# Monetary Policy with Diverse Private Expectations

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**Abstract:** We study the impact of diverse beliefs on conduct of monetary policy. Individual belief is modeled by a state variable that defines an individual's perceived laws of motion. We use a New Keynesian Model that is solved with a quadratic approximation hence individual decisions are quadratic functions. Aggregation renders the belief distribution an aggregate state variable. Although the model has standard technology and policy shocks, diverse expectations change materially standard results about a smooth trade-off between inflation volatility and output volatility. Our main results are summed up as follows:

(i) *The policy space contains a curve of singularity which is a collection of policy parameters that divides the space into two sub-regions. Some trade-off between output and inflation volatilities exists within each region and some across regions.* (ii) *The singularity causes volatility of variables to be non monotone in policy parameters. Policy-makers cannot assume a more aggressive policy will change outcomes in a predictable manner.* (iii) *When beliefs are diverse a central bank must also consider the volatility of individual consumption and the related volatility of financial markets. We show aggressive anti-inflation policy increases consumption volatility and aggressive output stabilization policy entails rising inflation volatility. Efficient central bank policy must therefore be moderate.* (iv) *High optimism about the future typically lowers aggregate output and increases inflation. This "stagflation" effect is stronger the stickier prices are. Policy response is muted since the effects of higher inflation and lower output on interest rates partially cancel each other. Effective policy requires targeting exuberance directly or its effects in asset markets. Central banks already do so with short term interventions.* (v) *The observed high serial correlation of 0.80 in policy shocks contributes greatly to market volatility and we show that a reduction in persistence of central bank's deviations from a fixed rule will contribute to stability.* (vi) *Belief dispersion is measured by cross sectional standard deviation of individual beliefs. An increased belief diversity is found to make policy coordination harder and results in lower aggregate output and lower rate of inflation. Bank policy can lower belief dispersion by being more transparent.*

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*1.     Introduction*

It is universally recognized private expectations are very important for the conduct of monetary policy and are often cited by central banks as a reason for taking one action or another. Persistence of inflationary expectations is a force central banks struggle with when aggressive anti-inflationary policy is advocated and pessimistic private expectations about future returns are often given as a reason for actions to boost investment demand. Expectations matter in an economy with random shocks and a discussion of expectations entails an assessment of the private response to these shocks. Yet, most work on monetary policy is based on Rational Expectations (in short, RE) under which expectations as such have no independent effect that is different from their exogenous mathematical expectations.

The term *"independent effect"* identifies effect of changes in the distribution of private expectations which are *different* from effects of changes in the shocks' true mathematical expectations. Deviations from true expectations may be due to irrational behavior but our view is that it is a normal fact of life for rational agents to form their best expectations without perfect knowledge of the true mathematical distribution. This lack of knowledge is due to on-going societal changes (i.e. continuous or discrete regime changes) that alter the true distributions and which cannot be learned with precision because of the short duration of each economic environment. As a result, agents form their best subjective private expectations with perfectly sensible but non-converging deviations from the true expectations. It stands to reason that since agents do not know the true distributions of shocks, one needs to take a flexible view of how a rational agent should form his own subjective expectations. In Section 2 we outline our approach to this question which is at the foundations of this paper.

When we leave the RE paradigm three expectation channels emerge with effects that monetary policy has to contend with. First, diverse beliefs imply diverse choice functions. This is a standard channel, causing individual endogenous variables to be different from their corresponding aggregates.

The second channel arises from a recognition by rational agents that private expectations alter market dynamics, making endogenous variables depend upon market belief. Hence, to forecast endogenous variables a rational agent must forecast future market belief. This channel is central to our work here. The third channel arises from market expectations about future actions of the central bank. Virtually all research on this issue either assumes bank's credibility and individuals' belief in commitment to a monetary rule or no credibility and bank's discretion. Here we adopt the first assumption although the importance of the problem is obvious. It surfaced in the debate about policy to extricate the economy from the Zero Lower Bound (ZLB) by the central bank "managing" market belief so as to generate private inflation expectations (e.g. Krugman (1998), Eggertsson and Woodford (2004), Eggertsson (2006) and Woodford (2007a,b), (2012)). We do not believe a central bank can manage market belief particularly when a large segment of private agents is hostile to the Fed's thinking and actions. The fact is that since the rate has been at zero the Fed has failed to generate inflationary expectations in spite of several QE programs and promises to keep the zero rate beyond the formal exit date from the ZLB.

In this paper we study the effects of diverse beliefs on the efficacy of monetary policy and the consequences of the interaction of private expectations with policy. Our economic environment is a New Keynesian Model (in short, NKM) developed along standard lines (e.g. Clarida et al. (1999), Woodford (2003), Galí (2008), Walsh (2010)) which is adapted to an economy with diverse beliefs. Private expectations in our model are rational in the sense that they are compatible with past data and cannot be contradicted by the evidence. We thus study Rational Belief Equilibria (in short, RBE) defined in Kurz (1994) and developed for NKM in Kurz (2012) and Kurz, Piccillo and Wu (2013) (in short KPW (2013)) whose focus was the problem of aggregation in a model under log linear approximation. The present paper focuses on the policy implications of the model of Kurz (2012) and KPW (2013) but we solve it with a second order approximation to quantify and analyze effect of belief heterogeneity on market performance and policy. A second order approximation is a natural setting to study the role of higher moments of the distributions of agent specific variables, with emphasis on studying the effects of changes in the cross-sectional standard deviations of agents' characteristics on economic performance and policy. We explain in Section 3 the method of approximation and report in Appendix B detailed tests of the errors in the Euler equations of the model.

We investigate the policy trade-offs between output and inflation volatility. This question is not new to the policy debate and results in support of a presence of such trade-off and its characterization are

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extensive (e.g. Taylor (1979), (1993a), (1993b), Fuhrer (1994), Svensson (1997), Ball (1999), Rotemberg and Woodford (1999), Rudebusch and Svensson (1999) and others). This work points to some trade-off and, technically speaking, the data is compatible with the hypotheses of some trade-off. However, the entire research program has been model based and since RE is assumed universally, the Bayesian foundations of the approach taken implies that compatibility of the hypotheses with the data is not a compelling proof that better explanations of the evidence are unavailable, hence it leaves open many important questions. As to the pure theory of the problem, the initial NKM with a single technological shock was criticized for exhibiting no trade-off between the two volatilities (see Blanchard and Galí (2010)), labeling this phenomenon "divine coincidence." It was somewhat resolved by adding a second ad-hoc shock to the Phillips Curve (for discussion see Galí (2008), Chapter 5). From a formal perspective Kurz (2012) and KPW (2013) show that with diverse beliefs, two random terms are added to the aggregated NKM: one to the IS curve and one to the Phillips Curve and this, by itself, shows that the existence of diverse beliefs changes the nature of this trade-off. Indeed, Kurz et al. (2005a) have already anticipated this result by showing diverse beliefs by themselves and complete price flexibility are sufficient to render monetary policy effective with some trade-off between inflation and output volatility.

The NKM with diverse beliefs studied in this paper has a technology shock, a monetary policy shock and other random terms that reflect the effects of beliefs. It may thus appear we should expect nothing but smooth trade-off between inflation and output volatility. We show that this simple view is incorrect: both the efficacy of monetary policy and the nature of the trade-off between inflation and output volatility changes drastically. The main results of the paper are:

• *The policy space contains a curve of singularity that divides the policy parameter space into two sub-regions with some trade-off within regions and some across regions. A cost push shock (with unknown source) may increase tradeoff but not alter the structure outlined. This singularity causes volatility outcomes of policy to be non monotonic. Hence, central banks cannot assume predictable results of higher policy intensity.*

• *Diverse individual consumption requires central banks to consider volatility of individual consumption and the associated volatility in financial markets. An aggressive policy faces either a rising consumption volatility or inflation volatility hence all efficient central bank policies are moderate.*

• *Market optimism about the future most likely lowers output and raises inflation and this "stagflation" limits policy response since the effects of higher inflation and lower output on the interest rate partially cancel each other. To be effective a central bank can target exuberance directly. Increased belief dispersion, measured by cross sectional standard deviation of beliefs results in lower output and lower rate of inflation.*

- *High persistence of policy shocks contributes greatly to market volatility hence a reduction in the persistence of central bank deviations from a fixed rule contributes to stability.*

## 2.    *The setting: a New Keynesian Model with diverse private beliefs*

### 2.1    *Decomposition Principle*

The objective of studying the effect of heterogeneity on market performance and policy makes it impossible to solve the problem with standard "black box" methods used to solve dynamic optimization problems of a representative agent model. Instead, we examine the structure of the problem, make simplifying assumptions to render the problem manageable and then develop an iterative procedure for deducing the aggregates implied by the agent-specific decision functions. An important decomposition principle accompanies us along this process. This paper is better understood with the aid of this concept.

A second order approximation amounts to using second order polynomials to define individual decision functions and aggregating them to impose market clearing. Hence, an equilibrium is a set of polynomial parameters that satisfy individual optimum conditions and market clearing. The principle of decomposition - a simple consequence of Taylor's theorem - says the solution of the non-linear model consists of a linear and a non-linear part but *the linear part is exactly the solution of the corresponding NKM linear model*. Hence, in developing a second order approximation the linear model is a vital first step which is a reference to all that we do. This is particularly important when we implement an iterative procedure to solve for the aggregates and impose market clearing. For this reason we spend the first part of the paper to briefly reviewing KPW (2013) but focusing on our extensions of the model and on explaining the belief structure and why market belief has an independent effect.

The decomposition principle has other implications. For example, an equilibrium in the log-linearized economy is a set of linear functions specifying individual decisions and aggregate variables, all functions of state variables. In general, these are solved simultaneously, but if the exogenous shocks $(v_t^1, v_t^2, \ldots, v_t^K)$ are independent, decomposition means we can solve for parameters of endogenous variables as functions of $(v_t^1, v_t^2, \ldots, v_t^K)$, *one at a time*. That is, we can first solve for parameters of $v_t^1$ then for $v_t^2$ etc. and these solutions can be deduced analytically. Decomposition does not hold with respect to belief state variables, and this fact is important for understanding the role of diverse beliefs in the model. We study some of these cases in the development below.

## 2.2    *The monopolistic competitors*

The economy is a standard NKM with a continuum of agents and products. Agent are consumer-producers and we keep track of their functions: as households we index them by superscripts $j \in [0,1]$ but as firms with distinct products we index them by subscripts $i \in [0,1]$. Household j manages firm j hence in equations that involve a household and a firm, the index j is in the subscript and superscript.

Firm i produces an intermediate good i sold at price $p_{it}$. The firms are monopolistic competitors who select optimal prices of their intermediate goods given demand, wage rate and production technology which uses only labor, defined by

(1)  $$Y_{it} = \zeta_t N_{it} \ , \ \zeta_t > 0 \text{ a random variable with } E^m(\zeta_{t+1}) = 1 .$$

With belief diversity we keep track of different model probabilities and note that  m in (1) is the stationary "empirical" probability deduced from past data hence is common knowledge. A belief of agent j is a model specifying how his subjective probability *differs from m.* These are explained later.

To generate final consumption household j purchases intermediate goods from all firms in the economy and produces its own final consumption via the transformation

$$C_t^j = [ \int_{[0,1]} (C_{it}^j)^{\frac{\theta-1}{\theta}} di]^{\frac{\theta}{\theta-1}} \ , \ \theta > 1 .$$

$P_t$ is the price of final consumption, which is also "The Price Level," defined in equilibrium by

$$P_t \equiv [ \int_{[0,1]} p_{it}^{1-\theta} di]^{\frac{1}{1-\theta}} \quad .$$

Now, the household maximizes an objective

(2a)  $$\text{Max } E_t^j \sum_{\tau=0}^{\infty} \beta^{t+\tau} \left( \frac{1}{1-\sigma}(C_{t+\tau}^j)^{1-\sigma} - \frac{1}{1+\eta}(L_{t+\tau}^j)^{1+\eta} + \frac{1}{1-b}(\frac{M_{t+\tau}^j}{P_{t+\tau}})^{1-b} - \frac{\tilde{\tau}_b}{2}(\frac{B_{t+\tau}^j}{P_{t+\tau}})^2 \right) , \sigma > 0, \eta > 0, b > 0 .$$

A small penalty on excessive borrowing is a substitute for transversality conditions. KPW (2013) show it is used to avoid a Ponzi equilibrium with unbounded borrowing. The budget constraint is defined by

(2b)  $$C_t^j + \frac{M_t^j}{P_t} + \frac{B_t^j}{P_t} + \frac{T_t^j}{P_t} = (\frac{W_t^N}{P_t})L_t^j + [\frac{B_{t-1}^j(1+r_{t-1}) + M_{t-1}^j}{P_{t-1}}](\frac{P_{t-1}}{P_t}) + \frac{1}{P_t}[p_{jt}Y_{jt} - W_t^N N_{jt}]$$

$(M_0^j, B_0^j)$ is given, all j. Initial debt $B_0^j = 0$ and $M_0^j$ are given.

C  is consumption, L is labor supplied, M is money holding, T are transfers, $W^N$ is nominal wage, B is one period debt held and r is a nominal interest rate. Markets for bonds and for labor are competitive and

wages are flexible. We derive Euler equations and log linearize them but ignore the money equation as the central bank supplies money to enforce an interest rate policy specified later.

If $\overline{\Xi}$ is a steady state value of $\Xi$, percent deviations from steady state are $\hat{\vartheta}_t = (\Xi_t - \overline{\Xi})/\overline{\Xi}$. The exception are $\hat{r}_t$ and $\hat{b}_t^j$, defined by

$$\hat{b}_t^j = \frac{B_t^j}{P_t} \quad , \quad \overline{r} = \frac{1-\beta}{\beta} \quad \text{and} \quad \hat{r}_t = r_t - \overline{r}.$$

Assume zero inflation central bank target hence zero rate in steady state hence $\overline{\pi} = 1$, $\hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$, and define the real wage by $\hat{w}_t = \hat{w}_t^N - \hat{p}_t$ to conclude that

(3a)
$$\hat{c}_t^j = E_t^j(\hat{c}_{t+1}^j) - (\frac{1}{\sigma})[\frac{\hat{r}_t}{1+\overline{r}} - E_t^j(\hat{\pi}_{t+1})] + \tau_b \hat{b}_t^j \quad , \quad \tau_b = \frac{\overline{\tau}_B \overline{Y}^\sigma}{\sigma}$$

(3b)
$$-\sigma(\hat{c}_t^j) + \hat{w}_t = \eta(\hat{\ell}_t^j).$$

$Y_t$ is the aggregate output level and the equilibrium conditions are

(3c)
$$\int_{[0,1]} \hat{c}_t^j dj = \hat{c}_t = \hat{y}_t \quad , \quad \int_{[0,1]} \hat{n}_t^j dj = \hat{n}_t = \hat{\ell}_t = \int_{[0,1]} \hat{\ell}_t^j dj \quad , \quad \int_{[0,1]} b_t^j dj = 0.$$

(3b)-(3c) imply $\hat{w}_t = \eta(\hat{n}_t) + \sigma(\hat{y}_t)$ and since $\hat{n}_t = \hat{y}_t - \hat{\zeta}_t$ we have that $\hat{w}_t = (\eta + \sigma)\hat{y}_t - \eta\hat{\zeta}_t$.

Condition (3c) shows that although these are optimum conditions of j, aggregates must be defined to enforce market clearing conditions. We then define $\overline{E}_t(\bullet)$ for any random variable v by

$$\overline{E}_t(v_{t+1}) = \int_{[0,1]} (E_t^j v_{t+1}) dj.$$

This operator does not obey the law of iterated expectations. Now aggregate (3a) to deduce

(4)
$$\hat{y}_t = \overline{E}_t(\hat{y}_{t+1}) + \int_{[0,1]} \left(E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1})\right) dj - (\frac{1}{\sigma})[\frac{\hat{r}_t}{1+\overline{r}} - \overline{E}_t(\hat{\pi}_{t+1})]$$

and (4) is an aggregated IS curve but not entirely a function of standard aggregates.

Optimal pricing by firms is based on Calvo (1983) and is more involved. The reader is referred to KPW (2013) for detailed derivations. To explain the solution note the demand function of producer i

$$Y_{it} = (\frac{p_{it}}{P_t})^{-\theta} Y_t.$$

Maximizing over labor input is the same as maximizing over $p_{it}$. The profit function is then defined by

$$\Pi_{it} = \frac{1}{P_t}[p_{it}Y_{it} - W_t^N N_{it}] = [(\frac{p_{it}}{P_t})Y_{it} - \frac{W_t^N}{P_t}(\frac{Y_{it}}{\zeta_t})] \quad \text{while} \quad Y_{it} = (\frac{p_{it}}{P_t})^{-\theta} Y_t.$$

Nominal marginal cost is $(W_t^N/\zeta_t)$ and real marginal cost is $\varphi_t = \dfrac{1}{\zeta_t}\dfrac{W_t^N}{P_t}$. Hence, $\hat{\varphi}_t = -\hat{\zeta}_t + \hat{w}_t$. We make three assumptions:

**Assumption 1:** In a Calvo (1983) pricing process the distribution of beliefs among firm-agents is the same for those who adjust prices as those who do not adjust prices.

**Assumption 2**: An agent-firm chooses an optimal price so as to maximize discounted future profits given his own belief and considers transfer a lump sum. Actual transfers made ensure all households receive the same real profits, avoiding income effect of Calvo pricing. Transfers to agent j then equal

$$\frac{T_t^j}{P_t} = \Pi_t - \Pi_t^j \quad , \quad \Pi_t = \int_{[0,1]} \Pi_{it}\,di.$$

Let $p_{it}^\star$ be the optimal price of i and let $q_{it} = \dfrac{p_{it}^\star}{P_t}$. KPW (2013) show that if $\omega$ is a probability a firm cannot adjust its price at date t, the relation between the aggregates is

(5a)
$$\hat{q}_t \equiv \int_{[0,1]} \hat{q}_{it}\,di \;=\; \frac{\omega}{1-\omega}\hat{\pi}_t$$

and pricing dynamics

(5b)
$$\hat{q}_{it} = (1-\beta\omega)[(\eta+\sigma)\hat{y}_t - (1+\eta)\hat{\zeta}_t] + \beta\omega E_t^i[\hat{q}_{i(t+1)} + \hat{\pi}_{t+1}]$$

(5c)
$$\hat{q}_t = (1-\beta\omega)[(\eta+\sigma)\hat{y}_t - (1+\eta)\hat{\zeta}_t] + (\beta\omega)\bar{E}_t(\hat{q}_{t+1} + \hat{\pi}_{t+1}) + (\beta\omega)\int_{[0,1]}\left(E_t^i\hat{q}_{i(t+1)} - E_t^i\hat{q}_{(t+1)}\right)$$

(6)
$$\hat{\pi}_t = -\kappa(1+\eta)\hat{\zeta}_t + \kappa(\eta+\sigma)\hat{y}_t + \beta\bar{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\int_{[0,1]}\left(E_t^i\hat{q}_{i(t+1)} - E_t^i\hat{q}_{(t+1)}\right)$$

where $\kappa = \dfrac{(1-\beta\omega)(1-\omega)}{\omega}$. (6) is the Phillips Curve but is not entirely a function of aggregates.

The bond holding equation is deduced from linearization of the budget constraint that implies

(7)
$$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + [1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}](\hat{y}_t - \hat{c}_t^j)$$

and the aggregation of this equation is natural.

**Assumption 3:** Monetary policy rule is subject to an additive exogenous random policy shock $u_t$ which is observed by all agents and has a Markov transition.

We use mostly a Taylor type policy rule of the form

$$(8) \qquad \hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y(\hat{y}_t - \hat{y}_t^F) + u_t \ .$$

$\hat{y}_t^F$ is output under flexible prices taken to be "potential" output[1], $u_t$ is a policy shock and the weights $(\xi_\pi > 1, \xi_y \geq 0)$ measure policy intensity. Larger values of $\xi_\pi$ are taken to be "more aggressive inflation stabilization policy" and larger values of $\xi_y$ as "more aggressive output stabilizing policy."

Neither the IS curve (4) nor the Phillips Curve (6) are functions of aggregates only. KPW (2013) solve the problem of aggregation for the linearized economy and we review their approach shortly. Here we study policy implications *under a quadratic approximation* which raises new aggregation problems. We thus briefly explain how aggregates are defined in the linear model and then proceed to discuss our method of constructing aggregates in the more complex non-linear economy. Since all problems related to aggregation arise from the structure of expectations, we turn next to explain the belief structure.

*2.3    The structure of belief*

We follow the "Rational Belief" (in short, RB) approach due to Kurz (e.g. Kurz (1994), (1997), (2009)). For beliefs to be diverse there must be something agents cannot know. Here it is an unobserved state that affects all state variables. Process of exogenous shocks is non-stationary with time dependent mean values. For simplicity we assume *one unobserved state variable* about which agents hold diverse beliefs. Agents have long past data used to deduce an empirical probability m on observed variables. We assume the empirical and stationary probability m implies $(\hat{\zeta}_t, u_t)$ have Markov transitions:

$$(9a) \qquad \hat{\zeta}_{t+1} = \lambda_\zeta \hat{\zeta}_t + \rho_{t+1}^\zeta \qquad\qquad \begin{pmatrix} \rho_{t+1}^\zeta \\ \rho_{t+1}^u \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_\zeta^2, & 0, \\ 0, & \sigma_u^2 \end{bmatrix} \right), \ \ \text{i.i.d.}$$

$$(9b) \qquad u_{t+1} = \lambda_u u_t + \rho_{t+1}^u$$

The truth is that both processes are subject to unknown shifts in structure, taking the true form

$$(10a) \qquad \hat{\zeta}_{t+1} = \lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^s s_t + \tilde{\rho}_{t+1}^\zeta \qquad\qquad \begin{pmatrix} \tilde{\rho}_{t+1}^\zeta \\ \tilde{\rho}_{t+1}^u \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \tilde{\sigma}_\zeta^2, & 0, \\ 0, & \tilde{\sigma}_u^2 \end{bmatrix} \right).$$

$$(10b) \qquad u_{t+1} = \lambda_u u_t + \lambda_u^s s_t + \tilde{\rho}_{t+1}^u$$

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[1] The flexible prices output level we consider and from which we derive $\hat{y}_t^F$ in its linear or quadratic approximation is well known and function of the productivity shock only: $Y_t^F = \left( \dfrac{\theta-1}{\theta} \right)^{\frac{1}{\sigma+\eta}} \zeta_t^{\frac{1+\eta}{\sigma+\eta}}$ (see Walsh (2010) p. 335).

Time varying parameters $\mathbf{s_t}$ are unobserved hence (9a)-(9b) are time averages of (10a)-(10b).

Individual belief are described by state variables $\mathbf{g_t^j}$ that are the agents' subjective conditional expectations of $\mathbf{s_t}$. The notation $(\hat{\zeta}_{t+1}^j, \mathbf{u}_{t+1}^j)$ expresses j's *perception* of t+1 shocks before observing them. By convention $(E_t^j \hat{\zeta}_{t+1}, E_t^j \mathbf{u}_{t+1})^2$ is the same as $(E_t^j \hat{\zeta}_{t+1}^j, E_t^j \mathbf{u}_{t+1}^j)$ since agent's expectation can be taken only with respect to perception. Agent j's then *perceives* $(\hat{\zeta}_{t+1}^j, \mathbf{u}_{t+1}^j)$ at date t to be distributed as

(11a) $\quad \hat{\zeta}_{t+1}^j = \lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g \mathbf{g}_t^j + \rho_{t+1}^{j\zeta}$

(11b) $\quad \mathbf{u}_{t+1}^j = \lambda_u \mathbf{u}_t + \lambda_u^g \mathbf{g}_t^j + \rho_{t+1}^{ju}$

$$\begin{pmatrix} \rho_{t+1}^{j\zeta} \\ \rho_{t+1}^{ju} \end{pmatrix} \sim N \left( \begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \hat{\sigma}_\zeta^2, & \hat{\sigma}_{\zeta u}, \\ \hat{\sigma}_{\zeta u}, & \hat{\sigma}_u^2 \end{bmatrix} \right)$$

(11a)-(11b) show that $\mathbf{g_t^j}$ specifies the *difference between j's date t forecast and the forecasts under m.*

KPW (2013) synthesize the RB approach with three rationality axioms on j's belief. These imply that $\mathbf{g_t^j}$ fluctuates over time with a dynamic law of motion which is shown to be

(12a) $\quad \mathbf{g}_{t+1}^j = \lambda_Z \mathbf{g}_t^j + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda_Z^u [\mathbf{u}_{t+1} - \lambda_u \mathbf{u}_t] + \rho_{t+1}^{jg}$ , $\rho_{t+1}^{jg} \sim N(0, \sigma_g^2)$ , $\rho_{t+1}^{jg}$ are correlated across j.

Several natural assumptions are made. First, each agent is anonymous in assuming his belief has no effect on the market. Second, although each $\mathbf{g_t^j}$ is not publically observed, the distribution of all $\mathbf{g_t^j}$ is observed hence mean market belief $Z_t = \int_{[0,1]} \mathbf{g}_t^j \mathbf{dj}$ is also observed. Like all observed variables, it has an empirical distribution and induces diverse beliefs about its future. The RB approach shows that agents forming beliefs about mean market belief expand their state spaces but does not trigger an infinite regress since $Z_t$ is common knowledge and its empirical distribution is deduced from (12a)

(12b) $\quad Z_{t+1} = \lambda_Z Z_t + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda_Z^u [\mathbf{u}_{t+1} - \lambda_u \mathbf{u}_t] + \tilde{\rho}_{t+1}^Z$ , $\tilde{\rho}_{t+1}^Z = \int_{[0,1]} \rho_{t+1}^{jg} \mathbf{dj}$.

Note $\tilde{\rho}_{t+1}^Z \neq 0$ since, due to correlation across agents, the law of large numbers does not hold and $\tilde{\rho}_{t+1}^Z$ may exhibit time dependence as well. Since correlation is not determined by individual rationality it is a *belief externality*. Agents' uncertainty about future market belief $Z_{t+1}$ *is central to our approach.*

In sum, agents have data on $\{(\hat{\zeta}_t, \mathbf{u}_t, Z_t), t = 1, 2, \dots \}$ and know their *joint empirical distribution.* We assume it is a Markov probability with transition functions described by

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[2] The notation $(\hat{\zeta}_{t+1}^j, \mathbf{u}_{t+1}^j)$ is used to highlight *perception* of the variables $(\zeta_{t+1}, \mathbf{u}_{t+1})$ by agent i before they are observed. In general, for an aggregate variable $\mathbf{x}_{t+1}$, there is no difference between $E_t^j \mathbf{x}_{t+1}^j$ and $E_t^j \mathbf{x}_{t+1}$ since j's expectations can be taken only with respect to j's perception. However, it is important to keep in mind the context. If in a discussion the variable $\mathbf{x}_{t+1}$ is assumed to be observed at t+1, then it cannot be perceived at that date. Hence, the notation $\mathbf{x}_{t+1}^j$ expresses *perception* of $\mathbf{x}_{t+1}$ by agent j before the variable is observed and $E_t^j \mathbf{x}_{t+1}$ expresses the expectations of $\mathbf{x}_{t+1}$ by j, *in accordance with his perception.* This procedure does not apply to j-specific variables such as $E_t^j \hat{e}_{t+1}^j$ which has a natural interpretation.

(13a) $\quad \hat{\zeta}_{t+1} = \lambda_\zeta \hat{\zeta}_t + \rho^\zeta_{t+1}$

(13b) $\quad u_{t+1} = \lambda_u u_t + \rho^u_{t+1}$

(13c) $\quad Z_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda^u_Z[u_{t+1} - \lambda_u u_t] + \rho^Z_{t+1}$

$$\begin{pmatrix} \rho^\zeta_{t+1} \\ \rho^u_{t+1} \\ \rho^Z_{t+1} \end{pmatrix} \sim N \left( \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma^2_\zeta, & 0, & 0 \\ 0, & \sigma^2_u, & 0 \\ 0, & 0, & \sigma^2_Z \end{bmatrix} \right), \text{ i.i.d.}$$

Agents who do not believe (13a)-(13c) are the truth, formulate their own beliefs. An agent's perception model $(\hat{\zeta}^j_{t+1}, u^j_{t+1}, Z^j_{t+1}, g^j_{t+1})$ is then described by the transition functions of the form

(14a) $\quad \hat{\zeta}^j_{t+1} = \lambda_\zeta \hat{\zeta}_t + \lambda^g_\zeta g^j_t + \rho^{j\zeta}_{t+1}$

(14b) $\quad u^j_{t+1} = \lambda_u u_t + \lambda^g_u g^j_t + \rho^{ju}_{t+1}$

(14c) $\quad Z^j_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z[\hat{\zeta}^j_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda^u_Z[u^j_{t+1} - \lambda_u u_t] + \lambda^g_Z g^j_t + \rho^{jZ}_{t+1}$

(14d) $\quad g^j_{t+1} = \lambda_Z g^j_t + \lambda^\zeta_Z[\hat{\zeta}^j_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda^u_Z[u^j_{t+1} - \lambda_u u_t] + \rho^{jg}_{t+1}$

$$\begin{pmatrix} \rho^{j\zeta}_{t+1} \\ \rho^{ju}_{t+1} \\ \rho^{jZ}_{t+1} \\ \rho^{jg}_{t+1} \end{pmatrix} \sim N \left( \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}, \begin{bmatrix} \hat{\sigma}^2_\zeta, & 0, & 0, & 0 \\ 0, & \hat{\sigma}^2_u, & 0, & 0 \\ 0, & 0, & \hat{\sigma}^2_Z, & \hat{\sigma}_{Zg} \\ 0, & 0, & \hat{\sigma}_{Zg}, & \sigma^2_g \end{bmatrix} \right)$$

## 2.4 Defining aggregates in the linear model

To see how aggregation depends upon the structure of beliefs we review KPW (2013) who show that in equilibrium of the log linearized economy optimal individual decision functions are of the form

(15a) $\quad \hat{c}^j_t = A^Z_y Z_t + A^\zeta_y \hat{\zeta}_t + A^u_y u_t + A^b_y \hat{b}^j_{t-1} + A^g_y g^j_t \equiv A_y \bullet (Z_t, \hat{\zeta}_t, u_t, \hat{b}^j_{t-1}, g^j_t)$

(15b) $\quad \hat{q}^*_{jt} = \frac{\omega}{1-\omega}[A^Z_\pi Z_t + A^\zeta_\pi \hat{\zeta}_t + A^u_\pi u_t + A^b_\pi \hat{b}^j_{t-1} + A^g_\pi g^j_t] \equiv \frac{\omega}{1-\omega} A_\pi \bullet (Z_t, \hat{\zeta}_t, u_t, \hat{b}^j_{t-1}, g^j_t)$

(15c) $\quad \hat{b}^j_t = A^Z_b Z_t + A^\zeta_b \hat{\zeta}_t + A^u_b u_t + A^b_b \hat{b}^j_{t-1} + A^g_b g^j_t \equiv A_b \bullet (Z_t, \hat{\zeta}_t, u_t, \hat{b}^j_{t-1}, g^j_t).$

The insurance assumption 3 implies that diversity is expressed in (15a)-(15c) by bond holdings $\hat{b}^j_{t-1}$ and beliefs $g^j_t$ hence by agent specific state space over $(Z_t, \hat{\zeta}_t, u_t, \hat{b}^j_{t-1}, g^j_t)$. Imposition of market clearing conditions (3c) and (5a) is then a simple linear operation which implies that the aggregates must be

(15d) $\quad \hat{y}_t = A^Z_y Z_t + A^\zeta_y \hat{\zeta}_t + A^u_y u_t + A^b_y 0 + A^g_y Z_t \equiv A_y \bullet (Z_t, \hat{\zeta}_t, u_t, 0, Z_t)$

(15e) $\quad \hat{\pi}_t = A^Z_\pi Z_t + A^\zeta_\pi \hat{\zeta}_t + A^u_\pi u_t + A^b_\pi 0 + A^g_\pi Z_t \equiv A_\pi \bullet (Z_t, \hat{\zeta}_t, u_t, 0, Z_t)$

(15f) $\quad \hat{q}_t = \frac{\omega}{1-\omega}[A^Z_\pi Z_t + A^\zeta_\pi \hat{\zeta}_t + A^u_\pi u_t + A^b_\pi 0 + A^g_\pi Z_t] \equiv \frac{\omega}{1-\omega} A_\pi \bullet (Z_t, \hat{\zeta}_t, u_t, 0, Z_t).$

Aggregate variables are functions of $(Z_t, \hat{\zeta}_t, u_t)$ where $Z_t$ is the new factor added to the NKM. Agents take market belief as an observed variable, like prices, and forecast its value like any other variable.

For completeness we note KPW (2013) prove (Theorem 6) in equilibrium there are constants $(B_y, B_\pi)$ such that, relative to the output gap $\hat{x}_t = \hat{y}_t - \hat{y}^F_t$, the aggregates satisfy the NKM conditions

| IS Curve | $\hat{x}_t = E_t^m(\hat{x}_{t+1}) - (\frac{1}{\sigma})[\frac{\hat{r}_t}{1+\bar{r}} - E_t^m(\hat{\pi}_{t+1}) - \sigma(1-\lambda_\zeta)(\frac{1+\eta}{\sigma+\eta})\hat{\zeta}_t - \sigma B_y Z_t];$ |
| --- | --- |

Phillips Curve $\quad \hat{\pi}_t = \kappa(\eta+\sigma)\hat{x}_t + \beta E_t^m \hat{\pi}_{t+1} + \beta B_\pi Z_t$ , $\qquad\qquad \kappa = \dfrac{(1-\beta\omega)(1-\omega)}{\omega}$ ;

Monetary rule $\quad \hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y(\hat{y}_t - \hat{y}_t^F) + u_t$ .

Diverse beliefs affect aggregates via $Z_t$ in the IS and Phillips Curves. From a perspective of each agent $Z_t$ is the belief of "others" and $(g_t^j - Z_t)$ impacts model dynamics, as seen in (7),(15a)-(15f).

The solution above is the first building block of our equilibrium model under a second order polynomial approximation that promotes other moments of the cross sectional distribution of beliefs. The problems of aggregation and market clearing are much more complicated in that model.


## 2.5 Some properties of the belief structure

### 2.5.1 RBE restrictions and the role of learning feed-back

The RB principle (see Kurz (1994)) is a model of rational agents who deviate from  m  but reproduce it with sufficiently long data. An RB model of the exogenous shocks exhibits the same volatility as the empirical model and it implies the following restrictions (for details, see KPW (2013)):

$$\text{Var}[\lambda_\zeta^g g_t^j + \rho_{t+1}^{j\zeta}] = \text{Var}[\rho_{t+1}^\zeta] \quad \Rightarrow \quad (\lambda_\zeta^g)^2 \text{Var}(g) + \hat{\sigma}_\zeta^2 = \sigma_\zeta^2$$

(16) $\qquad \text{Var}[\lambda_u^g g_t^j + \rho_{t+1}^{ju}] = \text{Var}[\rho_{t+1}^u] \quad \Rightarrow \quad (\lambda_u^g)^2 \text{Var}(g) + \hat{\sigma}_u^2 = \sigma_u^2$

$$\text{Var}[\lambda_Z^g g_t^j + \rho_{t+1}^{jZ}] = \text{Var}[\rho_{t+1}^Z] \quad \Rightarrow \quad (\lambda_Z^g)^2 \text{Var}(g) + \hat{\sigma}_Z^2 = \sigma_Z^2$$

By normalization $\lambda_\zeta^g = 1$ , therefore the rationality conditions (16) imply

(17a) $\quad \text{Var}(g) \le \sigma_\zeta^2 \quad , \quad (\lambda_u^g)^2 \text{Var}(g) \le \sigma_u^2 \quad , \quad (\lambda_Z^g)^2 \text{Var}(g) \le \sigma_Z^2 \quad , \quad \tilde{\sigma}_\zeta \le \sigma_\zeta \; , \; \tilde{\sigma}_u \le \sigma_u \; , \; \tilde{\sigma}_Z \le \sigma_Z$

In addition, the variance of $\rho_{t+1}^Z$ is restricted by $\sigma_g^2$ and is specified as

(17b) $\qquad\qquad \sigma_Z^2 \le \sigma_g^2$ with $\sigma_Z = \rho\sigma_g$ and $\rho = \text{corr}(\rho_{t+1}^{jg}, \rho_{t+1}^Z) > 0$ .

The unconditional variance of $g_t^i$ is

(18a) $\qquad\qquad \text{Var}[g] = \dfrac{1}{(1-\lambda_Z^2)}[(\lambda_Z^\zeta)^2 \sigma_\zeta^2 + (\lambda_Z^u)^2 \sigma_u^2 + \sigma_g^2]$ .

The parameters $(\lambda_Z^\zeta, \lambda_Z^u)$ are important. They measure learning feed-back from current data with which agents deduce changes in estimated value of $s_t$ in (10a)-(10b) from forecast errors in (14a)-(14b) . This causes revisions of the belief index $g_t^j$ which is j's subjective conditional expectations of $s_t$. The variance of g which ignores such feed-back is therefore

(18b) $\qquad\qquad \text{Var}^{NF}[g] = \dfrac{\sigma_g^2}{(1-\lambda_Z^2)}$ .

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Interest in (18b) arises from the need to reconcile learning feed-back with the RB principle. Learning feed-back increases the variance of $g_t^j$. In addition, comparing (10a)-(10b) with perception (14a)-(14b) shows learning feed back causes $g_t^j$ to introduce into (14a)- (14b) correlation with observed data which does not exist in (9a)-(9b). Hence, on the face of it, a learning feed-back violates the RB principle. But a rational agent who adjusts his belief about exogenous shocks in response to most recent data knows there is no learning within the actual data of exogenous shocks hence he must purge (16)-(18a) from the effect of learning feed-back. This is the reason why we use (18b) and not (18a) as the basis for restricting individual beliefs. Sufficient conditions implied by this procedure and used in this paper are

(19a) $\qquad \sigma_g \leq \sigma_\zeta \sqrt{(1 - \lambda_Z^2)} \qquad , \quad \lambda_u^g \leq \dfrac{\sigma_u}{\sigma_\zeta} \quad , \quad \lambda_Z^g \leq \dfrac{\sigma_Z}{\sigma_\zeta} \leq \rho\sqrt{(1 - \lambda_Z^2)}.$

(19b) $\qquad (\lambda_\zeta^g)^2 \mathrm{Var}_g^2 + \hat{\sigma}_\zeta^2 \geq \sigma_\zeta^2 \qquad , \qquad (\lambda_u^g)^2 \mathrm{Var}_g^2 + \hat{\sigma}_u^2 \geq \sigma_u^2 \quad , \qquad (\lambda_Z^g)^2 \mathrm{Var}_g^2 + \hat{\sigma}_Z^2 \geq \sigma_Z^2.$

To clarify the effect of learning feed-back return to (14d) where future belief of j depends upon realized future data from which he will deduce $g_{t+1}^j$. Such dependence of belief upon current data amplifies the effect of beliefs. To see why suppose $g_t^j > 0$ thus j is optimistic today about larger $s_t$ and hence he perceives a larger $\hat{\zeta}_{t+1}^j$ and a larger forecast error $(\hat{\zeta}_{t+1}^j - \lambda_\zeta \hat{\zeta}_t)$ at t+1. With learning feed-back he also knows that he *expects* to interpret this larger forecast error as a larger value of $s_{t+1}$ as well, and the larger is the learning feed-back parameter $\lambda_Z^\zeta$ the larger is this secondary effect of $g_t^j > 0$ on the *expected value of* $s_{t+1}$. The mathematical result of this fact is seen by taking expectations of (14d) using (14a):

$$ E_t^j[g_{t+1}^j] = \lambda_Z g_t^j + \lambda_Z^\zeta g_t^j + \lambda_Z^u \lambda_u^g g_t^j = (\lambda_Z + \lambda_Z^\zeta + \lambda_Z^u \lambda_u^g) g_t^j. $$

Due to learning feed-back from current data the parameter $(\lambda_Z + \lambda_Z^\zeta + \lambda_Z^u \lambda_u^g)$ of $g_t^j$ in j's *expected* $g_{t+1}^j$ may exceed 1. Being only within the agent's model it does not have any effect on the actual dynamic movements of either $g_t^j$ or $Z_t$ hence has no effect on market instability or violation of Blanchard-Kahn conditions. It is merely a result of interaction between learning feed-back and persistence of beliefs.

Since we assume only one unobserved state variable, belief parameters must be oriented in sign so as to have comparable meaning. For technology it is clear being "optimistic" means $\lambda_\zeta^g > 0$ hence we set $\lambda_\zeta^g = 1$. As for policy shocks, we assume that an agent who is optimistic about stronger-than-usual future state expects the central bank will also expect a stronger-than-usual future state. Although such view results in higher nominal interest rates and lower output and inflation, it will all be in anticipation of stronger future state of the economy. Hence we have $\lambda_u^g > 0$ but note that expecting higher future state does not mean expecting higher output since when forming expectations optimistic agents take into

account the central bank reaction to a stronger future state when the bank may raise interest rates.

Empirical evidence on belief parameter discussed by KPW (2013) points to high persistence of mean market belief where $\lambda_Z$ is estimated with high confidence to be in the range of $0.6 \leq \lambda_Z \leq 0.8$ (see Kurz and Motolese (2011)) and also that $0 \leq \lambda_Z^\zeta \leq 0.70$, $0 \leq \lambda_Z^g \leq 0.25$, $\lambda_u^g \leq \dfrac{\sigma_u}{\sigma_\zeta}$, the last being based on earlier discussion. The open question is $\lambda_Z^u$. KPW (2013) and Wu (2014) provide empirical evidence that $(\sigma_u \leq 0.006$, $\lambda_u = 0.80)$ and $\lambda_u^g > 0$ postulates an association of agents' beliefs with policy shocks. However, agents believe the central bank knows as much as they do and do not believe a policy shock u is more informative about current state of the economy. We thus set $\lambda_Z^u = 0$.

### 2.5.2 *Decomposition and the interaction of diverse beliefs with policy*

Individual beliefs are about current unobserved state hence it is clear how an agent's belief has an effect on his optimal decisions. The problem is that interdependence makes it harder to pin down the effect of expectations on equilibrium output, inflation and other aggregates. To understand this we must explore in some details the mechanics by which market belief impact the equilibrium. To that end we use the decomposition principle introduced earlier.

To determine equilibrium parameters we use the decomposition principle to deduce the parameters of the technology shock $\zeta$. To do that insert (15a)-(15f) and the monetary rule into the linearized Euler equations (3a) and (5c). Next we use the perception model (14a)-(14d) to compute expectations of all state variables. Finally, since these are linear difference equations we match coefficients $(A_y^\zeta, A_\pi^\zeta)$ of $\hat{\zeta}_t$. The equations that determine these parameters are

(20a) $$\kappa(\eta + \sigma)A_y^\zeta - [1 - \beta\lambda_\zeta]A_\pi^\zeta = \kappa(1 + \eta).$$

(20b) $$[1 - \lambda_\zeta + \frac{\xi_y}{\sigma(1+\bar{r})}]A_y^\zeta + (\frac{\frac{\xi_\pi}{(1+\bar{r})} - \lambda_\zeta}{\sigma})A_\pi^\zeta = (\frac{\xi_y}{\sigma(1+\bar{r})})\frac{1+\eta}{\eta+\sigma}.$$

Now define

(21) $$T(\hat{\zeta}) = \begin{bmatrix} (1 - \lambda_\zeta + \frac{\xi_y}{\sigma(1+\bar{r})}), & \frac{\frac{\xi_\pi}{(1+\bar{r})} - \lambda_\zeta}{\sigma}, \\ \kappa(\eta + \sigma), & -(1 - \beta\lambda_\zeta) \end{bmatrix}$$

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and the solution of the above equations is

(22)
$$\begin{pmatrix} A_y^\zeta \\ A_\pi^\zeta \end{pmatrix} = T(\hat\zeta)^{-1} \begin{pmatrix} (\dfrac{\xi_y}{\sigma(1+\bar r)})\dfrac{1+\eta}{\eta+\sigma} \\ \kappa\Lambda\dfrac{1+\eta}{1-\alpha} \end{pmatrix}.$$

This solution exists and is unique since $|T(\hat\zeta)| = -[(1-\lambda_\zeta+\dfrac{\xi_y}{\sigma(1+\bar r)})(1-\beta\lambda_\zeta)+(\dfrac{\dfrac{\xi_\pi}{(1+\bar r)}-\lambda_\zeta}{\sigma})\kappa(\eta+\sigma)] < 0$ for all policies. But then, equilibrium parameters $(A_y^\zeta, A_\pi^\zeta)$ of $\hat\zeta_t$ depend only on the direct and indirect effects the exogenous shock $\hat\zeta_t$ has in the economy and do not depend upon the model's expectations.

The above points to a conclusion we labeled "decomposition." Equilibrium parameters of each exogenous shock are independent of expectations hence are the same in models with diverse beliefs and under RE. Also, they are solutions of the linear part in a higher order approximation of equilibrium under any expectation assumptions. This decomposition is an implication Taylor's theorem. But this result does not hold with respect to the effect of diverse belief on equilibrium. For simplicity we carry out computations for $(\zeta_t \neq 0, u_t = 0)$ but it will be clear how to modify computations for more shocks.

To determine $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ we follow the same procedure used for the technology shock and deduce a system of equations written in matrix form with $\Xi_1 = \dfrac{\xi_y}{\sigma} - (A_y^b + \tau_b)[1 + \dfrac{\theta-1}{\theta}\dfrac{\sigma}{\eta}]$,

$\Xi_2 = -(1-\lambda_Z-\lambda_Z^\zeta+(A_y^b+\tau_b))[1+\dfrac{\theta-1}{\theta}\dfrac{\sigma}{\eta}]$ and $\Xi_3 = \beta\omega(\lambda_Z+\lambda_Z^\zeta)+\beta(1-\omega)(\lambda_Z^\zeta+\lambda_Z^g)-1$ :

(23)
$$\begin{bmatrix} (1-\lambda_Z+\dfrac{\xi_y}{\sigma(1+\bar r)}) & \dfrac{\dfrac{\xi_\pi}{(1+\bar r)}-\lambda_Z}{\sigma} & \Xi_1 & \dfrac{\dfrac{\xi_\pi}{(1+\bar r)}-\lambda_Z}{\sigma} \\ \kappa(\eta+\sigma) & -(1-\beta\lambda_Z) & \kappa(\eta+\sigma) & \beta(1-\omega)\lambda_Z) \\ (\lambda_Z^\zeta+\lambda_Z^g) & \dfrac{(\lambda_Z^\zeta+\lambda_Z^g)}{\sigma} & \Xi_2 & \dfrac{(\lambda_Z^\zeta+\lambda_Z^g)}{\sigma} \\ 0 & \dfrac{(\lambda_Z^\zeta+\lambda_Z^g)}{\sigma} & 0 & \Xi_3 \end{bmatrix} \begin{pmatrix} A_y^Z \\ A_\pi^Z \\ A_y^g \\ A_\pi^g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -A_y^\zeta - \dfrac{A_\pi^\zeta}{\sigma} \\ -\beta A_\pi^\zeta \end{pmatrix}$$

Both (22) and (23) depend upon policy but there are two crucial differences between (22) and (23). The first explains why decomposition fails to hold here: the effect of belief works "through" the effects of exogenous shock. That is, since beliefs are about exogenous shocks their effect on equilibrium depends upon how exogenous shocks affect the economy. That is to assess the effect of beliefs one must first solve for $(A_y^\zeta, A_\pi^\zeta)$ which represent the effects of the exogenous shocks. These are then used on the

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right hand side of (23) to determine the effect of beliefs.

Second, the interaction of policy and beliefs is a crucial component of our theory. Whereas in (22) the matrix is non-singular and $(A_y^\zeta, A_\pi^\zeta)$ are *monotonic* in each of the policy instruments $(\xi_y, \xi_\pi)$, these two properties do not hold for (23): the matrix may be singular and the parameters are not monotonic with respect to $(\xi_y, \xi_\pi)$. Indeed, there exists a curve of $(\xi_y, \xi_\pi)$ along which the matrix is singular and as we approach this critical "ridge" in the policy space, volatility becomes unbounded. Also, equilibrium parameters $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ are not monotonic with respect to policy instruments $(\xi_y, \xi_\pi)$. These are departures from the common conclusions about policy trade-off under RE. To explain it further we first examine the interaction of policy with expectations and then specify when such interaction is reduced.

Private belief affect equilibrium via two different channels. First are the effects of $g_t^j$ and $Z_t$ on individual decisions and inspection of (15a)-(15c) show they are measured by $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$. Second, the effect on aggregates is more complex since they are functions of $Z_t$ only and (15d)-(15f) show they are measured by $(A_y^Z + A_y^g, A_\pi^Z + A_\pi^g)$. Also note that $(A_y^g, A_\pi^g)$ and $(A_y^Z, A_\pi^Z)$ play different roles in the agent's decision functions compared to their role in aggregates. In decision functions $(A_y^g, A_\pi^g)$ measure the effect of $g_t^j$ on j's current consumption and pricing decisions while $(A_y^Z, A_\pi^Z)$ measure an agent's *known* externality effect of market belief on the aggregates: output, inflation, interest rate and wage rate. More specifically, a more optimistic agent faces two *conflicting* incentives. Expected higher future wage causing income effect to increase today's consumption but also substitution effect to work less today and more tomorrow when the wage is higher. Similarly, a more optimistic monopolistic competitor expects lower future prices due to higher productivity (that exceeds the rise of wages due to labor supply elasticity). Risk of being unable to lower future prices motivates him to lower prices today. But expected higher future demand imply higher future prices and for the same sticky price effect an opposite motive exists to raise today's prices. These two private motives are based on an agent's forecasts of future aggregates that depend on $(A_y^Z + A_y^g, A_\pi^Z + A_\pi^g)$ which are impacted - *and this is the important point here*- by the policy in place.

With opposite private motives the dominant effect depends upon two factors. One is market interest rate since all choices are between today and the future. The second is the set of agent's forecasts of future aggregates such as wage rate and income. But both these factors depend upon the policy hence different policies will result in different $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ and what our theory shows is that when policy places higher relative weight on inflation stabilization or on output stabilization, the matrix in (23)

become singular and $(A_y^Z, A_\pi^Z, A_y^g, A_\pi^g)$ *change sign reflecting the change in the dominant effect discussed above*. But this establishes a deep interaction between market expectations and efficacy of policy. Market expectations may be supportive of the policy but a conflict may exist between policy and private expectations and this may result in unsatisfactory volatility outcomes of the aggregates.

We note that since signs of $(A_y^g, A_\pi^g)$ are typically opposite to the signs of $(A_y^Z, A_\pi^Z)$ and since aggregates are functions of $(A_y^Z + A_y^g, A_\pi^Z + A_\pi^g)$, the market carries out some cancellation of conflicting private motives. This does not prevent the sum of the parameters from changing sign as well.

What is the component of private expectations that accounts for the fact that the matrix in (23) may be singular? To answer this question we propose the following definition:

**Definition 1**: An economy has *no private beliefs about market belief (i)* if beliefs about market belief $Z_{t+1}$ are not diverse hence no agent uses his $g_t^j$ to forecast $Z_{t+1}$, and (ii) if there is no learning feedback from current data. These two require $\lambda_Z^\zeta = 0$, $\lambda_Z^u = 0$, $\lambda_Z^g = 0$ and hence $E_t^j[Z_{t+1}] = \lambda_Z Z_t$.

The following is the answer to the above question which we state without proof:

**Proposition 1:** If an economy has no private belief about market belief, the matrix in (23) is non singular for all feasible policies $\xi_y \geq 0$, $\xi_\pi \geq 1$ and therefore it does not change sign in the feasible policy space.

*3.     Constructing the quadratic approximation*

An important property of a linear model which is central to solving the aggregation problem in Section 2.4 is that the set of state variables *is closed under aggregation*. To that end note (15a)-(15c) are agent j's specific linear decision functions of individual and economy wide state variables. When these are aggregated, individual state variables aggregate to economy wide state variables and economic aggregates in (15d)-(15f) *are linear functions of economy wide state variables*. This last property is lost for higher order approximations due to the proliferation of moments. To understand how proliferation occurs consider an agent specific state variable $\hat{v}_t^j$ that aggregates to $\hat{v}_t$. Agent j's decisions are functions of $\hat{v}_t^j$ and in a second order approximation are also functions of $(\hat{v}_t^j)^2$ and $(\hat{v}_t)^2$ which are now state variables. But if we now aggregate $(\hat{v}_t^j)^2$ we must conclude that

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(24)
$$\int_{[0,1]} (\hat{v}_t^j)^2 dj \;=\; \int_{[0,1]} (\hat{v}_t^j - \hat{v}_t + \hat{v}_t)^2 dj \;=\; \sigma_{cs}^2(\hat{v})_t + (\hat{v}_t)^2$$

and *a new state variable* $\sigma_{cs}^2(\hat{v})_t$ *emerges* – the cross sectional variance of $\hat{v}_t^j$. Equation (24) explains why under a second order approximation our focus shifts to study the effect of variability of the cross sectional dispersion of heterogeneous variables. In other words, while the linear case examines only the aggregate effect of the mean of each individual variable, now we look at a more accurate index of diversity, which is the cross sectional dispersion of such variables that emerges from aggregation.

Dispersion of distributions are particularly important for quantifying the effect of belief diversity on market performance and policy. With a linear approximation the effect is measured with one coefficient. For example, $A_y^g$ measures the effect of $g^j$ on individual consumption. But these individual choices are subject to complex income and substitution effects leading one to expect a non-linear effect rather than a single signed effect. In fact, in the linear model we find that equilibrium parameters of belief variables change signs in response to changed policy parameters as seen in (23) and later in Tables 3a-3b where singularity reflects a change in net weight of income vs. substitution effect induced by changed policy parameters. A quadratic approximation enables a more subtle examination of these effects, in addition to a study of the manner in which policy alters them. However, in order to proceed we must resolve the question of how to select the state variables of the model.

*3.1    Selecting a set of state variables closed under aggregation: the role of cross sectional variances*

The argument of the previous section shows that with a higher order of approximation the only set of state variables closed under aggregation is the set of all infinite moments. Being infeasible it follows that any higher order approximation of finite order needs an ad-hoc decision to disregard higher moments at a degree that reflects an analytic choice of the model builder. Our approach is based on this same inevitable principle.

To explain our approach suppose $V^j$ is a vector of agent j specific state variables and $V$ is a vector of economy-wide state variables. Suppose decisions are quadratic in state variables then all decision functions are linear functions of $(V, V^2, VV^j, V^j, (V^j)^2)$ where squares and cross products are standard vector operations. To aggregate these decision functions one averages these terms over j. It is clear that

(25)    averaging over $(V, V^2)$ has no effect;

(26)    averaging over $(VV^j, V^j)$ leads to $(V\int_{[0,1]} V^j dj, V)$ which are already in (25);

(27)    averaging over $(V^j)^2$ introduces the cross sectional variances of these variables as in (24). If these

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moments are already in the vector $\mathbf{V}$ then aggregation does not add any new state variables. As briefly explained above, a set of state variables is closed under aggregation if aggregation does not add new state variables. Consistent aggregation then requires an economy to have a set of state variables which is closed under aggregation but a selection of such set is a modeling choice that reflects problems the model intends to study.

Our selection of state variables begins by noting that utility functions and beliefs are symmetric in our model, hence, the effects of asset holdings and beliefs are also symmetric in the sense that only the *distribution* of assets and beliefs impacts the equilibrium, not the identity of an agent who holds a given combination of assets and beliefs. Next, we identify the two individual specific state variables $\mathbf{V}^j = (\hat{b}_{t-1}^{\,j}, g_t^{\,j})$ the theory defined as causal in the sense they *cause and explain* the behavior of endogenous variables. They are essential to the theory and to attain consistent aggregation we must include all first and second moments of the joint distribution of $(\hat{b}_{t-1}^{\,j}, g_t^{\,j})$. This is done by identifying the implied second moments that must be included under aggregation in the set of economy-wide state variables:

$$(28) \qquad \int_{[0,1]} (g_t^{\,j})^2 dj = \sigma_{cs}^2(g)_t + Z_t^2$$

$$(29) \qquad \int_{[0,1]} (\hat{b}_{t-1}^{\,j})^2 dj = \sigma_{cs}^2(b)_t$$

$$(30) \qquad \int_{[0,1]} (\hat{b}_{t-1}^{\,j} g_t^{\,j}) dj = Cov(b,g)_t$$

Hence the economy wide state variables we need for the joint distribution of $(\hat{b}_{t-1}^{\,j}, g_t^{\,j})$ is then $(Z_t, (Z_t)^2, \sigma_{cs}^2(g)_t, \sigma_{cs}^2(b)_t, Cov(b,g)_t)$. Since we study the effects of asset holdings and beliefs, our economy wide state variables will then be

$$(31) \qquad \mathbf{V} = (Z_t, \hat{\zeta}_t, u_t, \sigma_{cs}^2(g)_t, \sigma_{cs}^2(b)_t, Cov(b,g)_t).$$

However, even after including $(\sigma_{cs}^2(g)_t, \sigma_{cs}^2(b)_t, Cov(b,g)_t)$, the set of state variables is still not closed under aggregation since proliferation of moments continues for (31). For example, evaluating $\mathbf{V}^2$ imply that $\sigma_{cs}^2(g)_t$ generates a new variable $(\sigma_{cs}^2(g)_t)^2$ as in (24). To define a set of state variables closed under aggregation when one uses a second order approximation we follow Preston and Roca (2007) and Den Haan and Rendahl (2010), and assume the following:

**Assumption 4:** In addition to state variables $(\mathbf{V}, \mathbf{V}^j)$, moments of state variables included are

$$(32a) \qquad \mathbf{V}^2 = ((Z_t)^2, (\hat{\zeta}_t)^2, (u_t)^2, Z_t \hat{\zeta}_t, Z_t u_t, \hat{\zeta}_t u_t)$$

$$(32b) \qquad (\mathbf{V}^j)^2 = ((\hat{b}_{t-1}^{\,j})^2, (g_t^{\,j})^2, \hat{b}_{t-1}^{\,j} g_t^{\,j})$$

(32c)
$$VV^j = (\hat{b}^j_{t-1}Z_t, \hat{b}^j_{t-1}\hat{\zeta}_t, \hat{b}^j_{t-1}u_t, g^j_t Z_t, g^{j}_t\hat{\zeta}_t, g^j_t u_t).$$

The set is closed under aggregation. All higher order moments of quadratic approximation are ignored.

Assumption 4 ensures decision functions and economic aggregates are approximated by strictly second order polynomials. For instance, aggregate income is approximated by the following polynomial

$$\hat{y}_t = P^y(\hat{\zeta}_t, u_t, Z_t, \sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, Cov(b,g)_t, \hat{\zeta}^2_t, u^2_t, Z^2_t, \hat{\zeta}_t u_t, \hat{\zeta}_t Z_t, u_t Z_t; \Theta^y)$$

where $\Theta^y$ is the polynomial's parameter vector. In $P^y$ terms like $(\hat{\zeta}_t \sigma^2_{cs}(g)_t, (\sigma^2_{cs}(b)_t)^2)$ are ignored.

Assumption 4 also determines ways of computing high moments and cross sectional variances. For example, to compute cross sectional variance of individual consumption, recall that it satisfies

(33)
$$\sigma^2_{cs}(c)_t = \int_{[0,1]} (\hat{c}^j_t)^2 dj - (\hat{y}_t)^2$$

hence the problem is how to compute the integral of squared consumption. Squaring the full second order polynomial that describes individual consumption means including in (33) terms of order higher than 2 which, by Assumption 4, are ignored. This means terms squared in (33) are only the linear terms of the consumption function. By the principle of decomposition it is equivalent to taking the solution of the linear model, squaring it and integrating as in (33). This shows again the use of decomposition.

Given a set of state variables a consistent aggregation also requires that all decision functions are functions of state variables whose laws of motion are specified. Hence, we now need to show how to deduce or construct transition functions for all state variables. This is our next task.

*3.2    Formulating Transition Functions for* $(\sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, Cov(b,g)_t)$

Transition functions of state variables are deduced either analytically from the stochastic properties assumed or via an approximation which, by Assumption 4, is carried out by using the linear solution and deducing from it the needed transition. Our discussion exhibits examples of both cases.

We start with the transition function of $\sigma^2_{cs}(g)_t$, derived from the stochastic structure of the random term $\rho^{jg}_{t+1}$ that was unspecified in the transition function of $g^j_{t+1}$ in (14d). In the computational model we specify the correlation across agents and make the following simplified assumption:

**Assumption 5:**  $\rho^{jg}_{t+1} = \Upsilon_{t+1}(1+\varepsilon^{jg}_{t+1})$   $\Rightarrow$   $\rho^Z_{t+1} = \Upsilon_{t+1}$ where $\Upsilon_{t+1}$ is a sequence of i.i.d. random variables

with mean 0 and variance $\sigma_Z^2$. In addition, $\varepsilon_{t+1}^{jg}$ are i.i.d. with mean 0 and variance $\sigma_\varepsilon^2$ and are uncorrelated across agents, independent of $\Upsilon_{t+1}$. Both are uncorrelated with $(g_t^{jg}, Z_t)$.

To use Assumption 5 apply (24) and compute

$$\int_{[0,1]} (g_t^{\,j})^2 dj = \sigma_{cs}^2(g)_t + Z_t^2 \ .$$

To deduce $\sigma_{cs}^2(g)_t$ consider the relation between $\rho_{t+1}^{jg}$ and the mean $\rho_{t+1}^Z = \int_{[0,1]} \rho_{t+1}^{jg} dj$. Use (12a)-(12b) to conclude

$$(g_{t+1}^{\,j} - Z_{t+1}) = \lambda_Z (g_t^{\,j} - Z_t) + (\rho_{t+1}^{jg} - \rho_{t+1}^Z) \ .$$

By Assumption 5

(34)
$$(g_{t+1}^{\,j} - Z_{t+1}) = \lambda_Z (g_t^{\,j} - Z_t) + \Upsilon_{t+1} \varepsilon_{t+1}^{jg}$$

take squares and integrate. With notation $\sigma_{cs}^2(g)_t$, we find this variance is a Markov process with transition

(35)
$$\sigma_{cs}^2(g)_{t+1} = \lambda_Z^2 \sigma_{cs}^2(g)_t + \Upsilon_{t+1}^2 \sigma_\varepsilon^2$$

and this is the desired transition function. Note also that here $\sigma_{cs}^2(g)_t$ fluctuates with a known transition but *it is not normally distributed*![3]

We now turn to the transition function of $\sigma_{cs}^2(b)_t$ which is approximated from the linear solution as follows. Square (15c) and integrate over agents to deduce

$$\int_{[0,1]} (\hat{b}_t^{\,j})^2 dj = \int_{[0,1]} [(A_b^b)^2 (\hat{b}_{t-1}^{\,j})^2 + 2A_b^b A_b^g (\hat{b}_{t-1}^{\,j} g_t^{\,j}) + 2A_b^b A_b^u (\hat{b}_{t-1}^{\,j} u_t) + 2A_b^b A_b^Z (\hat{b}_{t-1}^{\,j} Z_t)$$
$$+ 2A_b^b A_b^\zeta (\hat{b}_{t-1}^{\,j} \hat{\zeta}_t) + (A_b^g)^2 (g_t^{\,j})^2 + 2A_b^g A_b^u (g_t^{\,j} u_t) + 2A_b^g A_b^Z (g_t^{\,j} Z_t)$$
$$+ 2A_b^g A_b^\zeta (g_t^{\,j} \hat{\zeta}_t) + (A_b^u)^2 (u_t)^2 + 2A_b^u A_b^Z (u_t Z_t) + 2A_b^u A_b^\zeta (u_t \hat{\zeta}_t)$$
$$+ (A_b^Z)^2 (Z_t)^2 + 2A_b^Z A_b^\zeta (Z_t \hat{\zeta}_t) + (A_b^\zeta)^2 (\hat{\zeta}_t)^2] dj \ .$$

Now apply (3c) and (28)-(30) we conclude

(36)
$$\sigma_{cs}^2(b)_t = (A_b^b)^2 \sigma_{cs}^2(b)_{t-1} + 2A_b^b A_b^g Cov(b,g)_t + (A_b^g)^2 \sigma_{cs}^2(g)_t + [(A_b^g)^2 + 2A_b^g A_b^Z + (A_b^Z)^2](Z_t)^2$$
$$+ 2A_b^u(A_b^g + A_b^Z)(Z_t u_t) + 2A_b^\zeta(A_b^g + A_b^Z)(Z_t \hat{\zeta}_t) + (A_b^u)^2 (u_t)^2 + 2A_b^u A_b^\zeta (u_t \hat{\zeta}_t) + (A_b^\zeta)^2 (\hat{\zeta}_t)^2$$

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[3] To compute $\sigma_\varepsilon^2$ in (35) keep in mind that by Assumption 5 the variance of $\rho_{t+1}^{jg}$ is $\sigma_g^2 = Var(\Upsilon_{t+1}(1+\varepsilon_{t+1})) = \sigma_Z^2(1+\sigma_\varepsilon^2)$ while the covariance between individual and mean market belief is $Cov(\rho_{t+1}^{jg}, \rho_{t+1}^Z) = E((1+\varepsilon_{t+1}^{jg})\Upsilon_{t+1}^2) = \sigma_Z^2$. Given $\rho = corr(\rho_{t+1}^{jg}, \rho_{t+1}^Z)$, it follows that $\rho = Cov(\rho_{t+1}^{jg}, \rho_{t+1}^Z)/(\sigma_g \sigma_Z) = 1/\sqrt{1+\sigma_\varepsilon^2} \implies \sigma_\varepsilon^2 = 1/\rho^2 - 1$.

The third transition of $\mathbf{Cov(b,g)}_t$ is also approximated with the linear solution for bond holdings specified in (15c). Now multiply (15c) by (34) and integrate over $j$ to have

$$\int_{[0,1]} \hat{b}_t^{\,j}(g_{t+1}^{\,j}-Z_{t+1})dj = \int_{[0,1]}[A_b^b \hat{b}_{t-1}^{\,j}\lambda_Z(g_t^{\,j}-Z_t)+A_b^b \hat{b}_{t-1}^{\,j}\Upsilon_{t+1}\varepsilon_{t+1}^{jg}]dj + \int_{[0,1]}[A_b^Z Z_t\lambda_Z(g_t^{\,j}-Z_t)+A_b^Z Z_t\Upsilon_{t+1}\varepsilon_{t+1}^{jg}]dj$$

$$+\int_{[0,1]}[A_b^\zeta \hat{\zeta}_t\lambda_Z(g_t^{\,j}-Z_t)+A_b^\zeta \hat{\zeta}_t\Upsilon_{t+1}\varepsilon_{t+1}^{jg}]dj + \int_{[0,1]}[A_b^u u_t\lambda_Z(g_t^{\,j}-Z_t)+A_b^u u_t\Upsilon_{t+1}\varepsilon_{t+1}^{jg}]dj$$

$$+\int_{[0,1]}[A_b^g g_t^{\,j}\lambda_Z(g_t^{\,j}-Z_t)+A_b^g g_t^{\,j}\Upsilon_{t+1}\varepsilon_{t+1}^{jg}]dj.$$

Therefore, by (28)

(37) $$\mathbf{Cov(b,g)}_{t+1} = \lambda_Z A_b^b \mathbf{Cov(b,g)}_t + \lambda_Z A_b^g \sigma_{cs}^2(g)_t$$

which is the desired transition function.

The transition functions of the cross-sectional moments (35), (36), and (37) provide valuable tools for a study of the effect of the distributions of bond holdings and beliefs on market performance. They also provide an added tool for the further study of the role of private expectations on economic aggregates and policy efficacy.

*3.3    Final set of equations that define an equilibrium*

The final system of equations that defines an equilibrium before any approximation, is now stated. The form which we use is somewhat different from the system described for the linear model.

(38a)   optimal bond holding    $\tilde{\tau}_b b_t^{\,j} + (C_t^{\,j})^{-\sigma}\dfrac{1}{1+r_t} = E_t^{\,j}[\beta(C_{t+1}^{\,j})^{-\sigma}\dfrac{1}{\pi_{t+1}}],$

(38b)   optimal labor supply    $(C_t^{\,j})^{-\sigma}W_t = (L_t^{\,j})^{\eta},$

(38c)   budget constraint    $C_t^{\,j} + \dfrac{b_t^{\,j}}{1+r_t} = W_t L_t^{\,j} + \dfrac{b_{t-1}^{\,j}}{\pi_t} + [Y_t - W_t L_t],$

(38d)   optimal pricing    $q_{jt} = \dfrac{\theta}{\theta-1}\dfrac{S_{j,t}}{V_{j,t}},$

(38e)         $S_{j,t} = (C_t^{\,j})^{-\sigma}Y_t\varphi_t + \beta\omega E_t^{\,j}[S_{j,t+1}\pi_{t+1}^{\theta}],$

(38f)         $V_{j,t} = (C_t^{\,j})^{-\sigma}Y_t + \beta\omega E_t^{\,j}[V_{j,t+1}\pi_{t+1}^{\theta-1}],$

(38g)   inflation identity    $1 = (1-\omega)\int_{[0,1]}q_{jt}^{(1-\theta)}dj + \omega(\pi_t)^{-(1-\theta)},$

(38h)   a monetary rule    $\hat{r}_t = \xi_\pi(\pi_t-\bar{\pi}) + \xi_y(\hat{y}_t-\hat{y}_t^F) + u_t,$

(38i)   market clearing    $\int_{[0,1]} C^{\,j}dj = C_t = Y_t \;,\; \int_{[0,1]} N_t^{\,j}dj = N_t = L_t = \int_{[0,1]} L_t^{\,j}dj \;,\; \int_{[0,1]} b_t^{\,j}dj = 0.$

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To clarify developments below note the difficulties of solving for an equilibrium with heterogenous agents. Standard optimization techniques can be used to solve agent j's problem *if the agent knows the equilibrium map of the aggregates and of the prices*. Distinct from an optimization, an equilibrium requires two additional conditions: aggregation of individual decision functions that define equilibrium aggregates and second, an imposition of market clearing conditions that define prices. Here it amounts to the wage rate and the inflation rate. These are the two central problems we address next.

### 3.4    *Procedure for solving equilibrium under second order approximation*

Our general equilibrium solution consists of two parts: the first is a solution of all individual decision functions employing a second order approximation of the optimum conditions given hypothetical parameter values of equilibrium prices and aggregate variables. One can consider this a "conjectured" equilibrium. Second, we aggregate the solved individual decision functions, using the market clearing conditions as restrictions on the aggregation. We then deduce from the aggregates implied new parameter values of the maps of prices and aggregate variables. With these new parameters we repeat the first part and seek convergence of the sequence of conjectured equilibria to equilibrium for which no further iterations improve the estimates. The algorithm to compute an equilibrium can be described as follows:

*Step (i)*    Solve the linear model in (15a)-(15f) and store the coefficients;

*Step (ii)*    deduce from step (i) coefficients of the cross-sectional variances (35), (36), and (37);

*Step (iii)*    set initial values for coefficients of the second order terms of aggregate variables and other cross-sectional moments all needed for our postulated law of motion of the iterative procedure;

*Step (iv)*    solve (38a)-(38h) for agent j decision functions using perturbation method;

*Step (v)*    aggregate over all j decision functions to update the coefficients of the law of motion of aggregate variables and other cross-sectional moments of step (iii);

*Step (vi)*    iterate until convergence.

In order to solve for agent decision functions in step (iv) we need to specify the law of motion required by step (iii) and such information is provided in detail in Appendix A.

### 4.    *Monetary policy with diverse beliefs.*

After giving a summary of the parameter choice, we illustrate the simulation results of the model.

In particular, we focus on the aggregate effects of changes to the mean market belief and to its cross-sectional standard deviation. We do not aim at a precise calibration of the model, but we provide examples that can highlight some of the qualitative equilibrium features about the interaction between diverse private expectations and monetary policy. Of course, our results go beyond those of a single representative agent economy and add more complexity to the interaction between the private sector and monetary policy.

## 4.1    The choice of parameters

The quarterly model parameters are set according to standard values in the literature (e.g. Galí (2008), Walsh (2010)): $\beta=0.99, \sigma=0.9, \eta=1.0, \tilde{\tau}_b=10^{-3}, \omega=2/3, \theta=6, \lambda_\zeta=0.90$. Unlike the standard Real Business Cycle (in short, RBC) assumption about the technology shock being $\sigma_\zeta=0.0072$ (as measured by the Solow residual) we set it to $\sigma_\zeta=0.0045$ on the ground that such measure contains other endogenous factors including diverse private beliefs. Parameters of the monetary policy rule are also in accord with standard range used in the literature $(\xi_y \geq 0, \xi_\pi > 1)$[4]. As noted KPW (2013) and Wu (2014) provide parameter estimates of the monetary policy shock transition function (13b) based on quarterly data on Federal Reserve policy choices. We then set $\lambda_u=0.8$ and $\sigma_u=0.002$ (see also Rudebusch (2002)). Finally, the belief parameters, most of which have been discussed and motivated earlier, are set as follows: $\lambda_Z=0.8, \lambda_Z^\zeta=0.35, \lambda_Z^u=0, \lambda_Z^g=0.05, \lambda_\zeta^g=1, \lambda_u^g=\dfrac{\sigma_u}{\sigma_\zeta}, \sigma_g=0.0025, \rho=0.2$. In what follows we shall refer to the above parameterization as the basic parameter choice and label it as Basic Model. Any change to it will be specifically noted. All results reported are statistics of model simulations over 10,000 periods.

## 4.2    Aggregate volatility and non-monotonicity of Monetary Policy

We proved in Section 2.5.2 the policy space contains a curve of singularity. This set of policy parameters divides the space into two sub-regions and our key finding is that policy trade-off is not a smooth curve in this parameter space. It takes diverse forms *within each sub-region* and a trade-off exists *between the two sub-regions* as explained below and illustrated in Tables 2a-2c. One of the main effects of this singularity is that the volatility outcome of a more aggressive policy action is strongly non

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[4] As customary, all values of the policy parameter $\xi_y$ are reported on an annual basis. Therefore in all simulations of the quarterly model the effective parameter value is given by $\xi_y/4$.

monotonic. The non-monotonicity and the singularity results are direct consequences of the diversity of private expectations.

To better illustrate the issues consider the standard RE single representative agent version of our model under which (9a)-(9b) are believed to be the truth. All preferences and exogenous processes parameters are as specified in Basic Model parameterization. Table 1 illustrates a trade-off faced by the central bank between output and inflation stabilization. If the monetary authority acts more aggressively to stabilize inflation, output becomes more volatile but inflation volatility falls. In our parameter configurations an aggressive output stabilization reduces the volatility of output but fails to control inflation. The response of these aggregate measures is mostly monotonic, making the effects of central bank's policy relatively straightforward to predict.

**Table 1:** Output and inflation volatility trade-off and monotonicity under Rational Expectations

| | | $\sigma_{\hat{y}}$ | | | | | | | $\sigma_{\pi}$ | | | | |
| | | | $\xi_y$ | | | | | | | $\xi_y$ | | | |
| | | 0.25 | 0.75 | 1 | 1.5 | 2 | 5 | 0.25 | 0.75 | 1 | 1.5 | 2 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **1.2** | 1.050 | 1.041 | 1.038 | 1.034 | 1.032 | 1.033 | 2.666 | 2.334 | 2.197 | 1.967 | 1.780 | 1.135 |
| | **1.3** | 1.048 | 1.042 | 1.040 | 1.037 | 1.036 | 1.036 | 2.224 | 1.986 | 1.886 | 1.713 | 1.569 | 1.044 |
| $\xi_\pi$ | **1.5** | 1.046 | 1.043 | 1.042 | 1.041 | 1.040 | 1.040 | 1.677 | 1.537 | 1.475 | 1.366 | 1.272 | 0.902 |
| | **1.6** | 1.046 | 1.044 | 1.043 | 1.042 | 1.042 | 1.042 | 1.494 | 1.382 | 1.332 | 1.242 | 1.164 | 0.846 |
| | **2** | 1.047 | 1.047 | 1.046 | 1.046 | 1.046 | 1.046 | 1.042 | 0.986 | 0.960 | 0.912 | 0.869 | 0.677 |
| | **5** | 1.055 | 1.055 | 1.055 | 1.055 | 1.055 | 1.055 | 0.320 | 0.314 | 0.312 | 0.306 | 0.301 | 0.274 |

When considering the diverse beliefs case the interaction of private expectations with monetary policy adds much more complexity and results in a departure from conclusions about policy trade-off under RE. Such an interaction of policy with beliefs is a crucial component of our theory and results in relevant policy implications which we review in more detailed discussions in the following Sub-Sections.

Why do we have a more complex scenario under diverse beliefs than under the standard RE case of Table 1? Changes in expectations alter agents' demand for consumption, their motives to work and make investment decisions in financial markets. When agents hold diverse expectations, aggregate dynamics are altered by the distribution of beliefs and financial markets exhibit significant trading volumes. Also changes in expectations alter individual intertemporal allocation of consumption and labor between date t and future dates. Thus altering the aggregate supply of labor, aggregate output and the volume of borrowing in the bond market. We note that although central banks are concerned with excessive borrowing and asset prices, these play no role in the dynamics of the representative RE agent in

Table 1 since such an agent does not borrow.

We illustrate in Tables 2a-2c the more complex volatility outcome of the diverse beliefs economy. Tables 2a-2b exhibit the standard deviations of output ($\sigma_{\hat{y}}$) and inflation ($\sigma_\pi$) under Basic Model specification. Tables 2a-2b reveal a more complex structure of non-monotonicity than the one observed in Table 1. In each Table the gray cells approximate the curve of policy parameters $(\xi_y, \xi_\pi)$ along which the matrix is singular. Approaching this "ridge" results in rising volatility which in some cases becomes unbounded. Thus, the outcome of any monetary policy action is no longer monotonic and straightforward to predict! The tables are divided into two sub-regions[5] that reflect the two dominant interaction effects of expectations with policy explained in detail in section 2.5.2: income effects and intertemporal substitution effect. In policy sub-region 1 (at the bottom left) the income effect dominates while in policy sub-region 2 (at the upper right) the intertemporal substitution effect dominates. Within sub-region 1 any aggressive output stabilization policy utilizing larger values of $\xi_y$ is self defeating since it causes an increased output volatility rather than a decrease. However, in that sub-region there is no trade-off between inflation and output volatility: a more aggressive inflation stabilizing policy reduces both volatility of output and of inflation. In spite of this, such a policy is subject to limitations. Its effect on output volatility is bounded below: as $\xi_\pi \to \infty$, output volatility $\sigma_{\hat{y}} \to 1.06$ which is bounded away from zero. The second but very important limitation is the volatilities of the bond market and individual consumption. This arises only in heterogeneous agents models and is absent in any single representative agent NKM.

Table 2c reports the standard deviation of individual consumption ($\sigma_{\hat{c}^j}$) which exhibits volatility patterns that differ from those of aggregate income. This results from the fact that an individual's consumption responds to that individual's belief as well as to economy wide measures of belief diversity but aggregation ensures that aggregate income responds to economy wide belief variables with the same parameters as an individual's consumption. In such an economy trading volume in financial markets is large and we use a penalty as a proxy for credit regulations to ensure equilibrium exists (see Kurz and Motolese (2001) and Kurz et al. (2005b)). Monetary policy decisions or fluctuations of interest rates impact their bond holdings and give rise to the higher volatility of individual consumption with respect to aggregate income. Table 2c shows that although within sub-region 1 any anti inflation policy reduces both the volatility of output and inflation, it faces the limitation of an increasing volatility of individual

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[5] See Appendix B for an analysis of the approximation error in the Euler Equations along the simulation path in both sub-regions.

consumption (marked by the cells in italic bold font in Table 2c). Since a more volatile individual consumption entails more volatile financial markets, such a limitation restricts the adoption of aggressive anti-inflation policies. Therefore, such policies are not suitable and only moderate policies are desirable.

**Table 2a:** Output volatility under diverse beliefs (Basic Model)

| $\sigma_{\hat{y}}$ | | | | | | | $\xi_y$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
| | 1.2 | 3.214 | 7.792 | 20.126 | 3e+05 | 2.758 | 0.963 | 0.916 | 0.961 | 1.001 | NaN | 1.058 | 1.051 |
| | 1.3 | 1.876 | 2.051 | 2.699 | 11.671 | 1e+03 | 1e+02 | 1.184 | 0.833 | 0.889 | 0.940 | 0.976 | 1.000 |
| | 1.4 | 1.621 | 1.655 | 1.725 | 1.932 | 16.298 | 6.595 | NaN | 4.260 | 0.863 | 0.822 | 0.887 | 0.934 |
| $\xi_\pi$ | 1.5 | 1.503 | 1.513 | 1.531 | 1.565 | 1.974 | 4.371 | 3.392 | 35.631 | 5e+04 | 1.625 | 0.795 | 0.833 |
| | 1.6 | 1.430 | 1.433 | 1.440 | 1.450 | 1.507 | 2.316 | 2.607 | 2.410 | 4.246 | 4e+11 | 10.899 | 1.020 |
| | 1.7 | 1.380 | 1.380 | 1.383 | 1.387 | 1.405 | 1.471 | 4.410 | 1.990 | 1.967 | 2.573 | 9.989 | 3e+17 |
| | 2 | 1.289 | 1.288 | 1.287 | 1.287 | 1.289 | 1.293 | 1.302 | 1.324 | 1.600 | 1.638 | 1.464 | 1.497 |
| | 5 | 1.121 | 1.121 | 1.120 | 1.120 | 1.120 | 1.120 | 1.119 | 1.119 | 1.119 | 1.119 | 1.118 | 1.118 |

**Table 2b:** Inflation volatility under diverse beliefs (Basic Model)

| $\sigma_\pi$ | | | | | | | $\xi_y$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
| | 1.2 | 14.562 | 32.185 | 50.663 | 1e+02 | 18.706 | 7.846 | 5.360 | 4.373 | 4.334 | NaN | 4.679 | 3.075 |
| | 1.3 | 3.405 | 4.032 | 4.771 | 46.649 | 1e+02 | 74.805 | 12.642 | 6.435 | 4.576 | 3.694 | 3.180 | 2.872 |
| | 1.4 | 2.089 | 2.202 | 2.408 | 2.925 | 47.875 | 36.792 | 9e+02 | 29.946 | 9.136 | 5.379 | 3.994 | 3.273 |
| $\xi_\pi$ | 1.5 | 1.662 | 1.663 | 1.711 | 1.827 | 3.696 | 17.120 | 17.842 | 71.513 | 1e+02 | 16.542 | 7.061 | 4.600 |
| | 1.6 | 1.438 | 1.408 | 1.402 | 1.427 | 1.681 | 6.202 | 9.727 | 10.675 | 24.844 | 3e+02 | 45.348 | 10.894 |
| | 1.7 | 1.288 | 1.249 | 1.225 | 1.218 | 1.278 | 1.699 | 17.216 | 6.447 | 7.185 | 12.909 | 45.829 | 86.933 |
| | 2 | 1.003 | 0.970 | 0.942 | 0.918 | 0.887 | 0.883 | 0.924 | 1.120 | 3.445 | 4.032 | 2.826 | 3.183 |
| | 5 | 0.324 | 0.320 | 0.316 | 0.313 | 0.306 | 0.299 | 0.292 | 0.286 | 0.280 | 0.275 | 0.269 | 0.265 |

**Table 2c:** Individual Consumption volatility under diverse beliefs (Basic Model)

| $\sigma_{\hat{c}j}$ | | | | | | | $\xi_y$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
| | 1.2 | 41.536 | 3e+02 | 2e+03 | 1e+15 | 79.443 | 9.844 | 3.828 | 2.567 | 2.809 | NaN | 4.596 | 2.697 |
| | 1.3 | 5.065 | 7.573 | 12.897 | 26.599 | 2e+06 | 7e+04 | 27.122 | 6.730 | 3.409 | 2.525 | 2.272 | 2.236 |
| | 1.4 | 2.442 | 2.829 | 3.473 | 4.552 | 42.134 | 1e+02 | NaN | 2e+02 | 13.769 | 4.971 | 3.008 | 2.416 |
| $\xi_\pi$ | 1.5 | 1.921 | 1.995 | 2.121 | 2.327 | 3.218 | 10.805 | 26.473 | 4e+03 | 4e+10 | 43.623 | 8.479 | 3.935 |
| | 1.6 | 1.793 | 1.803 | 1.824 | 1.864 | 2.043 | 2.808 | 5.925 | 11.259 | 66.585 | 1e+25 | 9e+02 | 18.822 |
| | 1.7 | 1.758 | 1.753 | 1.751 | 1.753 | 1.784 | 1.885 | 4.448 | 3.970 | 6.419 | 18.359 | 5e+02 | 5e+33 |
| | 2 | *1.746* | *1.738* | *1.731* | *1.724* | *1.710* | *1.698* | *1.688* | *1.688* | 1.879 | 2.192 | 2.119 | 2.545 |
| | 5 | 1.750 | 1.749 | 1.749 | 1.748 | 1.747 | 1.746 | 1.745 | 1.744 | *1.743* | *1.742* | *1.741* | *1.740* |

Turning to sub-region 2 (at the top right) in Tables 2a-2c, observe there are policy configurations that can achieve lower volatility of output compared to sub-region 1, but then at the cost of higher volatility of inflation and of individual consumption. Note that there is a trade-off *between the sub-regions themselves*. For instance, select the monetary policy configuration ($\xi_y$=0.5, $\xi_\pi$=1.4) in sub-region 1. In order to achieve a lower output volatility while keeping a non aggressive inflation policy we have to move

to sub-region 2 in Table 2a and choose for example a policy configuration where $3.5 \leq \xi_y \leq 4.5$, and $\xi_\pi = 1.4$ with the monetary authority being extremely aggressive in targeting the output gap: this is possible only at the cost of higher volatility of inflation and of individual consumption.

Tables 2a-2c show that in an economy with diverse beliefs monetary policy must find a balance between stabilization of inflation, output and individual consumption. While this choice is absent from a single agent economy, it is understandable under diverse beliefs since now the central bank acts in a more complex world, where its policies are amplified or muffled by expectations of agents (see also Motolese (2003)). There exists a deep interaction between market expectations and efficacy of policy. Market expectations may be supportive of the policy but a conflict may exist between policy and private expectations and this may result in unsatisfactory aggregate volatility outcomes. The next two sub-sections examine the interaction of market beliefs with policy actions. We start by studying the effect of changes in the mean market belief.

### 4.3    The effect of the mean market belief

How does excessive market exuberance in the form of high optimism about future productivity impact aggregate output and inflation?  How does it interact with monetary policy?  To answer these we report in Tables 3a-3b the equilibrium elasticities of aggregate output and inflation with respect to mean market belief variable $Z_t$ under Basic Model specification. The impact of optimism (a positive shift to $Z_t$) is not uniform across sub-regions. As explained earlier shifts to $Z_t$ change the intertemporal allocation of consumption and labor between date t and future dates. But, what is the direction of such changes? In sub-region 1 optimism about future productivity boosts aggregate output while in sub-region 2 almost everywhere it lowers it.  In sub-region 1 the income effect from expected higher future income prevails over the intertemporal substitution effect and increases present consumption, pushing aggregate output higher. In sub-region 2 the intertemporal substitution effect prevails. However, the intensity of its impact depends upon the monetary policy configuration: in sub-region 1 aggressive inflation stabilization policies are successful in reducing aggregate output volatility. Thus crushing also the additional volatility induced by shifts in the mean market belief.

Turning to the effect of $Z_t$ on inflation we note that it is uniformly positive in sub-region 2 with a magnitude which can explain the higher volatility of inflation in the same sub-region in Table 2b. In sub-region 1, however, its effect is mostly negative, it is monotonically declining with $\xi_y$ and rising with $\xi_\pi$ to

a point that for high enough intensity of $\xi_\pi$ the sign of the effect changes again and becomes positive. In sum, the sign of the response of inflation to market optimism varies with policy.

**Table 3a:** Equilibrium coefficients of aggregate output with respect to mean market belief $Z_t$ (Basic Model)

|  |  | $\xi_y$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  |  | **0.25** | **0.5** | **0.75** | **1** | **1.5** | **2** | **2.5** | **3** | **3.5** | **4** | **4.5** | **5** |
|  | **1.2** | 5.49 | 7.64 | 16.13 | -34.93 | -3.36 | -1.35 | -0.66 | -0.34 | -0.16 | -0.06 | 0.01 | 0.06 |
|  | **1.3** | 3.12 | 3.40 | 3.88 | 4.74 | 12.62 | -8.49 | -2.47 | -1.22 | -0.70 | -0.42 | -0.26 | -0.15 |
|  | **1.4** | 2.30 | 2.39 | 2.52 | 2.71 | 3.46 | 5.83 | 119.93 | -4.68 | -1.95 | -1.09 | -0.69 | -0.45 |
| $\xi_\pi$ | **1.5** | 1.86 | 1.89 | 1.94 | 2.01 | 2.25 | 2.74 | 3.89 | 8.80 | -13.63 | -3.19 | -1.61 | -0.99 |
|  | **1.6** | 1.57 | 1.58 | 1.61 | 1.64 | 1.74 | 1.93 | 2.27 | 2.95 | 4.80 | 22.93 | -6.19 | -2.41 |
|  | **1.7** | 1.37 | 1.37 | 1.38 | 1.39 | 1.45 | 1.54 | 1.69 | 1.93 | 2.38 | 3.36 | 6.77 | -29.05 |
|  | **2** | 0.99 | 0.99 | 0.99 | 0.99 | 0.99 | 1.01 | 1.04 | 1.08 | 1.14 | 1.22 | 1.35 | 1.53 |

**Table 3b:** Equilibrium coefficients of inflation with respect to mean market belief $Z_t$ (Basic Model)

|  |  | $\xi_y$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  |  | **0.25** | **0.5** | **0.75** | **1** | **1.5** | **2** | **2.5** | **3** | **3.5** | **4** | **4.5** | **5** |
|  | **1.2** | -4.73 | -9.41 | -26.69 | 74.72 | 11.30 | 6.85 | 5.13 | 4.19 | 3.58 | 3.14 | 2.81 | 2.55 |
|  | **1.3** | -1.08 | -1.93 | -3.13 | -5.07 | -21.16 | 20.57 | 8.29 | 5.56 | 4.31 | 3.59 | 3.10 | 2.75 |
|  | **1.4** | -0.20 | -0.58 | -1.03 | -1.59 | -3.39 | -8.36 | -235.92 | 12.15 | 6.55 | 4.69 | 3.74 | 3.16 |
| $\xi_\pi$ | **1.5** | 0.13 | -0.09 | -0.34 | -0.62 | -1.35 | -2.52 | -4.99 | -14.95 | 29.61 | 8.67 | 5.41 | 4.07 |
|  | **1.6** | 0.27 | 0.12 | -0.04 | -0.21 | -0.62 | -1.17 | -1.99 | -3.49 | -7.29 | -43.53 | 14.37 | 6.76 |
|  | **1.7** | 0.34 | 0.23 | 0.11 | -0.01 | -0.28 | -0.60 | -1.03 | -1.64 | -2.65 | -4.69 | -11.56 | 59.72 |
|  | **2** | 0.38 | 0.32 | 0.27 | 0.21 | 0.09 | -0.04 | -0.18 | -0.34 | -0.53 | -0.76 | -1.07 | -1.49 |

We report in Figures 1 and 2 impulse responses of output gap, inflation, wage rate and nominal interest rate to a 0.1% shock to mean market belief $Z_t$ and to productivity $\hat{\zeta}_t$. The policy parameters in the figures are representative points in the two sub-regions: $(\xi_y=0.5, \xi_\pi=1.4)$ and $(\xi_y=4.5, \xi_\pi=1.4)$. The difference between Figures 1 and 2 is due to two factors: the dominant income effect in sub-region 1 vs. the substitution effect in sub-region 2 and the interaction of these effects with $\hat{\zeta}_t$, whose impact is augmented or shrunk via the positive learning feedback in the mean market belief $Z_t$. This accounts for the sign reversal of the impulse response of the output gap to a $\hat{\zeta}_t$ shock.

In Figures 1 and 2 we maintain the assumption that optimistic agents incorporate a belief of the central bank's over-reaction to a stronger future state and adjust their expectations of future policy shocks accordingly (i.e. $\lambda_u^g > 0$). What happens if agents disregard future policy over-reaction and $\lambda_u^g = 0$? To answer the above we simulate the economy of Basic Model by setting $\lambda_u^g = 0$. Figures 3 and 4 display responses under the new parameters choice at the points $(\xi_y=0.5, \xi_\pi=1.4)$ and $(\xi_y=4.5, \xi_\pi=1.4)$.

**Figure 1**: Effects of a 0.1% shock to productivity (blue-solid) and mean market belief (red-dashed) at ($\xi_y$=0.5, $\xi_\pi$=1.4) in sub-region 1 (Basic Model).



**Figure 2**: Effects of a 0.1% shock to productivity (blue-solid) and mean market belief (red-dashed) at ($\xi_y$=4.5, $\xi_\pi$=1.4) in sub-region 2 (Basic Model).

While in Figure 1(Figure 2) optimism leads to higher(lower) output, in Figure 3(Figure 4) it leads to lower(higher) output. Comparison of Figures 1 and 3 reveals that when agents take into account future policy over-reaction, their aggregate exuberance is being tempered by their expected change in the interest rate which then alters the weights they place on income and intertemporal substitution effects. In Figure 1 this makes the income effect arising from higher future income prevail, in sub-region 1, over the intertemporal substitution effect and that increases present consumption, lowers inflation expectations and current nominal rate. But when aggregate exuberance does not lead to expected policy over-reactions and higher future interest rate, the intertemporal substitution effect prevails. Agents work less and postpone consumption at future dates depressing current aggregate output, increase inflation and current nominal

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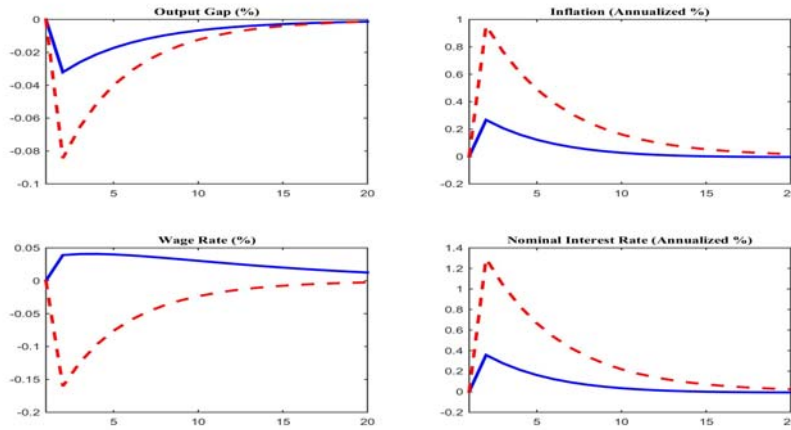
**Figure 3**:  Effects of a 0.1% shock to productivity (blue-solid) and mean  market belief (red-dashed) at ($\xi_y$=0.5, $\xi_\pi$=1.4) in sub-region 1 when $\lambda_u^g = 0$.
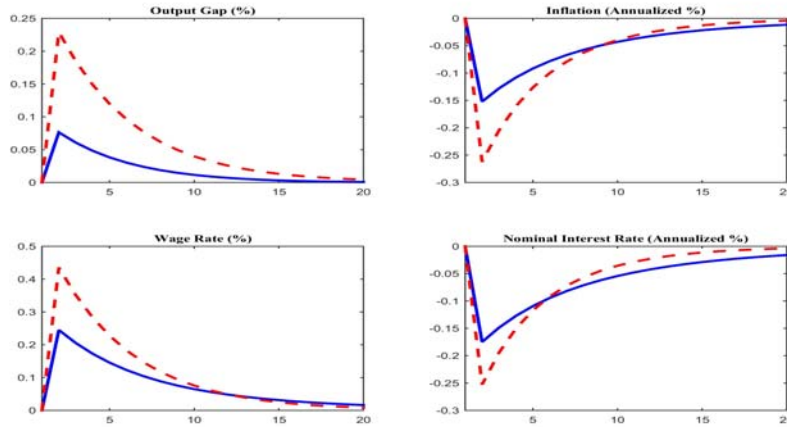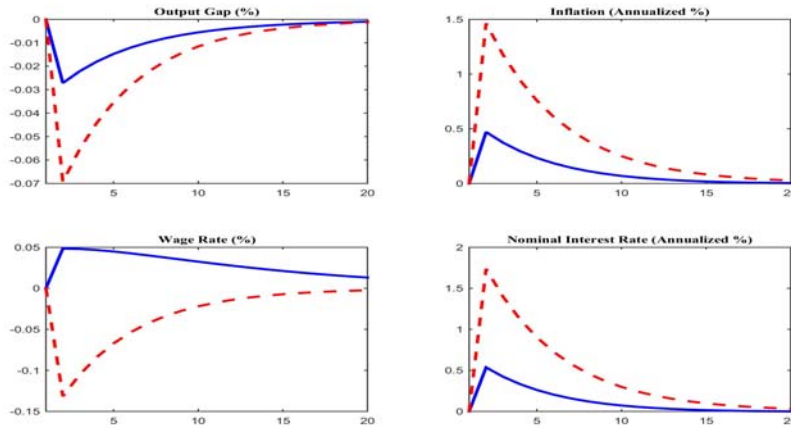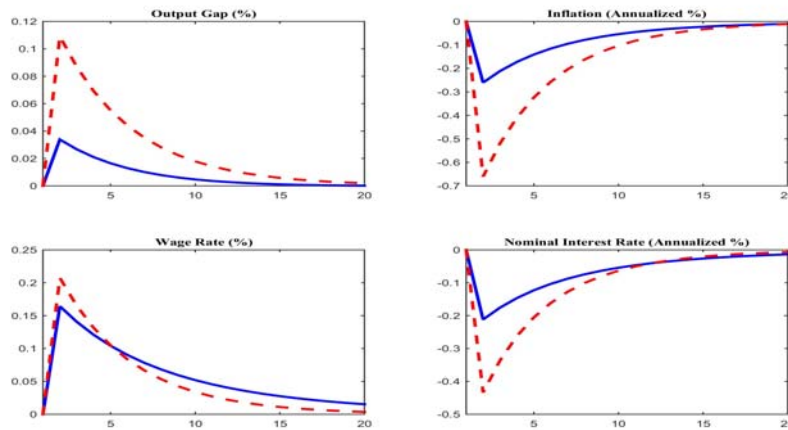


**Figure 4**:  Effects of a 0.01% shock to productivity (blue-solid) and mean market belief (red-dashed) at ($\xi_y$=4.5, $\xi_\pi$=1.4) in sub-region 2 when $\lambda_u^g = 0$.

rate. In Figures 2 and 4 the converse occurs. Note that an analogous reasoning applies to a positive technology shock: the positive learning feedback in the mean market belief $Z_t$ interacts with $\hat{\zeta}_t$ and amplifies or reduces the effect of increased productivity. Also, the effects exhibited in Figures 1-4 are stronger the more sticky prices are.

It is important to observe that the response functions in Figures 1-4 do not demonstrate strong policy trade-off at the policy parameters under study. Indeed, they demonstrate wide circumstances when the economy exhibits a *stagflation response* to a shock when output and inflation respond with opposite signs. In such circumstances policy is muted since the effects of higher inflation and lower output on the nominal interest rates are partly canceled. In Figures 1-4 inflation dominates and the nominal rate changes

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in the same direction as inflation but there are policy configurations where both inflation and output are away from target but policy will not respond to such economic distortion. These suggests a central bank must target directly such force as exuberance or its effects in asset markets.

## 4.4    *The effect of beliefs heterogeneity*

As explained earlier, in our model the cross-sectional distribution of individual beliefs is a state variable. We therefore report in Tables 4a-4b the computed equilibrium elasticities of aggregate output and inflation with respect to the cross-sectional variance of beliefs $\sigma^2_{cs}(g)_t$ under Basic Model specification.  It is natural therefore to ask what the effect of increasing disagreement across agents is. Tables 4a-4b reveal that the impact of beliefs heterogeneity is uniform across the policy space in sub-regions 1 and 2: higher diversity of market belief measured by the cross sectional standard deviation $\sigma^2_{cs}(g)_t$ makes policy coordination harder and results in lower aggregate output and deflation.

**Table 4a:** Equilibrium coefficients of aggregate output with respect to $\sigma^2_{cs}(g)_t$ (Basic Model)

|  | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | $\xi_y$ | | | | | |
| | **1.2** | -2162.5 | -5194.9 | -2.7 | -0.4 | -2383.5 | -626.7 | -270.2 | -144.1 | -85.0 | -26.2 | -48.0 | -33.5 |
| | **1.3** | -547.3 | -862.6 | -1717.0 | -9460.7 | -2004.7 | -4944.2 | -720.4 | -275.7 | -143.1 | -87.2 | -59.0 | -43.5 |
| | **1.4** | -233.5 | -314.9 | -463.7 | -800.0 | 7701.1 | 1215.6 | 0.1 | -877.0 | -270.8 | -133.3 | -79.9 | -53.8 |
| $\xi_\pi$ | **1.5** | -124.9 | -156.1 | -204.5 | -287.1 | -896.0 | 2106.7 | 911.7 | 1595.2 | -302.9 | -211.4 | -108.6 | -66.8 |
| | **1.6** | -75.7 | -90.4 | -111.4 | -142.7 | -284.7 | -1241.9 | 1083.8 | 646.1 | 913.0 | 1e+04 | 270.3 | -37.3 |
| | **1.7** | -49.5 | -57.5 | -68.2 | -83.0 | -137.4 | -301.9 | -2877.8 | 677.0 | 472.1 | 591.9 | 1654.4 | 0.1 |
| | **2** | -17.8 | -19.7 | -22.1 | -25.1 | -33.5 | -48.0 | -76.6 | -151.4 | -612.2 | 543.8 | 259.3 | 219.6 |

**Table 4b:** Equilibrium coefficients of inflation with respect to $\sigma^2_{cs}(g)_t$ (Basic Model)

|  | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | $\xi_y$ | | | | | |
| | **1.2** | -3120.5 | -7467.4 | -3119.4 | -2e+04 | -3335.9 | -750.0 | -195.2 | 65.2 | 371.6 | 3503.5 | -733.4 | -351.5 |
| | **1.3** | -469.4 | -554.3 | -428.8 | 5886.6 | -3e+04 | -2e+04 | -1887.9 | -578.9 | -225.3 | -75.0 | 13.8 | 88.7 |
| | **1.4** | -163.0 | -169.1 | -146.1 | 0.8 | -9514.4 | -7393.0 | -2e+05 | -5604.4 | -1164.4 | -436.4 | -197.3 | -88.6 |
| $\xi_\pi$ | **1.5** | -77.8 | -77.1 | -68.5 | -38.9 | 394.4 | -3041.8 | -3205.7 | -2e+04 | -4e+04 | -2639.6 | -778.5 | -335.1 |
| | **1.6** | -44.0 | -42.8 | -38.6 | -28.2 | 52.5 | 900.3 | -1647.4 | -1788.1 | -4534.6 | -1e+05 | -8860.9 | -1508.1 |
| | **1.7** | -27.7 | -26.7 | -24.3 | -19.6 | 7.5 | 129.0 | 2619.5 | -1047.0 | -1136.1 | -2142.9 | -9034.5 | -1e+04 |
| | **2** | -10.0 | -9.6 | -9.0 | -8.1 | -4.7 | 3.2 | 22.9 | 84.6 | 520.9 | -640.5 | -400.2 | -428.8 |

Heterogeneity of beliefs is an aggregate negative belief externality induced by the lack of coordination across agents. This is also illustrated in Figure 5 which reports the impulse responses of the aggregates to a one-standard-deviation (about 0.00085%) quarterly increase to dispersion of private expectations $\sigma^2_{cs}(g)_t$ at the policy $(\xi_y=0.5, \xi_\pi=1.4)$ which is close to actual policy parameters according to

Taylor (1993a,b).

Why does increased diversity bring about deflation? As just explained, it is important to think of the variable $\sigma_{cs}^2(g)_t$ as a market externality that changes aggregate risk in the market. When disagreement among agents about future business conditions increases, aggregate market risk increases and hence all forecasts become less reliable: agents are now faced with increased uncertainty which is endogenously generated by the distribution of beliefs. Given that higher uncertainty depresses output it is riskier for a firm that is allowed to optimally set prices at date t to be stuck with too high a price at future dates and, ceteris paribus, prices are lowered in anticipation of bad times ahead. But as the monetary authority removes inflation uncertainty by adopting a more aggressive inflation stabilization policy the effect of disagreement among agents dissipates in sub-region 1, as can be seen in Tables 4a-4b. However, variations in policy parameters have a strong non-linear effect that can be seen across the space.
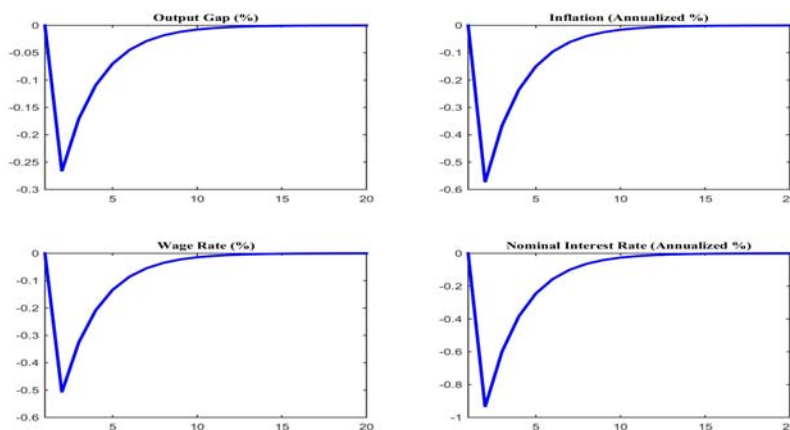


**Figure 5**: Effects of a one-standard-deviation quarterly increase to dispersion of private expectations at ($\xi_y$=0.5, $\xi_\pi$=1.4) (Basic Model).

## 4.5    *The adverse effects of persistent policy shocks*

Basic Model includes estimated policy shock parameter values $(\lambda_u=0.8, \sigma_u=0.0020)$ that reflect the practice of US central banking. To explore effects of such shocks on the aggregates, Figure 6 presents impulse response functions to policy shocks. Since $\sigma_u = 0.002$, the figure shows a one percentage point policy shock changes output by only 0.8% but its effect on inflation is stronger: it changes annualized inflation by 5%! This is also seen in the fact that when general equilibrium effects are accounted for, such a shock changes the nominal rate by 3.5 percentage points. A one percentage point change in $u_t$ is of size $5\sigma_u$ hence it is, practically speaking, too large. But our results here demonstrate that policy shocks have

strong effects and policy makers may find our assessment of these effects useful for improving central bank practice. With this in mind, we first ask what is the meaning of a central bank's deviation from a fixed and common knowledge policy rule? This is a controversial question about which many opinions have been expressed in the past.
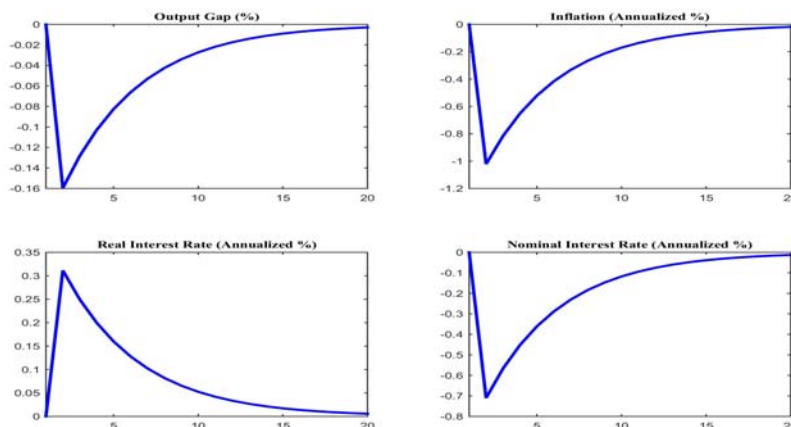


**Figure 6**: Effects of a one-standard-deviation increase to monetary shock $u_t$ at ($\xi_y$=0.5, $\xi_\pi$=1.4) (Basic Model).

An interpretation of what policy shocks mean should be deduced from what is common knowledge among agents about these shocks and what agents expect will happen after such shocks occur. Since there is a known policy rule to which the central bank will return after a shock, viewing policy shocks as "policy discretion" is not plausible. If a policy maker uses discretion then there is no known fixed policy rule. The shocks must then arise from unusual circumstances when a policy maker deviates from the known rule for reasons that are unrelated to expected, present or past values of observed variables. This explains why these shocks are exogenous and unpredictable except for their own persistence. Being unpredictable means there is no fixed rule to explain them and this requires us to consider classes of examples when a central bank deviates from a policy rule but *markets believe the bank will return to the recognized policy rule*.

One example is an October 1987 size fall in stock prices in response to which the Fed provided massive liquidity to prevent stress in financial markets. The event was unexpected and markets knew the Fed's deviation was of emergency nature and expected the central bank will return to the known rule. A

second example is the unexpected collapse of Lehman Brothers which caused a run on the financial sector and created a short term need for reasonably priced liquidity. These two examples reflect a general set of cases when a central bank reacts to financial stress. It provides needed liquidity for emergency reasons and markets recognize them to be temporary and therefore not altering the known monetary rule in place. A third generic example is the response of a central bank to unexpected political pressure that requires it to follow a course of action needed to attain a limited objective and when markets recognize it is for a short term duration.

A fourth generic case applies to all previous cases but when the event is not observed but is, instead, *anticipated* to happen by the bank. In such cases a policy shock reflects the bank's *belief in future unusual circumstances*. An example is a changed random and unknown date for a central bank to raise the nominal rate and exit the ZLB *later* (*earlier*) than specified by a known policy rule. This is accomplished with a positive (negative) policy shock *but with assumed markets' conviction* that the bank's action is a temporary deviation from the rule and that it will return to the known rule when the rate is raised.

To sharpen the argument observe there is an obvious difference between a central bank that formally makes a permanent change in its policy rule and a bank that responds to emergency by deviating from its known rule and markets know it will return to the old known rule. This difference is about what markets believe a central bank is doing today and will do in the future. *This is the unifying principle needed for a plausible interpretation of what policy shocks are*. It also replies to a common view which insists that "no discretion" means a bank loses credibility if it ever deviates from its rule. The idea of policy shocks opens the door to those circumstances when a central bank deviates for emergency reasons but is committed to the known rule *and markets believe it*.

Shocks to discount rate are often used as proxies for demand shocks and are actually shocks to the natural rate. But in the linearized bond demand equation (i.e. aggregate IS curve) such shocks and policy shocks *are mathematically equivalent* if a zero nominal rate is not binding. In addition, policy shocks cannot be "nominal interest rate shocks" since the nominal rate is an endogenous variable, not a state variable. But then, how should we think of policy shocks? The best answer is that policy shocks are persistent unpredictable shocks to the natural rate caused by special events like those described earlier. The central bank announces each shock and adds its value to the known policy rule.
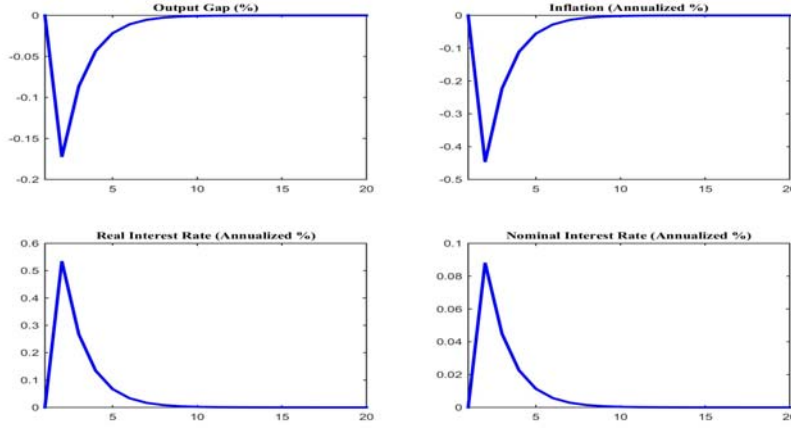
**Figure 7**: Effects of a one-standard-deviation increase to monetary shock $u_t$ at ($\xi_y$=0.5, $\xi_\pi$=1.4) when the persistence parameter $\lambda_u$=0.5.

To further study the effects of such shocks we present in Figure 7 impulse response functions when $\lambda_u = 0.5$ which is much smaller than the persistence level in the data. This value is chosen since some papers that study policy shocks report simulations with this value (e.g. Galí (2008) and Walsh (2008)). We now observe that Figure 6 exhibits a *negative* general equilibrium response of the nominal rate to a *positive* policy shock while in Figure 7 the signs of the shock and the response are the same. If one considers $u_t = +\sigma_u$ a positive shock to the nominal rate then Figure 6 is puzzling since it suggests a shock that *increases* the nominal rate results in a general equilibrium *decreased* rate. But, if one considers $u_t$ a shock to the natural rate then changes in the nominal rate are $\dfrac{\partial \hat{r}_t}{\partial u_t} = \xi_y \dfrac{\partial \hat{y}_t}{\partial u_t} + \xi_\pi \dfrac{\partial \hat{\pi}_t}{\partial u_t} + 1$ and note that $\dfrac{\partial \hat{y}_t}{\partial u_t}$ , $\dfrac{\partial \hat{\pi}_t}{\partial u_t}$ *are exactly the derivatives of output and inflation with respect to a change in the natural rate.* In all cases we have $\dfrac{\partial \hat{y}_t}{\partial u_t} < 0$ , $\dfrac{\partial \hat{\pi}_t}{\partial u_t} < 0$ hence the following is our result regarding the change in the nominal rate:

(39a) $\qquad\qquad \lambda_u = 0.80 \quad \Rightarrow \quad \dfrac{\partial \hat{r}_t}{\partial u_t} < 0 \quad \text{since} \quad |\xi_y \dfrac{\partial \hat{y}_t}{\partial u_t} + \xi_\pi \dfrac{\partial \hat{\pi}_t}{\partial u_t}| > 1$

(39b) $\qquad\qquad \lambda_u = 0.50 \quad \Rightarrow \quad \dfrac{\partial \hat{r}_t}{\partial u_t} > 0 \quad \text{since} \quad |\xi_y \dfrac{\partial \hat{y}_t}{\partial u_t} + \xi_\pi \dfrac{\partial \hat{\pi}_t}{\partial u_t}| < 1 .$

Once $u_t$ is considered a shock to the natural rate the results above can be better explained. A positive $u_t$ shock generates a deflation but the size of this effect depends upon the persistence of the

shock. If the persistence is high, expected deflation intensifies with a feed-back effect that increases the intensity of today's deflation. With a strong enough feed-back, a rise in the natural rate causes such a deflation that the resulting change in the equilibrium nominal rate becomes negative. However, the two figures also show that once we focus on the *real rate* then there is no ambiguity: the sign of the shock to the natural rate is always the same as the sign of its effect on the real rate. Further comparison of Figures 6 and 7 also shows that the higher degree of persistence results in a more intense response of the endogenous variables which is explained by the same mechanism just outlined.

We have now provided an empirical as well as a conceptual interpretation of policy shocks. But then we also draw the practical conclusion that policy shocks have a large man-made component. That is, we suggest there is a component of choice in the values policy shocks take since both intensity and persistence of shocks may be altered by human response. We are not suggesting the shock is a choice variable since we have argued it reflects events in response to which a central bank is compelled to act. But, in responding to actual or expected events a central bank must also make some judgment in the sense that a bank decides in advance on general principles that guide how it would respond to such special events. Hence, it is plausible to conclude that in the long run there may be some room for modulating how strongly to act and how fast to return to the norm. The reason for such long term choice is that it has dramatic effects. We now show that the most powerful effect of central bank's response to unexpected events *is the speed at which it returns to the old policy*. The long term effect on market volatility is dominated by the persistence parameter $\lambda_u$.

<p align="center"><strong>Table 5:</strong> the effect of the persistence of Monetary shock</p>

| | | $\xi_y \rightarrow$ | 0.5 | | | 3.5 | | |
|---|---|---|---|---|---|---|---|---|
| | | $\lambda_u \rightarrow$ | 0.7 | 0.8 | 0.9 | 0.7 | 0.8 | 0.9 |
| | | $\sigma_{\hat{y}}$ | 1.472 | 1.655 | 1.997 | 0.828 | 0.863 | 1.863 |
| | 1.4 | $\sigma_{\pi}$ | 1.363 | 2.202 | 4.194 | 4.844 | 9.136 | 17.222 |
| $\xi_\pi$ | | $\sigma_{\hat{e}^j}$ | 1.767 | 2.829 | 7.388 | 4.445 | 13.770 | 62.340 |
| | | $\sigma_{\hat{y}}$ | 1.312 | 1.380 | 1.462 | 1.500 | 1.967 | 3.927 |
| | 1.7 | $\sigma_{\pi}$ | 0.843 | 1.249 | 2.157 | 3.126 | 7.185 | 19.189 |
| | | $\sigma_{\hat{e}^j}$ | 1.656 | 1.753 | 1.860 | 2.558 | 6.419 | 25.335 |

Table 5 reports results of fixing all parameter of Basic Model as in Tables 2a-2c but varying the value of $\lambda_u$ over the three value of 0.7, 0.8 and 0.9 thus permitting only small changes in this value. The

Table shows such changes in $\lambda_u$ have small effects on output volatility when the policy parameter $\xi_y$ is small but somewhat stronger effect if $\xi_y$ takes large values. The effect on consumption volatility is stronger but can be muted by a large inflation stabilization parameter $\xi_\pi$. The largest effect of higher persistence of policy shocks is the increased volatility of inflation. Altering the value of $\lambda_u$ from the estimated level of 0.8 to 0.9 increases inflation volatility *drastically* for all values of policy parameters while a reduction of $\lambda_u$ from 0.8 to 0.7 modulates inflation volatility significantly. The explanation for these results is the one we have already provided above, in addition to the standard statistical fact that smaller values of $\lambda_u$ imply a smaller long term unconditional variance of $u_t$ hence lower volatility. We suggest that in the long run the central bank does have some choice as to the response to policy shocks. It follows from our analysis that, apart from special conditions when slow adjustment is explicitly justified, policy performance would improve if persistence of policy shocks is lowered and after a shock the central bank returns to long run policy more rapidly.

*5.* *Conclusions and policy implications*

Our NKM allows for diverse beliefs about technology and policy shocks and was solved under a quadratic approximation. Aggregating quadratic decision functions naturally turns diversity of market belief into a new state variable. But since the model has two exogenous shocks in addition to random terms associated with diverse beliefs, it appears we should expect nothing but smooth trade-off between inflation and output volatility. We show that this simple view is incorrect: both efficacy of monetary policy and the nature of this trade-off change drastically. We sum up the main results of our study but focus mainly on policy implications, all of which *are direct results of the diversity of beliefs*:

(i) *The policy space contains a curve of singularity which is a collection of policy parameters that divides the space into two sub-regions. Some trade-off between output and inflation volatility exists within each region and some across these regions. Hence, important trade-off may be accomplished with a discontinuous jump in policy choice rather than by small adjustments in existing policy. Adding a cost push shock (with unknown source) may improve trade-off but we proved in Section 2.5.2 that it will not change the singularity and hence will not change the results outlined here.*

(ii) *The singularity causes volatility outcomes of policy to be complex and non monotonic. Hence a policy-maker cannot assume that a more aggressive policy parameter will change outcomes in a smooth and monotone manner predicted by a single agent RE model.*

(iii) *Belief diversity implies diverse individual consumption and diverse bond holdings. Central bank policy must therefore consider volatility of individual consumption, in addition to volatility of output and inflation. We show that aggressive anti-inflation policy increases consumption volatility and hence financial markets' fluctuations while aggressive output stabilization policy entails rising inflation volatility. The implication is that just a consideration of the three objectives involved, efficient central bank policy must be moderate avoiding aggressive move in any direction.*

(iv) *Market exuberance need not be irrational and high optimism about future business conditions will very likely lower aggregate output and increase inflation. This "stagflation" effect is stronger the more sticky prices are. The policy response is then muted by the fact that effects of higher inflation and lower output on the interest rate partially cancel each other. To be more effective a central bank may need to target such exuberance directly or its effect in asset markets. We believe that in practice central banks do respond to asset price movements even if temporarily and not systematically.*

(v) *Policy shocks exhibit a high serial correlation of 0.80 in US data. It partly reflects a bank's belief which motivates deviations from the fixed policy rule. We show this high persistence contributes greatly to market volatility and conclude a change in a central bank's own belief or assessment can improve efficiency. A reduction in the persistence of central bank deviation from a fixed rule reduces volatility and contributes to stability. Reduced persistence requires only a more rapid return to a known policy rule after temporarily deviating from it. We recognize the importance of the deviations themselves.*

(vi) *Dispersion of market belief is measured by cross sectional standard deviation of individual indexes belief developed in the text. We find that an increased belief diversity in the market makes policy coordination harder and results in lower aggregate output and lower rate of inflation. Bank policy can contribute to lowering dispersion across diverse beliefs by being as transparent as possible.*

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## Appendix A: More details about the second order approximation and solution approach.

In order to solve the micro-economic system on step (iv) in Section 3.3 we need to specify the laws of motion required by step (iii). We aggregate over the second order expansion of the labor supply equation (38b) and obtain:

(A1) $$-\sigma \hat{y}_t + \hat{w}_t + \frac{\sigma(1+\sigma)}{2}\int_{[0,1]}(\hat{c}_t^j)^2 dj - \sigma \hat{y}_t \hat{w}_t \;=\; \eta \hat{\ell}_t + \frac{\eta(\eta-1)}{2}\int_{[0,1]}(\hat{\ell}_t^j)^2 dj,$$

where the integrals, by decomposition and by Assumption 4, are computed by aggregating the squared linear approximation of the individual choice function. Aggregate income and aggregate optimal price are respectively approximated by the following second order polynomials:

(A2) $$\hat{y}_t \;=\; P^y(\hat{\zeta}_t, u_t, Z_t, \sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, \text{Cov}(b,g)_t, \hat{\zeta}_t^2, u_t^2, Z_t^2, \hat{\zeta}_t u_t, \hat{\zeta}_t Z_t, u_t Z_t; \Theta^y);$$

(A3) $$\hat{q}_t \;=\; P^q(\hat{\zeta}_t, u_t, Z_t, \sigma^2_{cs}(g)_t, \sigma^2_{cs}(b)_t, \text{Cov}(b,g)_t, \hat{\zeta}_t^2, u_t^2, Z_t^2, \hat{\zeta}_t u_t, \hat{\zeta}_t Z_t, u_t Z_t; \Theta^q).$$

Aggregating over the second order expansion of the labor demand equation implied by firm i production function in (1) we get:

(A4) $$\hat{y}_t - \frac{1}{2\theta}(\hat{y}_t)^2 \;=\; \hat{\zeta}_t - \frac{1}{2\theta}(\hat{\zeta}_t)^2 + \hat{\ell}_t - \frac{1}{2\theta}\int_{[0,1]}(\hat{n}_{it})^2 di + \frac{\theta-1}{\theta}\hat{\zeta}_t \hat{\ell}_t.$$

Our last task is to define the relationship between aggregated optimal price $\hat{q}_t$ in (A3) and inflation $\hat{\pi}_t$. We aggregate prices across firms i, as in KPW (2013) (see equation (9)), by

(A5) $$\int_{S_t}(q_{it})^{1-\theta}di + \omega(\frac{1}{\pi_t})^{1-\theta}=1,$$

where $S_t$ is the set of firms in [0,1] which can change prices and has measure (1- ω). The second order approximation of (A5) leads to:

(A6)
$$\int_{S_t} [\hat{q}_{it} - \frac{\theta}{2}(\hat{q}_{it})^2] di = \omega[\hat{\pi}_t + \frac{\theta-2}{2}(\hat{\pi}_t)^2].$$

Sticky prices in this economy is assumed to occur as a result of Calvo pricing. Calvo pricing creates an artificial heterogeneity in the optimal pricing of the firms, which is in addition to the heterogeneity arising from expectations. However, Assumption 1 ensures that the integral on the left-hand side of (A6) is independent of sets $S_t$. For all random sets $S_t$ in [0,1],

$$\int_{S_t} [\hat{q}_{it} - \frac{\theta}{2}(\hat{q}_{it})^2] di = (1-\omega)\int_{[0,1]} [\hat{q}_{it} - \frac{\theta}{2}(\hat{q}_{it})^2] di,$$

therefore, we can rewrite (A6) as

(A6')
$$\hat{q}_t - \frac{\theta}{2}\int_{[0,1]} (\hat{q}_{it})^2 di \;=\; \frac{\omega}{1-\omega}[\hat{\pi}_t + \frac{\theta-2}{2}(\hat{\pi}_t)^2].$$

Hence, equations (A1)-(A4) and (A6') specify the laws of motion required by step (iii).

To solve the system of equations (38a)-(38h) given (A1)-A4) and (A6') we use the iterative procedure described in Section 3.3 and compute solutions at each iteration by perturbation method. To initiate iteration process we pick initial values for the set of coefficients $\Theta = (\Theta^y, \Theta^q)$ in (A2) and (A3). Aggregate variables in agent j's world are taken as given. The algorithm iterates on the unknown coefficients $\Theta = (\Theta^y, \Theta^q)$ of the polynomials (A2) and (A3) until equilibrium conditions are satisfied. The iterative procedure can set coefficients at each iteration k using fixed-point iteration $\Theta_{k+1} = \delta\Theta_k + (1-\delta)\Theta^*$, where $\delta$ is a damping parameter and $\Theta^*$ are the coefficients obtained after aggregating the optimal solution of agent j given aggregate parameters $\Theta_k$. An alternative procedure, which we follow, is that of using the embedded iterative procedure of a Gauss-Newton non-linear equations solver. Convergence is achieved when equilibrium conditions are satisfied with an error of less than $10^{-8}$.

As extensively pointed out by Schmitt-Grohé and Uribe (2004) and Preston and Roca (2007) when using a second-order perturbation approximation the presence of uncertainty affects the constant terms of the decision rules. This requires a risk correction to the mean of individual and aggregate variables which is computed according to Schmitt-Grohé and Uribe (2004) and adapting to our case their Matlab routines.

We also use their routines to compute all other coefficients of our second-order approximation. All simulations and impulse response analysis conducted in the text take into account such a correction in the mean.

**Appendix B: Errors in the Euler Equations**.

We measure the quality of a candidate solution by computing the mean, the max and the standard deviation of the errors in the Euler Equations. If the economic significance of these errors is small, we accept the solution. We test the accuracy of the approximation solution obtained from the second order iteration procedure according to Judd (1998): errors in the Euler equations of agent's consumption and labor decisions are constructed as unit-free measurement by a division of the marginal utility of consumption. For the optimal pricing equation of the firms, we do not measure it in units of consumption, since price ratio already provides a comparison baseline of 1.

$$(\text{B1}) \qquad \tilde{\varepsilon}_\tau^C \equiv \frac{\tilde{\tau}_b(b_\tau^j) + (C_\tau^j)^{-\sigma}\dfrac{1}{1+r_\tau} - \beta E_\tau^j[(C_{\tau+1}^j)^{-\sigma}\dfrac{1}{\pi_{\tau+1}}]}{(C_\tau^j)^{-\sigma}};$$

$$(\text{B2}) \qquad \tilde{\varepsilon}_\tau^\ell \equiv \frac{(C_\tau^j)^{-\sigma}w_\tau - (L_\tau^j)^\eta}{(C_\tau^j)^{-\sigma}};$$

$$(\text{B3}) \qquad \tilde{\varepsilon}_\tau^q \equiv q_\tau^j - \frac{\theta}{\theta-1}\frac{S_\tau^j}{V_\tau^j}.$$

All errors are measured in 10-based logarithm of the absolute value of the errors: $\varepsilon_\tau^C \equiv \log_{10}(|\tilde{\varepsilon}_\tau^C|)$, $\varepsilon_\tau^\ell \equiv \log_{10}(|\tilde{\varepsilon}_\tau^\ell|)$, $\varepsilon_\tau^q \equiv \log_{10}(|\tilde{\varepsilon}_\tau^q|)$. For each point, we need to compute the expectations in (B1) to measure the errors. We do this by using the monomial integration rule with $2N^2+1$ nodes according to Judd (1998) and the Matlab code provided by Judd et al. (2011). Here, we adapt it to the case of correlated random variables using the change of variables via Cholesky decomposition. Since, according to Assumption 5, $(\Upsilon, \varepsilon)$ are the primitive shocks following normal distributions that are used to construct $\rho^{jg}$ and the stochastic component of $\sigma_{cs}^2(g)_t$, we use the perception model with the random normal variables $(\rho^{j\zeta}, \rho^{ju}, \rho^{jZ}, \Upsilon, \varepsilon)$ to measure the errors in the Euler equations. Note that expectations are formed for each agent j. The variance-covariance matrix of the perception model is specified through the rationality

constraints discussed in Section 2.5.1 and by the further conditions explained in footnote 3.

**Table B1:** mean, max and standard deviation of errors (B1) in optimal bond holding equation (38a)

| $\varepsilon^C$ | | | $\xi_y$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
| | 1.2 | Mean | -1.48 | -0.53 | 0.23 | 6.68 | -1.47 | -2.42 | -2.32 | -2.31 | -2.21 | NaN | -2.14 | -2.59 |
| | | Max | -0.29 | 1.56 | 3.48 | 25.37 | 0.39 | -1.25 | -2.16 | -2.27 | -2.14 | NaN | -1.71 | -2.16 |
| | | St. Dev. | 0.25 | 0.42 | 0.67 | 3.38 | 0.46 | 0.33 | 0.08 | 0.01 | 0.01 | NaN | 0.05 | 0.04 |
| | 1.3 | Mean | -3.34 | -2.76 | -2.20 | -1.39 | 2.47 | 1.12 | -2.37 | -2.62 | -2.56 | -2.59 | -2.60 | -2.58 |
| | | Max | -1.95 | -1.67 | -1.31 | -1.23 | 9.95 | 7.25 | -0.50 | -1.52 | -2.21 | -2.51 | -2.54 | -2.53 |
| | | St. Dev. | 0.42 | 0.34 | 0.31 | 0.23 | 1.49 | 0.97 | 0.70 | 0.30 | 0.10 | 0.04 | 0.01 | 0.01 |
| | 1.4 | Mean | -3.16 | -2.86 | -2.55 | -2.19 | -0.08 | -0.66 | NaN | -0.84 | -2.84 | -2.73 | -2.70 | -2.74 |
| | | Max | -2.75 | -2.66 | -2.46 | -2.13 | 0.39 | 0.76 | NaN | 1.30 | -0.99 | -1.76 | -2.35 | -2.63 |
| | | St. Dev. | 0.15 | 0.12 | 0.09 | 0.06 | 0.07 | 0.24 | NaN | 0.40 | 0.59 | 0.24 | 0.11 | 0.04 |
| | 1.5 | Mean | -3.26 | -3.05 | -2.85 | -2.62 | -2.01 | -1.25 | -1.35 | 0.70 | 4.49 | -1.66 | -2.95 | -2.82 |
| | | Max | -3.09 | -2.94 | -2.76 | -2.55 | -1.98 | -0.80 | -0.60 | 4.22 | 18.90 | -0.14 | -1.33 | -1.97 |
| | | St. Dev. | 0.06 | 0.04 | 0.04 | 0.03 | 0.02 | 0.05 | 0.13 | 0.65 | 2.36 | 0.34 | 0.43 | 0.22 |
| $\xi_\pi$ | 1.6 | Mean | -3.38 | -3.23 | -3.07 | -2.91 | -2.51 | -1.83 | -1.64 | -1.72 | -1.08 | 13.11 | -0.21 | -2.33 |
| | | Max | -3.16 | -3.07 | -2.96 | -2.81 | -2.44 | -1.82 | -1.27 | -1.15 | 0.09 | 41.22 | 2.91 | -0.77 |
| | | St. Dev. | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.02 | 0.04 | 0.08 | 0.22 | 5.71 | 0.51 | 0.42 |
| | 1.7 | Mean | -3.49 | -3.37 | -3.25 | -3.12 | -2.83 | -2.43 | -1.55 | -1.89 | -1.97 | -1.66 | -0.28 | 22.55 |
| | | Max | -3.21 | -3.16 | -3.09 | -2.99 | -2.73 | -2.37 | -1.50 | -1.57 | -1.50 | -0.88 | 1.89 | 73.01 |
| | | St. Dev. | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.01 | 0.10 | 0.03 | 0.06 | 0.14 | 0.40 | 8.33 |
| | 2 | Mean | -3.76 | -3.69 | -3.63 | -3.55 | -3.40 | -3.21 | -2.99 | -2.68 | -2.09 | -2.06 | -2.39 | -2.46 |
| | | Max | -3.30 | -3.29 | -3.27 | -3.24 | -3.16 | -3.03 | -2.85 | -2.58 | -2.06 | -1.78 | -2.11 | -2.09 |
| | | St. Dev. | 0.10 | 0.08 | 0.07 | 0.06 | 0.04 | 0.02 | 0.01 | 0.01 | 0.02 | 0.02 | 0.03 | 0.04 |
| | 5 | Mean | -4.50 | -4.50 | -4.51 | -4.51 | -4.52 | -4.53 | -4.54 | -4.55 | -4.56 | -4.56 | -4.57 | -4.57 |
| | | Max | -3.37 | -3.38 | -3.38 | -3.39 | -3.39 | -3.40 | -3.41 | -3.42 | -3.42 | -3.43 | -3.43 | -3.43 |
| | | St. Dev. | 0.52 | 0.52 | 0.52 | 0.52 | 0.51 | 0.51 | 0.51 | 0.51 | 0.50 | 0.50 | 0.50 | 0.49 |

**Table B2:** mean, max and standard deviation of errors (B3) in optimal pricing equation (38d)

| $\varepsilon^q$ | | | $\xi_y$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 0.25 | 0.5 | 0.75 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
| | 1.2 | Mean | -1.34 | -0.96 | 2.12 | 13.33 | -1.28 | -2.14 | -2.68 | -2.78 | -2.70 | NaN | -2.16 | -2.92 |
| | | Max | 2.71 | 3.00 | 24.61 | 157.53 | 2.12 | 1.74 | 1.98 | 0.61 | -0.60 | NaN | 2.44 | 1.64 |
| | | St. Dev. | 0.81 | 0.68 | 2.07 | 12.54 | 0.73 | 0.86 | 0.71 | 0.66 | 0.68 | NaN | 0.80 | 0.73 |
| | 1.3 | Mean | -2.71 | -2.25 | -1.87 | -0.20 | -0.22 | -0.53 | -1.50 | -2.29 | -2.88 | -3.23 | -3.31 | -3.27 |
| | | Max | -0.64 | -0.87 | -0.20 | 3.55 | 1.33 | 4.58 | 2.67 | 1.95 | 2.34 | 0.45 | -0.94 | -1.32 |
| | | St. Dev. | 0.58 | 0.61 | 0.63 | 0.53 | 0.23 | 0.52 | 0.88 | 1.07 | 0.94 | 0.73 | 0.66 | 0.68 |
| | 1.4 | Mean | -3.11 | -2.81 | -2.51 | -2.25 | -0.33 | -0.51 | NaN | -0.74 | -1.76 | -2.46 | -3.01 | -3.38 |
| | | Max | -1.26 | -1.27 | -0.92 | -0.48 | 0.03 | 0.02 | NaN | 2.36 | 2.44 | 2.10 | 1.68 | 0.85 |
| | | St. Dev. | 0.60 | 0.61 | 0.61 | 0.67 | 0.19 | 0.21 | NaN | 0.53 | 1.01 | 1.13 | 1.03 | 0.88 |
| | 1.5 | Mean | -3.42 | -3.21 | -2.99 | -2.76 | -1.89 | -0.54 | -0.49 | -0.35 | -0.19 | -1.02 | -1.99 | -2.63 |
| | | Max | -1.56 | -1.49 | -1.35 | -1.06 | -0.11 | 0.01 | 3.27 | 0.02 | 2.96 | 3.43 | 2.56 | 2.82 |
| | | St. Dev. | 0.61 | 0.62 | 0.63 | 0.64 | 0.63 | 0.17 | 0.30 | 0.21 | 0.31 | 0.73 | 1.10 | 1.16 |
| $\xi_\pi$ | 1.6 | Mean | -3.67 | -3.51 | -3.35 | -3.16 | -2.78 | -1.45 | -0.56 | -0.67 | -0.55 | -0.02 | -0.52 | -1.42 |
| | | Max | -1.80 | -1.70 | -1.67 | -1.46 | -0.94 | 0.29 | 2.59 | 2.61 | 2.85 | 2.12 | 1.75 | 2.38 |
| | | St. Dev. | 0.61 | 0.61 | 0.63 | 0.64 | 0.66 | 0.50 | 0.36 | 0.42 | 0.21 | 0.20 | 0.28 | 0.89 |
| | 1.7 | Mean | -3.88 | -3.76 | -3.62 | -3.48 | -3.15 | -2.73 | -0.83 | -0.93 | -1.00 | -0.61 | -0.47 | 8.83 |
| | | Max | -2.00 | -1.90 | -1.83 | -1.77 | -1.37 | -0.84 | 1.27 | 2.60 | 3.95 | 2.46 | 0.02 | 106.21 |
| | | St. Dev. | 0.60 | 0.61 | 0.62 | 0.64 | 0.65 | 0.71 | 0.45 | 0.59 | 0.62 | 0.39 | 0.21 | 8.34 |
| | 2 | Mean | -4.36 | -4.29 | -4.22 | -4.14 | -3.95 | -3.73 | -3.47 | -3.14 | -1.89 | -1.26 | -1.90 | -1.96 |
| | | Max | -2.47 | -2.40 | -2.32 | -2.25 | -2.12 | -1.94 | -1.61 | -1.18 | -0.13 | 2.73 | 1.86 | 2.75 |
| | | St. Dev. | 0.58 | 0.59 | 0.61 | 0.61 | 0.63 | 0.66 | 0.66 | 0.71 | 0.54 | 0.76 | 0.95 | 0.99 |
| | 5 | Mean | -6.31 | -6.41 | -6.50 | -6.55 | -6.57 | -6.55 | -6.48 | -6.35 | -6.16 | -6.00 | -5.88 | -5.77 |
| | | Max | -4.59 | -4.67 | -4.72 | -4.77 | -4.42 | -4.18 | -4.01 | -3.87 | -3.75 | -3.64 | -3.54 | -3.45 |
| | | St. Dev. | 0.60 | 0.62 | 0.63 | 0.64 | 0.68 | 0.74 | 0.81 | 0.82 | 0.75 | 0.70 | 0.68 | 0.68 |

While the max error of the optimal labor supply equation (38b) measured by (B2) is uniformly around $10^{-12}$ in all points away from the ridge, errors in the optimal bond holding equation (38a) measured by (B1) and in the optimal pricing equation (38d) measured by (B3) vary across the policy

parameters grid away from the ridge. Tables B1 and B2 report the mean, max and standard deviation of errors respectively in the optimal bond holding equation (38a) and in the optimal pricing equation (38d). In each table the grayed cells are either cells on or too close to the ridge or cells where the quadratic approximation failed to give a satisfactory accuracy in the Euler equations. Heterogeneous beliefs and the additional cross-sectional state variables introduce additional complexity which results in higher volatility and less accuracy. However, all policy implications as well as all impulse response analysis reported in the text are based on policy configurations where an average accuracy of less than $10^{-2}$ in (38a) and (38d) has been achieved.