accessibility. q' is *weakly U-accessible* from q iff there exists a sequence $q_0, \ldots, q_k$ such that $q = q_0$, $q' = q_k$, and either $q_i A_U q_{i-1}$ or $q_{i-1} A_U q_i$ for every $i \in [1, k]$. We write $qWA_U q'$.

The system (1) is *weakly controllable at* q_0 iff $WA(q_0) = \mathcal{C}$. If this is the case, the system is necessarily weakly controllable at any other point, because weak accessibility is an equivalence relation, and it is *weakly controllable.* Again, weak controllability is a global concept that has a useful local equivalent.

System (1) is *locally weakly controllable at* q_0 if for every neighborhood U of q_0, $WA_U(q_0)$ is a neighborhood of q_0. It is *locally weakly controllable* if this is true at every q_0.

Clearly, local (weak) controllability implies (weak) controllability. Furthermore, for symmetric systems, (local) weak controllability is equivalent to (local) controllability. Therefore, *for a symmetric system, local weak controllability implies controllability.*

3.2 Frobenius Theorem

Consider the set $X(\mathcal{C})$ of smooth vector fields on $\mathcal{C}$. Using (1), each constant control $u \in \Omega$ defines a vector field $X_u = f(., u)$ on $\mathcal{C}$. We denote by F the set of all the vector fields corresponding to the admissible values of the control:

$$F = \{X \in X(\mathcal{C}) | \exists u \in \Omega, X = f(., u)\}$$

We define an internal operation on $X(\mathcal{C})$ that defines a Lie algebra structure.

Let (X, Y) be any pair of vector fields. Given any configuration q, let us consider a path starting at q obtained by concatenating four consecutive paths:

- the first path follows the flow of X during δt;

- the second path follows the flow of Y during δt;

- the third path follows the flow of $-X$ during δt;

- the fourth path follows the flow of $-Y$ during δt.

We denote by q' the configuration reached at the end of these four paths. A straightforward Taylor expansion shows that:

$$\lim_{\delta t \to 0} \frac{q' - q}{\delta t^2} = dY \cdot X - dX \cdot Y.$$

The expression $dY \cdot X - dX \cdot Y$ determines a new vector field which is commonly denoted by $[X, Y]$ and called *Lie bracket* of X and Y.

By definition, the Control Lie Algebra associated with F, denoted by $CLA(F)$, is the smallest subalgebra of $X(\mathcal{C})$ which contains F. Stated otherwise, $CLA(F)$ is the subspace of $X(\mathcal{C})$ generated by all linear combinations of vector fields in F and all their Lie brackets recursively computed.

For each $q_0 \in \mathcal{C}$, let $CLA(F)(q_0)$ denote the subspace of tangent vectors spanned by the vector fields of $CLA(F)$ at q_0.

A connected submanifold $\mathcal{C}'$ of $\mathcal{C}$ is an *integral submanifold* of $CLA(F)$ if at each $q \in \mathcal{C}'$ the tangent space to $\mathcal{C}'$ is contained in $CLA(F)(q)$. $\mathcal{C}'$ is a *maximal integral submanifold* of $CLA(F)$ if it is not properly included in any other integral manifold.

The Frobenius integrability theorem can be stated as follows:

Theorem 1 (Frobenius Integrability Theorem)

If the dimension of $CLA(F)(q)$ has a constant value k for every $q \in \mathcal{C}$, there exists a partition of $\mathcal{C}$ into maximal integral submanifolds of $CLA(F)$ all of dimension k.

3.3 Controllability Rank Condition

The system (1) is said to satisfies the *Controllability Rank Condition at q_0* if the dimension of $CLA(F)(q_0)$ is n. If this is true for every q_0 then it satisfies the *Controllability Rank Condition.*

The following results, based on the work of [Chow 39], were derived by [Hermann 63], [Haynes and Hermes 70], and working independently by [Lobry 70], [Sussmann and Jurdjevic 72], [Krener 74].

Theorem 2 (The Controllability Rank Condition Theorem: Sufficient Condition)

If the system (1) satisfies the Controllability Rank Condition at q_0, then it is locally weakly controllable at q_0.

Theorem 3 (The Controllability Rank Condition Theorem: Necessary Condition)

If the system (1) is locally weakly controllable, then the Controllability Rank Condition is satisfied on an open dense subset of $\mathcal{C}$.

In particular, if we are only interested in symmetric systems for which the dimension of $CLA(F)(q)$ does not depend on q we can infer that local controllability (which implies controllability) is equivalent to the Controllability Rank Condition.

Another presentation of the Controllability Rank Condition Theorem based on the concept of *distribution* can be found in [Isidori 85]. Its relation to nonholonomic systems is analyzed in in [Li and Canny 89] and [Barraquand and Latombe 89b]. The formulation developed here allows to deal with non-linear equality/inequality constraints, whereas the alternative presentation using distributions is limited to linear equality constraints.

4 Nonholonomic Constraints

4.1 Terminology

We denote the robot by $\mathcal{A}$ and its workspace by $\mathcal{W}$. A *configuration* of $\mathcal{A}$ is a specification of the position of every point in $\mathcal{A}$ with respect to a Cartesian frame embedded in $\mathcal{W}$. The *configuration space* of $\mathcal{A}$ is the space, denoted by $\mathcal{C}$, of all the possible configurations of $\mathcal{A}$. The configuration space of a mechanical system made of rigid bodies is a smooth manifold [Arnold 78]. For instance, the configuration space of a two-dimensional rigid body translating and rotating in $\mathcal{W} = \mathbf{R}^2$ is $\mathcal{C} = \mathbf{R}^2 \times S^1$, where S^1 denotes the unit circle. In virtually any practical situation, the range of positions reachable by the robot's bodies can be bounded, making $\mathcal{C}$ into a compact manifold. Let n be the dimension of $\mathcal{C}$. A configuration q is represented as a list $(q_1, \ldots, q_n)$ of n generalized coordinates.

Now, suppose that a scalar constraint of the form:

$$F(q, t) = 0 \tag{2}$$

with $q \in \mathcal{C}$ and t denoting time, applies to the motion of $\mathcal{A}$. Let us further assume that F is smooth with non-zero derivative. Then, in theory, one could use the equation to solve for one of the generalized coordinates in terms of the other coordinates and time. Thus, equation (2) defines a $(n-1)$-dimensional submanifold of $\mathcal{C}$. This submanifold is in fact the actual configuration space[1] of $\mathcal{A}$ and the $n-1$ remaining coordinates its actual generalized coordinates. Constraint (1) is called a *holonomic* equality constraint. More generally, there may be k constraints of the form (2). If they are independent – i.e. their Jacobian matrix has full rank – they determine a $(n-k)$-dimensional submanifold of $\mathcal{C}$, which is the actual configuration space of $\mathcal{A}$.

A constraint of the form:

$$F(q,t) < 0 \quad \text{or} \quad F(q,t) \leq 0$$

where F is smooth with non-zero derivative, is an inequality constraint. It typically acts as a mechanical stop or an obstacle. It simply determines a subset of $\mathcal{C}$ having the same dimension as $\mathcal{C}$.

Constraint (2) is only a kinematic constraint of one sort. Another one is a scalar constraint of the form:

$$G(q,\dot{q},t) = 0 \tag{3}$$

with $\dot{q} \in T_q(\mathcal{C})$, the *tangent space* of $\mathcal{C}$ at q. The pair $(q,\dot{q})$ belongs to $TB(\mathcal{C})$, the *tangent bundle* associated with the manifold $\mathcal{C}$. The tangent space represents the space of the velocities of $\mathcal{A}$. The tangent bundle is also called the "phase space" in Physics and the "state space" in control theory. The tangent space of a smooth manifold is a vector space having the same dimension as the manifold. Hence, $T_q(\mathcal{C})$ has dimension n for every $q \in \mathcal{C}$. The tangent bundle $TB(\mathcal{C})$ is a smooth manifold of dimension $2n$.

A kinematic constraint of the form (3) is holonomic if it is integrable, i.e. $\dot{q}$ can be eliminated and the equation (3) rewritten in the form (2). Otherwise, the constraint is called a *nonholonomic* equality constraint. As we will see below, a nonholonomic equality constraint restricts the space of velocities achievable by $\mathcal{A}$ at any configuration q to a $(n-1)$-dimensional linear subspace of $T_q(\mathcal{C})$ without affecting the dimension of the configuration space. If there are k independent nonholonomic equality constraints of the form (3), the space of achievable velocities is a subspace of $T_q(\mathcal{C})$ of dimension $n-k$.

A nonholonomic equality constraint is generally caused by a rolling contact between two rigid bodies. It expresses the fact that the relative velocity of the two points of contact is zero. When the motion in contact combines rolling and sliding, the expression depends on the friction coefficient of the two bodies, and hence is nonlinear. When there is no sliding, the nonholonomic constraint is linear in $\dot{q}$. The second case,

[1] If constraint (2) depends on t, $\mathcal{A}$'s configuration space is time-dependent, otherwise it is time-independent. Many usual holonomic constraints – e.g., the prismatic and revolute joints of a manipulator arm – are time-independent.

though less general than the first, is much simpler and quite widespread in practice. For instance, in the car example, this corresponds to assuming no slipping of the wheels on the ground.

A constraint of the form:

$$G(q, \dot{q}, t) < 0 \quad \text{or} \quad G(q, \dot{q}, t) \leq 0$$

is a kinematic inequality constraint. It restricts the set of achievable velocities at any configuration q to a subset of $T_q(\mathcal{C})$ having the same dimension as $T_q(\mathcal{C})$. A constraint bounding the steering angle of a car is a typical kinematic inequality constraint.

When dealing with constraints of the form (3), two important questions arise:

- Are they integrable?

- If they are not integrable, do they restrict the range of achievable configurations?

We investigate these questions in the next two subsections. First, we show that under very general conditions the concept of kinematic constraint on a mechanical system is dual and equivalent to the concept of control system as defined in (1). This equivalence enables us to use results from nonlinear control theory to answer our questions. Using the Frobenius Integrability Theorem, we give a necessary and sufficient condition of holonomy (and non-holonomy) for equality constraints of the form (3). Then, using the Controllability Rank Condition Theorem we analyze the second question. We state a necessary and sufficient condition under which kinematic constraints, whether they are linear or nonlinear, equality or inequality, have no effect on the range of achievable configurations.

For simplicity, in the rest of the paper, we will assume that the kinematic constraints do not depend on time. However, all the results remain valid when constraints are time-dependent.

4.2 Kinematic Constraints and Control Systems

Let us consider a set of $k < n$ independant kinematic constraints of the form (3):

$$G(q, \dot{q}) = (G_1(q, \dot{q}), \ldots, G_k(q, \dot{q})) = (0, \ldots, 0)$$

For each q, $G_q = G(q, .)$ defines a function from $T_q(\mathcal{C})$ to $\mathbf{R}^k$. As the k constraints are independent, the Jacobian of this function has full rank. The subset of the tangent space verifying the kinematic constraints is simply $G_q^{-1}(0, \ldots, 0)$. According to the Generalized Implicit Function Theorem (also called Constant Rank Theorem), this subspace is a submanifold of $T_q(C)$ of dimension $n-k$. Let us denote a local coordinate chart of this manifold by $u = (u_{k+1}, \ldots, u_n)$, and define $f_q = u^{-1}$. We obtain the following relation:

$$\dot{q} = f_q(u) = f(q, u)$$

Under the additional assumption that f is smooth as a function of q, this relation locally defines a non-linear control system with $n-k$ controls $(u_{k+1}, \ldots, u_n)$. Assume that we impose inequality constraints in addition to equality constraints. These new constraints are transformed into inequalities on the controls by means of the inverse of the chart u. They define the shape of the set Ω of admissible controls.

Reciprocally, if we consider any control system of the type (1) such that $f(q,.) = f_q$ has full rank as a function of the control $u = (u_{k+1}, \ldots, u_n)$, then we can apply again the Constant Rank Theorem for every q and obtain a local atlas chart $G_q = (G_q^1, \ldots, G_q^n)$ verifying:

$$G_q^i(f_q(u)) = G_q^i(\dot{q}) = G^i(q, \dot{q}) = 0, \forall i \in [1, k]$$

and

$$G_q^i(f_q(u)) = G_q^i(\dot{q}) = u_i, \forall i \in [k+1, n]$$

The first k equalities given above precisely define k independent kinematic constraints. Furthermore, the inequalities on the controls that define the shape of the set Ω are transformed into inequalities on the velocity by means of the $G_q^i, i \in [k+1, n]$.

Therefore, in general, a mechanical system subject to a set of k independent kinematic constraints is locally equivalent to a control system containing $n-k$ controls for which the function f has full rank in u. Furthermore, any additional inequality constraints on the velocities are equivalent to inequality constraints on the controls.

In some practical cases, this local equivalence between kinematically constrained robots and control systems can be globalized. In particular, when there are only pure rolling constraints on the system, G is linear in $\dot{q}$ and the global equivalence can be formally proven (see [Barraquand and Latombe 89b]).

4.3 Nonholonomy and Controllability

Let us consider a mechanical system subject to a set of k independent kinematic equality constraints of the form (3). To answer the integrability question, we first compute as indicated above the *equivalent control system*, that is the function $f(q, u)$.

We can characterize the integrability of the constraints using Frobenius Theorem. For each configuration q the dimension r of $CLA(F)(q)$ is clearly greater than or equal to $n-k$.

If r takes values greater than $n - k$, then the maximal integral manifolds have a dimension greater than $n - k$, and the system is non-integrable, i.e. nonholonomic.

On the other hand, if r remains equal to $n - k$ at every q, then the Frobenius Theorem entails that the maximal integral manifolds of the system have dimension $n-k$. Therefore, the configurations of the system always stay in an $n-k$ dimensional submanifold of $\mathcal{C}$. As a consequence, the velocities always belong to the tangent space of this submanifold, which is precisely $CLA(F)(q)$. But these same velocities belong to a $n - k$ dimensional submanifold S of the tangent space of $\mathcal{C}$ at q defined by the constraints. Therefore, S is necessarily equal to $CLA(F)(q)$, hence linear. This implies that the k equality constraints are *linear* in $\dot{q}$. This result, though intuitively clear, is worth being outlined. To characterize holonomic constraints, we can therefore limit ourselves to those that are linear in the velocity parameters. In that case, we can replace the configuration space $\mathcal{C}$ by the maximal integral submanifold passing through the initial configuration, and get rid of the constraints. The equations defining this submanifold can be written locally in the form (2), i.e. $F(q) = 0$, which is the integral form of the constraints (3). By differentiating this last equation as a function of time, we find again the constraints on the velocity:

$$dF(q).\dot{q} = 0$$

which gives a more intuitive explanation of the fact that holonomic constraints are necessarily linear.

Proposition 1 (Non-holonomy of non-linear constraints) *Kinematic constraints that are properly non-linear as functions of the velocity are necessarily nonholonomic.*

Proposition 2 (Characterization of Holonomy) *A system subject to k independent equality constraints of the form (3) is holonomic if and only if the codimension $n - r$ of the Control Lie Algebra is equal to the number k of constraints. In such a case, the kinematic constraints are necessarily linear in the velocity parameters.*

The answer to the controllability question for robots subject to kinematic constraints is a direct consequence of the Controllability Rank Condition Theorem. As outlined above, given k independent constraints, we consider the equivalent control system with $n - k$ controls. Then we analyze the dimension r of the Control Lie Algebra by recursively computing the Lie brackets of constant control vector fields. If this number r is constant and equal to n, then the system is locally weakly controllable. Reciprocally, if the system is locally weakly controllable, then r is equal to n on an open dense subset of $\mathcal{C}$. If r varies on $\mathcal{C}$, complex phenomena may occur. The study of these phenomena forms the basis for the so-called Catastrophe Theory [Poston and Stewart 78].

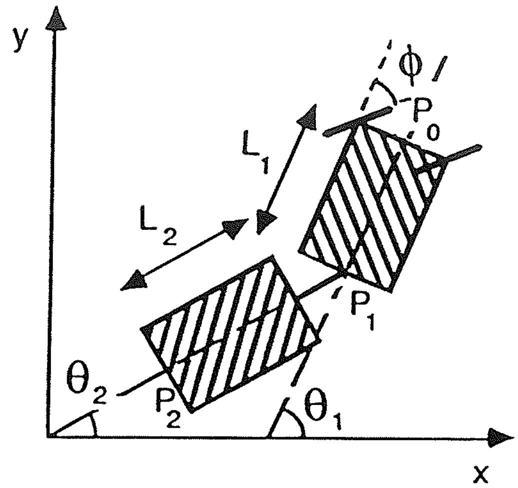

Figure 1: *A Two-Body Car*

Proposition 3 (Characterization of Controllability) *A system subject to kinematic constraints on the velocity – which may be linear or non-linear, equality or inequality – is locally weakly controllable if the dimension r of the Control Lie Algebra is maximal, i.e. equal to the dimension n of the configuration space.*

5 The Multi-Body Car System

Let us consider a front-wheel-drive four-wheel car[2]. We model the car by a two-dimensional object translating and rotating in the plane. The configuration space of the car is $D \times S^1$, where D is a compact domain of $\mathbf{R}^2$. We parameterize the car configuration by the coordinates X_1 and Y_1 of the midpoint P_1 between the two rear wheels and the angle θ_1 between the x axis of the Cartesian frame embedded in the plane and the main axis of the car. The velocity parameters are $\dot{X}_1$, $\dot{Y}_1$ and $\dot{\theta}_1$. The control parameters of the car are the velocity $v \in \mathbf{R}$ of the midpoint P_0 between the two front wheels (if $v > 0$, the car moves forward) and the steering angle ϕ measuring the orientation of the velocity of P_0 with respect to the main axis of the car.

Assuming a pure rolling contact between the wheels and the ground – i.e. no slipping – the velocity of P_1 is always along the main axis of the car. Hence, we have:

$$\dot{X}_1 = \lambda \cos\theta_1 \qquad \dot{Y}_1 = \lambda \sin\theta_1.$$

Eliminating λ, we get the following kinematic constraint on the velocity:

$$-\dot{X}_1 \sin\theta_1 + \dot{Y}_1 \cos\theta_1 = 0. \tag{4}$$

[2]Our presentation can easily be modified to treat other types of car-like robots.

The equivalent control system is easily computed:

$$\left.\begin{array}{rcl}\dot{X}_1 & = & v\cos\phi\cos\theta_1 \\ \dot{Y}_1 & = & v\cos\phi\sin\theta_1 \\ L_1\dot{\theta}_1 & = & v\sin\phi\end{array}\right\} \quad (5)$$

As v can take both positive and negative values, the system is *symmetric*. In such a case weak controllability is equivalent to controllability, and the results of the previous sections are applicable.

The multi-body car system consists of adding one or more bodies to be towed by the car. For example, the two-body car problem consists of analyzing the behaviour of the mechanical system defined by a car towing a single body. In this problem, there are two kinematic constraints: the velocity of the midpoint between the rear wheels of each body is tangent to the orientation of the body.

More generally, one can consider the n-body car system, which consists of a car towing $n-1$ bodies serially hooked (e.g., a luggage carrier in an airport). Figure 1 displays a schematic model of such a system. The midpoint between the rear wheels of the first body (the car) is denoted by P_1. The midpoint between the rear wheels of the k^{th} body is denoted by P_k. We therefore have n points $P_1, \ldots, P_n$, whose coordinates are denoted by $(X_1, Y_1), \ldots, (X_n, Y_n)$. The orientation of the k^{th} body with respect to the x axis of the Cartesian frame embedded in the plane is denoted by θ_k. The configuration space of the n-body car is $D \times (S^1)^n$, where D is a compact domain of $\mathbf{R}^2$. We parameterize the configuration by $(X_1, Y_1, \theta_1, \ldots, \theta_n)$. The velocity parameters are $\dot{X}_1, \dot{Y}_1, \dot{\theta}_1, ..., \dot{\theta}_n$. The control parameters are the same as for the car, that is, the velocity v and the steering angle ϕ.

There are n kinematic constraints, one for each body. To establish these constraints, it is convenient to represent the points $P_1, \ldots, P_n$ in the complex plane, i.e.: $P_k = X_k + iY_k$.L_k denoting the length of the k^{th} body, we can write the geometric constraint between the bodies $k-1$ and k as:

$$P_k = P_{k-1} - L_k \exp(i\theta_k)$$

which can be rewritten:

$$P_k = P_1 - \sum_{l=2}^{k} L_l \exp(i\theta_l) \quad (6)$$

The kinematic constraint of the k^{th} body is:

$$\dot{P}_k = \lambda_k \exp(i\theta_k)$$

which is equivalent to:

$$\Im(\exp(-i\theta_k)\dot{P}_k) = 0$$

where $\Im(z)$ denotes the imaginary part of the complex number z. Combining this characterization with the derivative of equation (6) and using the linearity of the $\Im$ operator, we obtain the k^{th} kinematic constraint:

$$-\dot{X}_1 \sin\theta_k + \dot{Y}_1 \cos\theta_k - \sum_{l=2}^{k} L_l \dot{\theta}_l \cos(\theta_l - \theta_k) = 0$$

In particular, we obtain for $k = 1$:

$$-\dot{X}_1 \sin\theta_1 + \dot{Y}_1 \cos\theta_1 = 0 \tag{7}$$

which is precisely the kinematic constraint (4) of the car problem.

For $k = 2$, we get:

$$-\dot{X}_1 \sin\theta_2 + \dot{Y}_1 \cos\theta_2 - L_2\dot{\theta}_2 = 0 \tag{8}$$

Equations (7) and (8) are the kinematic constraints of the two-body car problem.

Similarly, by combining $\dot{P}_k = \lambda_k \exp(i\theta_k)$ with the derivative of

$$|P_k - P_{k-1}|^2 = L_k^2$$

we get

$$\lambda_k = \cos(\theta_k - \theta_{k-1})\lambda_{k-1}$$

and by induction:

$$\dot{P}_k = v\cos(\phi)\left(\prod_{l=2}^{k}\cos(\theta_l - \theta_{l-1})\right)\exp(i\theta_k)$$

Hence, the equivalent control system of the n-body car is composed of equations (5) and

$$L_k\dot{\theta}_k = v\cos\phi\left(\prod_{l=2}^{k-1}\cos(\theta_l - \theta_{l-1})\right)\sin(\theta_{k-1} - \theta_k), \qquad \forall k \in [2, n]$$

Let $X(v, \phi)$ denote the vector field corresponding to the constant control (v, ϕ). We get:

$$X(v, \phi) = \cos(\phi)X(v, 0) + \sin(\phi)X(v, \pi/2)$$

If we take any two fields corresponding to two different values of the steering angle ϕ_1 and ϕ_2, we see that the Control Lie Algebra generated by $X(v, \phi_1)$ and $X(v, \phi_2)$ is the same as the one generated by $X(v, 0)$ and $X(v, \pi/2)$, because of the bilinearity of the Lie Bracket operation:

$$[X(v, \phi_1), X(v, \phi_2)] = \sin(\phi_1 - \phi_2)[X(v, 0), X(v, \pi/2)]$$

Therefore, the dimension of the Control Lie Algebra of the n-body car system is not affected by constraints on the steering angle. It has been shown in

[Barraquand and Latombe 89b] that this dimension is maximal for the single-body and the two-body car. Here we generalize this result to a three-body system.

The constant control vector field space is generated by the two following vector fields:

$$\begin{array}{llllllll} X_1 = & X(1,0) & = (& \cos\theta_1, & \sin\theta_1, & 0, & \frac{\sin(\theta_1-\theta_2)}{L_2}, & \cos(\theta_1-\theta_2)\frac{\sin(\theta_2-\theta_3)}{L_3}) \\ X_2 = & L_1 X(1,\frac{\pi}{2}) & = (& 0, & 0, & 1, & 0, & 0) \end{array}$$

whose Lie bracket is:

$$X_3 = [X_1, X_2] = (-\sin\theta_1, \cos\theta_1, 0, \cos(\theta_1-\theta_2)/L_2, -\sin(\theta_1-\theta_2)\sin(\theta_2-\theta_3)/L_3)$$

We next compute:

$$X_4 = L_2[X_1, X_3] = (0, 0, 0, 1/L_2, -\cos(\theta_2-\theta_3)/L_3)$$

and

$$\begin{aligned} X_5 &= \cos(\theta_1-\theta_2)[X_1, X_4] - \sin(\theta_1-\theta_2)[X_3, X_4] \\ &= (0, 0, 0, 1/L_2^2, -1/L_3^2 - \cos(\theta_2-\theta_3)/(L_2 L_3)) \end{aligned}$$

Finally:

$$\det(X_1, X_2, X_3, X_4, X_5) = -\frac{1}{L_2 L_3^2} \neq 0$$

We can state:

Proposition 4 *A single-body, two-body, or three-body car system is controllable whenever there are at least two different admissible positions of the steering wheel. In particular:*

1) *A n-body ($n < 4$) car system is controllable even if the steering angle is limited.*

2) *A n-body ($n < 4$) car system that can only turn left is still maneuverable on the right.*

3) *If there is a path for a n-body ($n < 4$) car system with limited steering angle in a given environment, then there is another path that uses only the extremal values of the steering angle.*

These statements are direct consequences of the fact that the dimension of the Control Lie Algebra is not affected by the choice of the steering angles.

The controllability problem of the general multi-body car system ($n \geq 4$) could be solved by computing the dimension of the Control Lie Algebra as it has been done above for $n < 4$. However, the symbolic computations of Lie Brackets and determinants are non-trivial in higher dimension and we have not been able to find a general recurrence formula for the determinant so far. Investigations are under way to determine if symbolic computation programs such as MACSYMA can facilitate this task.

6 Planning with nonholonomic Constraints

Let the workspace $\mathcal{W}$ of the robot $\mathcal{A}$ be populated by some stationary obstacles $\mathcal{B}_i$, $i = 1, ..., q$. These obstacles map in the configuration space $\mathcal{C}$ of $\mathcal{A}$ to regions $\mathcal{CB}_i$ called *C-obstacles* and defined by:

$$\mathcal{CB}_i = \{q \in \mathcal{C} \ / \ \mathcal{A}(q) \cap \mathcal{B}_i \neq \emptyset\}$$

where $\mathcal{A}(q)$ denotes the region of $\mathcal{W}$ occupied by $\mathcal{A}$ at configuration q. The subset $\mathcal{C}_{free} = \mathcal{C} \setminus \bigcup_{i=1}^{q} \mathcal{CB}_i$ is called *free space.* If both $\mathcal{A}$ and the $\mathcal{B}_i$'s are modelled as closed regions, the $\mathcal{CB}_i$'s are closed; $\mathcal{C}_{free}$ is an open subset of $\mathcal{C}$, hence a manifold of dimension n.

Given two configurations q_1 and q_2 in $\mathcal{C}_{free}$, the path planning problem is to construct a path connecting q_1 to q_2 and lying in $\mathcal{C}_{free}$, i.e. a continuous map $\tau : s \in [0,1] \mapsto \tau(s) \in \mathcal{C}_{free}$, such that $\tau(0) =$q$_1$ and $\tau(1) =$q$_2$. In the presence of nonholonomic constraints, the tangent to this path, $\frac{d\tau}{ds}$, must lie in the subset of the tangent space of $\mathcal{C}$ selected by the constraints.

We have implemented a general-purpose path planner based on the following main ideas [Barraquand and Latombe 89b]:

The configuration space is decomposed into an array of small rectangloids and the planner searches a directed graph whose nodes are these rectangloids. Two rectangloids are adjacent in this graph if there is a feasible free path between a configuration lying in the first rectangloid and a configuration lying in the second. The arcs of the graph are constructed by discretizing the control parameters of the robot and integrating the equations of motion. The graph is classically searched using Bellman Optimality Principle. Our implementation uses an uninformed A^* algorithm (Dijkstra's algorithm) with the number of reversals as the cost function. As any global motion planning technique, the approach is limited to robots with small configuration spaces. Indeed, it requires time $O(r^n \log r)$ and space $O(r^n)$, where r is the size of the decomposition along each axis of the configuration space.

The list of nodes whose successors have already been generated, called CLOSED, is simply represented by marking the corresponding cells in a large n-dimensional array. Hence, the access time to CLOSED is constant. The list of nodes whose successors have not been generated yet, called OPEN, is represented as a heap. Every modification and access to OPEN is therefore made in logarithmic time. These two implementation details are critical for the efficiency of the method.

Given a configuration q, the planner generates at most six successors of q by setting the two control parameters v and ϕ to the six values in:

$$\{-v_0, v_0\} \times \{-\phi_{max}, 0, +\phi_{max}\}$$

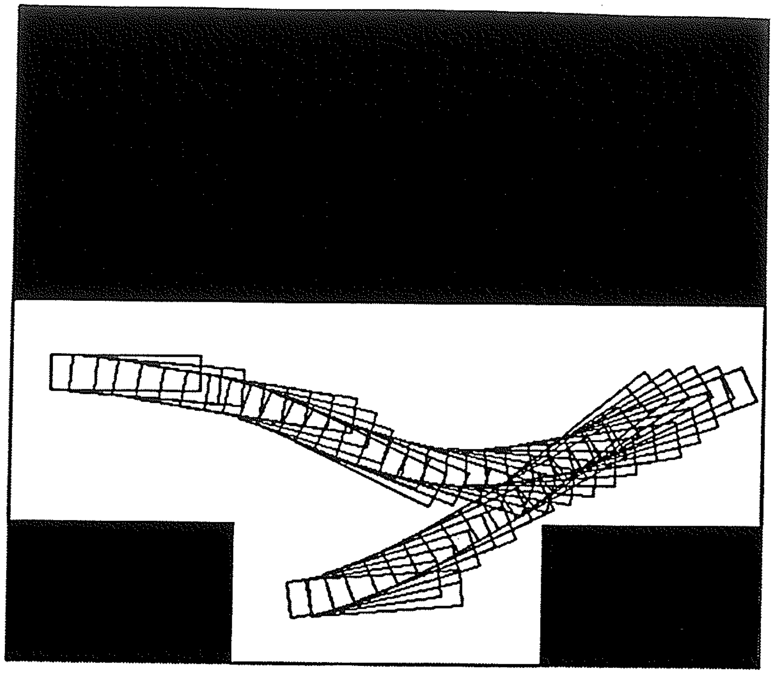

Figure 2: *A Parallel Parking Maneuver*

and integrating the velocity parameters of the robot along a "short" distance using the differential equations established above. By short distance, we mean that the integration time is 1 and v_0 is equal to the diameter of a cell. The goal is to avoid staying in the same cell as q, while not moving further than a neighboring cell. The three equations of a single-body car can be integrated analytically, because the integral curves are simply arcs of circles. However, the remaining $n-3$ equations of a general n-body car have to be integrated numerically. We implemented a fourth-order Runge-Kutta method for this integration.

This discretization of the control parameters derives from the discussion of section 3. If there exists a feasible free path between two given configurations, then there exists another feasible free path between these configurations that uses only the extremal values of the steering angle. The inclusion of $\phi = 0$ in the discretization set is aimed at allowing the robot to move along a straight path and hence reducing the total length of the generated path.

In this implementation, collisions are checked by intersecting the robot at every attained configuration with the obstacles in the workspace. The workspace is represented as a bitmap anf the test of intersection uses a divide-and-conquer technique (see [Barraquand and Latombe 89b]).

Despite its conceptual simplicity, this planner outputs quasi-optimal solutions to very tricky planning problems in reasonable amounts of time, as illustrated by the following experimental results.

We first experimented with the planner using a simulated car with various values of the maximal steering angle ϕ_{max} and several workspace arrangements.

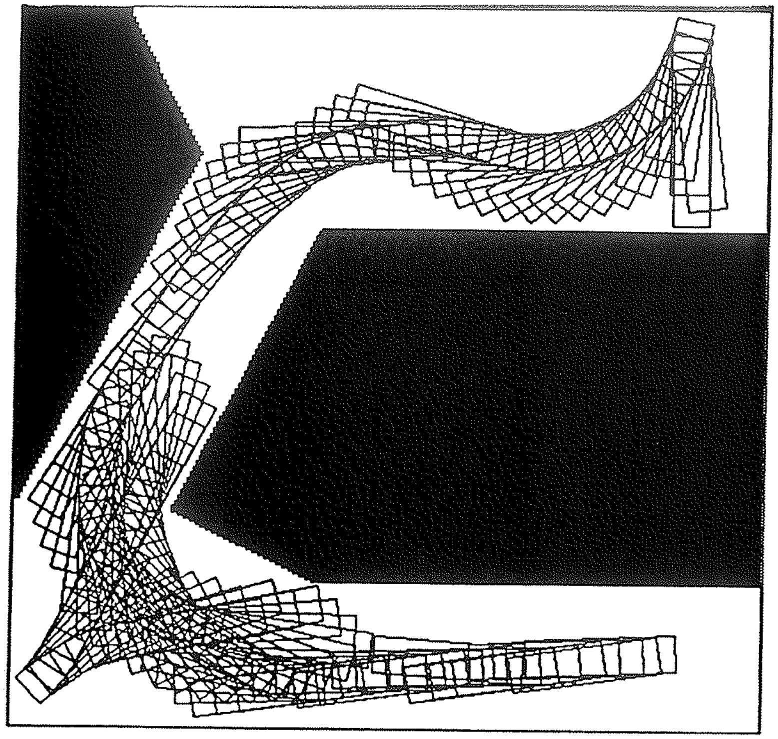

Figure 3: *Maneuvering in a Cluttered Workspace*

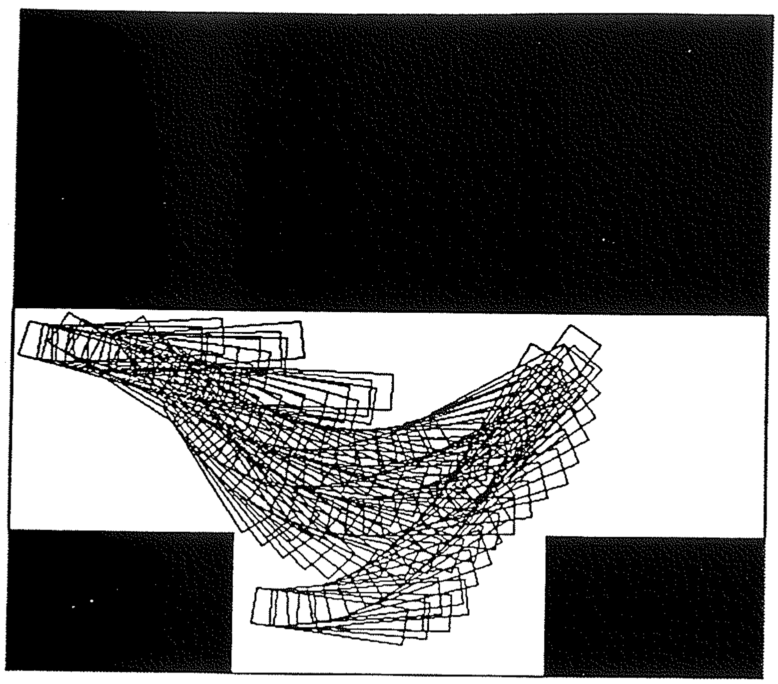

Figure 4: *Maneuvering of a Car That Can Only Turn Left*

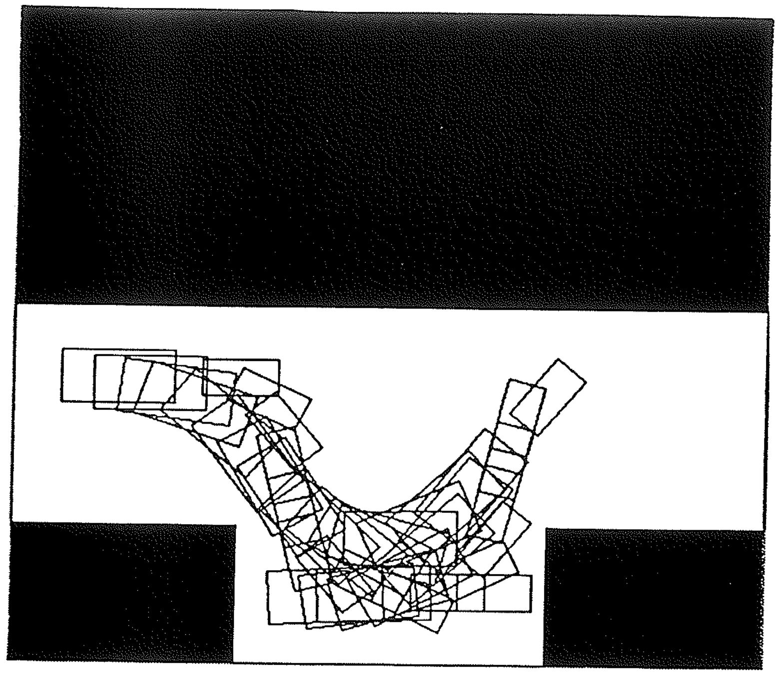

Figure 5: *Parking a Trailer*

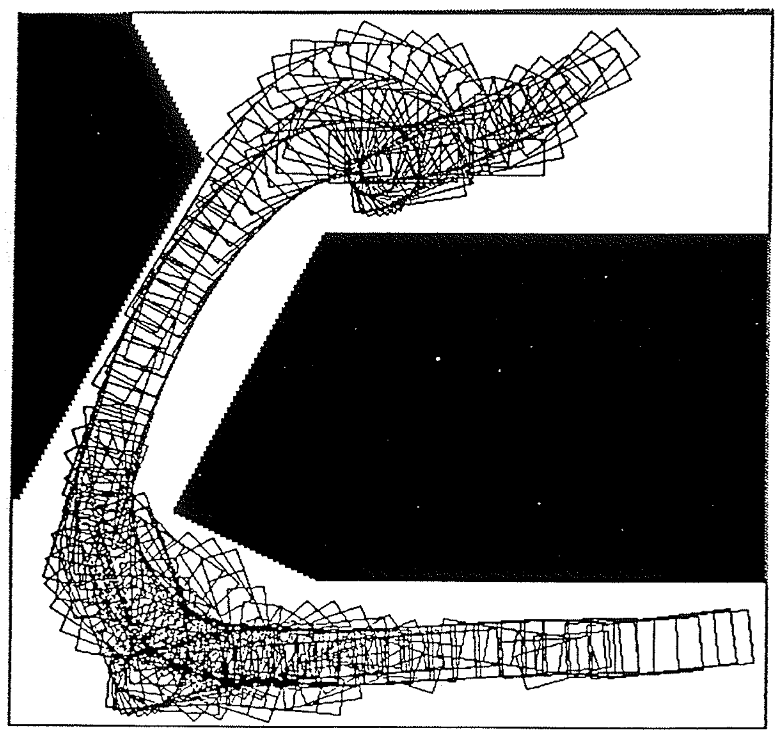

Figure 6: *Trailer Maneuvering in a Cluttered Workspace*

Figure 2 shows an example of the parallel parking problem with a very limited steering angle ($\phi_{max} = 30$ degrees). The running time for that example was 20 seconds[3].

Figure 3 shows an example with backing-up maneuvers in a cluttered workspace when the maximal steering angle ϕ_{max} is 45 degrees. The running time was about 30 seconds. Four maneuvers (i.e. changes of the sign of v) were necessary in this example.

Figure 4 shows an example of maneuvering for a car that can only turn left. The two extremal values of the steering angle are $\phi_{min} = 22.5$ degrees on the left and $\phi_{max} = 45$ degrees on the left. Seventeen maneuvers were necessary in this example.

We also conducted several experiments with a simulated two-body car.

Figure 5 shows an example where the two-body car has to be parked with a very limited steering angle ($\phi_{max} = 30$ degrees). The running time was 2 minutes.

Figure 6 shows an example where the trailer has to maneuver in a cluttered workspace with a maximal steering angle ϕ_{max} equal to 45 degrees. The running time was about 10 minutes.

7 Conclusion

In this paper, we presented a systematic technique for studying the controllability of robot systems subject to non-linear and/or inequality constraints on the velocity. In particular, it has been shown that the single-body, two-body, and three-body car systems are controllable whenever there are at least two different admissible positions of the steering wheel. We also derived the differential equations of motion for any multi-body car system. However, the controllability of the general multi-body car system has not been proven, although we conjecture it.

We have illustrated these results using a general path planner initially presented in [Barraquand and Latombe 89b]. This planner generates paths for multi-body car systems with minimal number of maneuvers at the resolution of the configuration space discretization. The running time of the planner is quite reasonable for single-body and two-body cars, even in very cluttered workspaces.

However, as the search algorithm requires exponential time and exponential space in the number of bodies, the planner is not practical for more than two bodies. Planning efficient motions for cars with three or more bodies is still an open problem.

[3] Our planner runs on a DEC3100 workstation

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