Ph.D. Qualifying Exam, Real Analysis

September 2006, part I

Do all the problems.

1 Let G be an unbounded open set in $(0, \infty)$. Define

$$D = \{x \in (0, \infty) : nx \in G \quad \text{for infinitely many } n\}.$$

Prove that D is dense in $(0, \infty)$.

2 Suppose that $f \in L^1([0,1])$ but $f \notin L^2([0,1])$. Find a complete orthonormal basis $\{\phi_n\}$ for $L^2([0,1])$ such that each $\phi_n \in \mathcal{C}^0([0,1])$ and such that

$$\int_0^1 f(x)\phi_n(x)\, dx = 0 \qquad \forall\, n.$$

3 Let H_1 and H_2 be two separable Hilbert spaces. Suppose that $A : H_1 \to H_2$ is a continuous injective linear map. Suppose that $\{v_j\}$ is a bounded sequence in H_1 such that Av_j converges strongly in H_2 to some element w. Prove that there exists an element $v \in H_1$ such that v_j converges weakly to v and $Av = w$.

4 Some problems about linear functionals:

a. Let $T : \mathcal{C}^0([0,1]) \longrightarrow \mathbb{C}$ be defined by $T(f) = f(1/2)$. Is T continuous with respect to the L^2 norm? Explain why or why not.

b. Let P denote the set of polynomials of arbitrary degree, and consider their restrictions to $[0,1]$ so as to consider $P \subset \mathcal{C}^0([0,1])$. Define the linear functional on P, $T_k(f) = a_k$ where a_k is the coefficient of x^k. Does T_k extend as a continuous linear functional to all of $\mathcal{C}^0([0,1])$?

Hint: Consider $f_n(x) = (1-x)^n$, $n \geq k$.

5 Let $Q = [0,1] \times [0,1]$ and denote by X be the set of all closed nonempty subsets of Q. Define

$$d(A,B) = \inf\{\delta > 0 : A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where for any $C \in X$, $C_\delta = \{x \in Q : \mathrm{dist}\,(x, C) < \delta\}$. Prove that (X, d) is a compact metric space.

Hints: First prove that the subset of elements $A \in X$ where A is finite is dense in X. Next, if $\{A_n\}$ is a decreasing nested sequence of closed subsets of Q, $\cap A_n = A$, prove that $A_n \to A$. Finally, if $\{B_n\}$ is an arbitrary Cauchy sequence in (X, d), consider $A_n = \cup_{k \geq n} A_k$.

Ph.D. Qualifying Exam, Real Analysis

September 2006, part II

Do all the problems.

1 Let H be a Hilbert space with an orthogonal decomposition into finite dimensional subspaces, $H = \bigoplus_{j=1}^{\infty} H_j$. Thus each $v \in H$ can be written uniquely as $v = \sum_{j=1}^{\infty} v_j$ with $v_j \in H_j$. Let $c = (c_1, c_2, \ldots)$ where each $c_j > 0$, and define the subset

$$A_c = \{v : ||v_j|| \leq c_j\} \subset H.$$

a. Prove that $c \in \ell^2$ if and only if A_c is compact in H.

b. Prove that every compact subset $K \subset H$ is contained in some A_c for some $c \in \ell^2$.

2 Let μ be a finite measure on $\mathbb{R}$. Define its Fourier transform $\widehat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} d\mu(x)$. Prove that

$$|\mu(\{x\})| \leq \limsup_{|\xi|\to\infty} |\widehat{\mu}(\xi)|.$$

3 Suppose that $p, q, r \in [1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Prove that for every $f \in L^p(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, their convolution is an element of $L^r(\mathbb{R})$, and that

$$\int_{-\infty}^{\infty} |f \star g|^r \, dx \leq \left(\int_{-\infty}^{\infty} |f|^p \, dx \right)^{\frac{r}{p}} \left(\int_{-\infty}^{\infty} |g|^q \, dx \right)^{\frac{r}{q}}.$$

Hint: Use interpolation.

4 Let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that $g(x+1) = g(x)$ for every x. Define

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} g(2^k x).$$

Show that there exists $A > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq A\, |x-y| \, \big| \log |x-y| \big|.$$

Hint: If $2^{-n-1} \leq |x-y| \leq 2^{-n}$, divide the series for f into two parts.

5 Suppose that f is a function from the natural numbers $\mathbb{N}$ to $\mathbb{R}^+$ such that for every value of $n, m \in \mathbb{N}$, $f(n+m) \leq f(n) + f(m)$. Prove that

$$\lim_{n\to\infty} \frac{f(n)}{n}$$

exists and equals

$$\inf_{n>0} \frac{f(n)}{n}.$$