A General Framework for Systemic Risk

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Joint work with Chen Chen and Garud Iyengar.

Systemic Risk

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Examples:

- firms in an economy
- business units in a company
- suppliers, sub-contractors, etc. in a supply chain network
- generating stations, transmission facilities, etc. in a power network
- flood walls, pumping stations, etc. in a levee system

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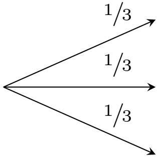
Systemic risk refers to the risk of catastrophic collapse of the entire system. Involves:

- the simultaneous analysis of outcomes across all entities in a system
- the possibility of complex interactions between components

Joint Distribution of Outcomes

- 3 firms in 3 future scenarios (equally likely)
- Loss matrix:



Scenario	Firm 1	Firm 2	Firm 3
ω_1	$+50$	-40	$+20$
ω_2	-40	$+50$	-40
ω_3	$+20$	$+20$	$+50$

(+ 'loss'; − 'profit')

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- Interbank derivatives exposures (e.g., AIG)
- Transmission of illiquidity, 'bank runs' (e.g., Lehman)
- Fire sales, asset price contagion (e.g., CDOs)

Contributions

- A general, axiomatic framework for coherent systemic risk
  analyzes joint distribution of outcomes
  allows for *some* endogenous mechanisms of contagion
  subsumes many recently proposed systemic risk measures

- A structural decomposition of systemic risk

- A dual representation for systemic risk measures
  'shadow price of risk'

- A mechanism for systemic risk attribution & decentralization

- Methodology extends to a much broader class of risk functions

Literature Review

- Axiomatic theory of single-firm risk measures:
  Artzner et al., (2000); see survey of Schied (2006)

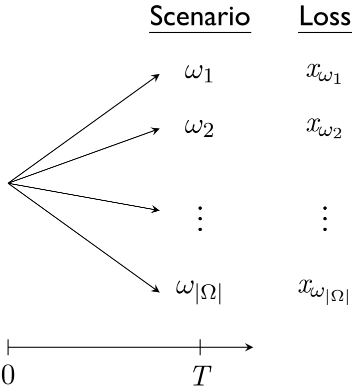
- Systemic risk measures: portfolio approach
  Gauthier et al., (2010); Tarashev et al., (2010); Acharya et al., (2010); Brownlees & Engle (2010); Adrian & Brunnermeier (2009)

- Systemic risk measures: deposit insurance / credit approach
  Lehar (2005); Huang et al., (2009); Giesecke & Kim (2011)

- Structural models of contagion & systemic risk:
  Acharya et al., (2010); Staum (2011); Liu & Staum (2010); Cont et al., (2011); Bimpikis & Tahbaz-Salehi (2012)

- Portfolio attribution:
  Denault (2001); Buch & Dorfleitner (2008)

Single-Firm Risk Measures



$\Omega =$ set of scenarios

$x \in \mathbb{R}^{\Omega}$

$x_{\omega} =$ loss in scenario ω

Coherent Risk Measures

Definition. A coherent single-firm risk measure is a function $\rho\colon \mathbb{R}^\Omega \to \mathbb{R}$ that satisfies, for all $x, y \in \mathbb{R}^\Omega$:

(i) Monotonicity: if $x \geq y$, then $\rho(x) \geq \rho(y)$

(ii) Positive homogeneity: for all $\alpha \geq 0$, $\rho(\alpha x) = \alpha\rho(x)$

(iii) Convexity: for all $0 \leq \alpha \leq 1$,

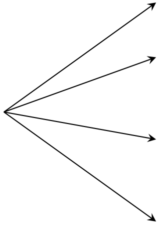
$$\rho(\alpha x + (1 - \alpha y)) \leq \alpha\rho(x) + (1 - \alpha)\rho(y)$$

(iv) Cash-invariance: for all $\alpha \in \mathbb{R}$,

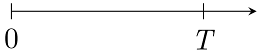
$$\rho(x + \alpha\mathbf{1}_\Omega) = \rho(x) + \alpha$$

[Artzner et al., 2000]

Systemic Risk Measures



Scenario	Firm 1	Firm 2		Firm $\|\mathcal{F}\|$
ω_1	X_{1,ω_1}	X_{2,ω_1}	$\cdots$	$X_{\|\mathcal{F}\|,\omega_1}$
ω_2	X_{1,ω_2}	X_{2,ω_2}	$\cdots$	$X_{\|\mathcal{F}\|,\omega_2},$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\omega_{\|\Omega\|}$	$X_{1,\omega_{\|\Omega\|}}$	$X_{2,\omega_{\|\Omega\|}}$	$\cdots$	$X_{\|\mathcal{F}\|,\omega_{\|\Omega\|}}$



$\Omega =$ set of scenarios

$\mathcal{F} =$ set of firms (entities in the system)

$X \in \mathbb{R}^{\Omega \times \mathcal{F}}$

$X_{i,\omega} =$ loss for firm i in scenario ω

Systemic Risk Measures: Definition

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Definition. A systemic risk measure is a function $\rho\colon \mathbb{R}^{\Omega \times \mathcal{F}} \to \mathbb{R}$ that satisfies, for all economies $X, Y, Z \in \mathbb{R}^{\Omega \times \mathcal{F}}$:

(i) *Monotonicity:* if $X \geq Y$, then

$$\rho(X) \geq \rho(Y)$$

(ii) *Positive homogeneity:* for all $\alpha \geq 0$,

$$\rho(\alpha X) = \alpha \rho(X)$$

(iii) *Normalization:* $\rho(\mathbf{1}_{\mathcal{E}}) = |\mathcal{F}|$

Systemic Risk Measures: Definition

Definition. (con't.) Given $x, y \in \mathbb{R}^{\mathcal{F}}$, define the ordering $x \succeq_\rho y$ by

$$x \succeq_\rho y \quad \Longleftrightarrow \quad \rho(x, \ldots, x) \geq \rho(y, \ldots, y)$$

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$$\begin{array}{l|c|c|c|l}
\text{Scenario} & \omega_1 \quad \ldots & \omega & \ldots \quad \omega_{|\Omega|} & \\
\text{Firm } 1 & X_{1,\omega_1} & X_{1,\omega} & X_{1,\omega_{|\Omega|}} & \\
\vdots & \vdots & \vdots & \vdots & = X \\
\text{Firm } |\mathcal{F}| & X_{|\mathcal{F}|,\omega_1} & X_{|\mathcal{F}|,\omega} & X_{|\mathcal{F}|,|\Omega|} & \\
 & & X_\omega \succeq_\rho Y_\omega \quad \forall\, \omega & & \Rightarrow \rho(X) \geq \rho(Y) \\
\text{Firm } 1 & Y_{1,\omega_1} & Y_{1,\omega} & Y_{1,\omega_{|\Omega|}} & \\
\vdots & \vdots & \vdots & \vdots & = Y \\
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\end{array}$$

Systemic Risk Measures: Definition

Definition. (con't.)

(v) *Convexity:* for all $0 \leq \alpha \leq 1$, $\bar{\alpha} = 1 - \alpha$

(a) *Outcome convexity:* if

$$Z = \alpha X + \bar{\alpha}\, Y$$

then, $\rho\big(Z\big) \leq \alpha\rho(X) + \bar{\alpha}\rho(\,Y)$

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(b) *Risk convexity:* if for all scenarios $\omega \in \Omega$,

$$\rho(Z_\omega, \ldots, Z_\omega) = \alpha\rho(X_\omega, \ldots, X_\omega) + \bar{\alpha}\rho(Y_\omega, \ldots, Y_\omega)$$

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Two different notions of **diversity**

Systemic Risk Measures: Definition

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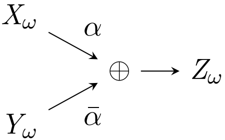
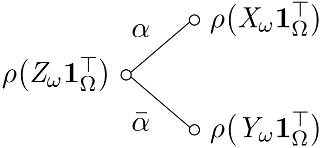
1. Outcome convexity: Increasing diversification reduces risk



$$\Rightarrow \quad \rho(Z) \leq \alpha\rho(X) + \bar{\alpha}\rho(Y)$$

2. Risk convexity: Removing randomness reduces risk



$$\Rightarrow \quad \rho(Z) \leq \alpha\rho(X) + \bar{\alpha}\rho(Y)$$

Structural Decomposition

Definition. An aggregation function is a function $\Lambda\colon \mathbb{R}^{\mathcal{F}} \to \mathbb{R}$ that is monotonic, positively homogeneous, convex, and normalized so that $\Lambda(\mathbf{1}_{\mathcal{F}}) = |\mathcal{F}|$.

Aggregation function: aggregates risk across firms in a given scenario

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Aggregation function: aggregates risk across firms in a given scenario

Theorem. A function $\rho\colon \mathbb{R}^{\Omega\times\mathcal{F}} \to \mathbb{R}$ is a systemic risk measure with $\rho(-\mathbf{1}_{\mathcal{E}}) < 0$ iff there exists

- an aggregation function Λ
- coherent single-firm base risk measure ρ_0 such that

$$\rho(X) = (\rho_0 \circ \Lambda)(X) \triangleq \rho_0\left(\Lambda(X_1), \Lambda(X_2), \ldots, \Lambda(X_{|\Omega|})\right)$$

Example: Economic Systemic Risk Measures

- $\mathcal{F} =$ firms in the economy
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Example. (Systemic Expected Shortfall)

$$\Lambda_{\text{total}}(x) \triangleq \sum_{i \in \mathcal{F}} x_i, \qquad \rho_{\text{SES}}(X) \triangleq (\text{CVaR}_\alpha \circ \Lambda_{\text{total}})(X)$$

[Acharya et al., 2010; Brownlees, Engle 2010]

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Example. (Deposit Insurance)

$$\Lambda_{\mathsf{loss}}(x) \triangleq \sum_{i \in \mathcal{F}} x_i^+, \qquad \rho_{\mathsf{DI}}(X) \triangleq \mathsf{E}\left[\Lambda_{\mathsf{loss}}(X_\omega)\right] = \mathsf{E}\left[\sum_{i \in \mathcal{F}} X_{i,\omega}^+\right]$$

[e.g., Lehar, 2005; Huang et al., 2009]

Example: Investing with Performance Fees

- $\mathcal{F} =$ a collection of hedge funds or portfolio managers
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- $\gamma_i \in [0,1]$ is the performance fee of fund i

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- $\gamma_{\mathsf{FoF}} \in [0,1]$ is the performance fee of the fund-of-funds manager

Example. (Fund-of-Funds Investor)

$$\Lambda_{\mathsf{FoF}}(x) \triangleq \sum_{i \in \mathcal{F}} \left(x_i + \gamma_i x_i^-\right) + \gamma_{\mathsf{FoF}} \left(\sum_{i \in \mathcal{F}} \left(x_i + \gamma_i x_i^-\right) \right)^-$$

Example: Resource Allocation

- $\mathcal{A} =$ a set of activities
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Consider the aggregation function:

$$\Lambda_{\mathsf{RA}}(x) \triangleq \begin{array}{ll} \underset{u}{\text{minimize}} & \sum_{a \in \mathcal{A}} c_a u_a \\ \text{subject to} & \sum_{a \in \mathcal{A}} b_{ia} u_a \geq x_i, \quad \forall\ i \in \mathcal{F} \\ & u \in \mathbb{R}^{\mathcal{A}} \end{array}$$

where

- $u_a =$ reduction in level of activity a (decision variable)
- $c_a =$ per-unit cost of reductions in activity a
- $b_{ia} =$ per-unit consumption of resource i by activity a

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$$\Lambda_{\mathsf{CM}}(x) \triangleq \begin{array}{ll} \underset{y \in \mathbb{R}_+^{\mathcal{F}},\ b \in \mathbb{R}_+^{\mathcal{F}}}{\text{minimize}} & \sum_{i \in \mathcal{F}} y_i + \gamma \sum_{i \in \mathcal{F}} b_i \\ \text{subject to} & b_i + y_i \geq x_i + \sum_{j \in \mathcal{F}} \Pi_{ji} y_j, \quad \forall\ i \in \mathcal{F} \end{array}$$

where

- loss x_i is covered by firm i reducing the payments by an amount y_i, or relying on an injection from the regulator in the amount b_i
- reminiscent of Eisenberg & Noe (2001)

“General” Aggregation Function

Given:

- $c \in \mathbb{R}^N_+$, $A \in \mathbb{R}^{K \times \mathcal{F}}_+$, $B \in \mathbb{R}^{K \times N}$
- $\mathcal{K} \subset \mathbb{R}^N$ a convex cone, such that $\exists\ \bar{y} \in \mathcal{K}$ with $B\bar{y} > \mathbf{0}$

Define:

$$\Lambda_{\mathsf{OPT}}(x) \triangleq \begin{array}{ll} \underset{y}{\text{minimize}} & c^\top y \\ \text{subject to} & Ax \le By \\ & y \in \mathcal{K} \end{array}$$

- Λ_{OPT} is monotonic, positively homogeneous, and convex
- if $\Lambda_{\mathsf{OPT}}(\mathbf{1}_{\mathcal{F}}) > 0$, it can also be normalized
- allows for general endogenous mechanisms for ‘co-operative’ contagion

Structural Decomposition: Proof Sketch

'If' part is not hard. 'Only if' part:

- Define $\Lambda(x) \triangleq \rho(x\mathbf{1}_\Omega^\top), \; \forall \; x \in \mathbb{R}^{\mathcal{F}}$
- Define $\rho_0(z) \triangleq \rho(X), \; \forall \; z \in \mathcal{Q}^\Omega$, for some X: $\Lambda(X_\omega) = z_\omega, \; \forall \; \omega$

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- **Step 1**: ρ_0 is well-defined. Suppose X, Y have $\Lambda(X_\omega) = \Lambda(Y_\omega), \; \forall\, \omega \in \Omega$. Preference consistency of ρ implies

$$\Lambda(X_\omega) \geq \Lambda(Y_\omega), \; \forall\, \omega \in \Omega \quad \Longrightarrow \quad \rho(X) \geq \rho(Y),$$

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Thus, $\rho(X) = \rho(Y)$

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- **Step 2**: Derive other properties: monotonicity, convexity, homogeneity of Λ and ρ_0.

$$\rho = (\rho_0 \circ \Lambda)(X) \triangleq \rho_0 \left(\Lambda(X_1), \Lambda(X_2), \ldots, \Lambda(X_{|\Omega|}) \right)$$

Acceptance Sets and Primal Representation

Theorem. Any systemic risk measure $\rho = (\rho_0 \circ \Lambda)$ can be expressed as

(PRIMAL)
$$\begin{aligned} \rho(X) = \quad & \underset{m,\ell}{\text{minimize}} && m \\ & \text{subject to} && (m,\ell) \in \mathcal{A}, \\ &&& (\ell_\omega, X_\omega) \in \mathcal{B}, \qquad \forall\, \omega \in \Omega, \\ &&& m \in \mathbb{R},\ \ell \in \mathbb{R}^\Omega. \end{aligned}$$

where acceptance sets $\mathcal{A}$ and $\mathcal{B}$ can be taken as the epigraphs of ρ_0 and Λ, i.e.,

$$\mathcal{A} \triangleq \left\{ (m,z) \in \mathbb{R} \times \mathbb{R}^\Omega \ :\ m \geq \rho_0(z) \right\},$$

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- m = cash position required to support ℓ

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Dual Representation

Theorem. Any systemic risk measure ρ can be expressed as

(DUAL)

$$\begin{aligned} \rho(X) = \quad & \underset{\bar{\pi},\Xi}{\text{maximize}} && \sum_{i\in\mathcal{F}}\sum_{\omega\in\Omega} \Xi_{i,\omega} X_{i,\omega} \\ & \text{subject to} && (1,\bar{\pi}) \in \mathcal{A}^* \\ & && (\bar{\pi}_\omega, \Xi_\omega) \in \mathcal{B}^*, \ \forall\, \omega \in \Omega \\ & && \bar{\pi} \in \mathbb{R}^\Omega, \ \Xi \in \mathbb{R}^{\mathcal{F}\times\Omega} \end{aligned}$$

where $\mathcal{A}^* \subset \mathbb{R} \times \mathbb{R}^\Omega$, $\mathcal{B}^* \subset \mathbb{R} \times \mathbb{R}^\mathcal{F}$ are (up to a sign change) the dual cones to epi(ρ_0), epi(Λ).

Further, $(\bar{\pi}, \Xi)$ must satisfy

$$\bar{\pi} \geq \mathbf{0}_\Omega, \quad \mathbf{1}_\Omega^\top \bar{\pi} \leq 1, \quad \Xi \geq \mathbf{0}_\mathcal{E}, \quad \mathbf{1}_\mathcal{F}^\top \Xi \leq |\mathcal{F}| \bar{\pi}^\top$$

Robust optimization interpretation: $\rho(X)$ is worst-case expected loss of a rescaled economy over a set of probability distributions and scaling functions

Shadow Price of Risk

$$\text{(DUAL)} \qquad \begin{aligned} \rho(X) = \quad & \underset{\bar{\pi},\Xi}{\text{maximize}} && \sum_{i\in\mathcal{F}} \sum_{\omega\in\Omega} \Xi_{i,\omega} X_{i,\omega} \\ & \text{subject to} && (1,\bar{\pi}) \in \mathcal{A}^* \\ &&& (\bar{\pi}_\omega, \Xi_\omega) \in \mathcal{B}^*, \ \forall\, \omega \in \Omega \\ &&& \bar{\pi} \in \mathbb{R}^\Omega, \ \ \Xi \in \mathbb{R}^{\mathcal{F}\times\Omega} \end{aligned}$$

Corollary. If $(\bar{\pi}^*, \Xi^*)$ is an optimal solution to (DUAL), then Ξ^* is a subgradient of ρ at X.

$\Xi^*_{i,\omega}$ is the shadow price of risk $\equiv$ the minimum marginal rate of increase of systemic risk given an increase of losses for firm i in scenario ω

Risk Attribution

Suppose ρ is a systemic risk measure, and Ξ^* is an optimal solution to (DUAL) at X. Define the risk attribution of firm i as

$$y_i^* = \sum_{\omega \in \Omega} \Xi_{i,\omega}^* X_{i,\omega}$$

By strong duality,

$$\rho(X) = \sum_{i \in \mathcal{F}} y_i^*$$

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Theorem. (No Undercut) Given $\alpha \in \mathbb{R}_+^{\mathcal{F}}$, define

$$r(\alpha) \triangleq \rho\big(\alpha_1 x_1; \ldots; \alpha_{|\mathcal{F}|} x_{|\mathcal{F}|}\big)$$

Then,

$$\alpha^\top y^* \leq r(\alpha)$$

Generalization of attribution scheme of Aumann & Shapley (1974) or Denault (2001); Buch & Dorfleitner (2008).

Decentralization

- $X^{(i)}$ = outcomes of firm i, $X \triangleq (X^{(1)}; X^{(2)}; \ldots; X^{(|\mathcal{F}|)})$
- $\mathcal{T}_i$ = set of outcomes of firm i, $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \ldots \times \mathcal{T}_{|\mathcal{F}|}$

Definition. (Social Optimality)
An economy $\bar{X} \in \mathcal{T}$ is socially optimal if it maximizes risk-adjusted welfare

$$\underset{X \in \mathcal{T}}{\text{maximize}} \sum_{i \in \mathcal{F}} U_i(X^{(i)}) - \tau \rho(X)$$

Here, $\tau > 0$ captures the systemic risk externality.

Decentralization

Theorem. Suppose that $\bar{X} \in \mathcal{T}$ is a socially optimal economy. There exists Ξ^* that is an optimal solution to the dual problem for $\rho(\bar{X})$ so that if we define, for each firm i, the tax function

$$t_i(X^{(i)}) \triangleq \tau \sum_{\omega \in \Omega} \Xi^*_{i,\omega} X_{i,\omega}$$

then, $\bar{X}^{(i)}$ is an optimal outcome for firm i, i.e.,

$$\bar{X}^{(i)} \in \underset{X^{(i)} \in \mathcal{T}^{(i)}}{\text{argmax}} \;\; U_i(X^{(i)}) - \tau \sum_{\omega \in \Omega} \Xi^*_{i,\omega} X_{i,\omega}$$

- Decentralized computation of optimal taxes possible

Decentralization

Planner's problem:

$$\underset{X \in \mathcal{T}}{\text{maximize}} \;\; \sum_{i \in \mathcal{F}} U_i(X^{(i)}) - \rho(X).$$

Decentralization scheme: apply proximal gradient method

$$\underset{X \in \mathcal{T}}{\text{maximize}} \; \sum_{i} U_i(X^{(i)}) - \left(\rho(\bar{X}) + \sum_{i} \left(X^{(i)} - \bar{X}^{(i)} \right)^{\top} \frac{\partial \rho(\bar{X})}{\partial X^{(i)}} + \frac{t}{2} \|X - \bar{X}\|_2^2 \right)$$

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Individual firm's problem:

$$X_i^* = \underset{X^{(i)}\in\mathcal{T}_i}{\text{argmax}} \left\{ U_i(X^{(i)}) - \left(X^{(i)} - \bar{X}^i\right)^\top \frac{\partial\rho(\bar{X})}{\partial X^{(i)}} - \frac{t}{2}\|X^{(i)} - \bar{X}^{(i)}\|_2^2 \right\}$$

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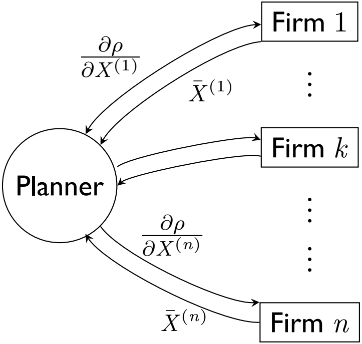
Information sent by the planner: $\frac{\partial\rho(\bar{X})}{\partial X^{(i)}}$

Information sent by the firm: $\bar{X}^{(i)}$

Decentralization

Communication between the planner and firms



Extensions

Homogeneous Systemic Risk Measures:

- monotone, +vely homogeneous, preference consistent, **not** convex
- structural decomposition exists
  homogeneous single-firm base risk measure
  homogeneous aggregation function

Convex Systemic Risk Measures:

- monotone, convex, preference consistent, **not** +vely homogeneous
- structural decomposition exists
  convex single-firm base risk measure
  convex aggregation function

Key idea: Preference consistency allows for the structural decomposition

Conclusions / Future Directions

- A general, axiomatic framework for coherent systemic risk
  analyzes joint distribution of outcomes
  allows for 'cooperative' endogenous forms of contagion
  potential applications in a broad array of engineering & economic systems
- A structural decomposition of systemic risk
- Mechanisms for systemic risk attribution & decentralization

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Future Directions:

- Statistical estimation of systemic risk
- Strategic mechanisms of contagion
- Is network structure important for systemic risk in financial markets?