The 'Boston' School Choice Mechanism

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February 4, 2010

Abstract

The Boston mechanism is a popular student placement mechanism in school choice programs around the world. We provide two characterizations of the Boston mechanisms. We introduce two new axioms, respect of preference rankings and rank-respecting Maskin monotonicity. A mechanism is the Boston mechanism for some priority if and only if it respects preference rankings and satisfies consistency, resource monotonicity, and rank-respecting Maskin monotonicity. In environments where each type of object has exactly one unit, as in house allocation, a characterization is given by respect of preference rankings, individual rationality, population monotonicity, and rank-respecting Maskin monotonicity.

Keywords: Mechanism design, matching, school choice, market design, Boston mechanism

JEL classification: C78, D78

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‡We are grateful to John William Hatfield for comments. Pete Troyan provided excellent research assistance. Ünver thanks NSF for the support of his research.

1 Introduction

School choice is a practice in which school children and their parents can express their preferences over schools, and the school system tries to accommodate their desires. School choice has become very popular in public school systems in the United States and around the world during the past few decades, and it has also become a hotly debated topic in public policy.

The so-called 'Boston' mechanism is a very popular student placement procedure. Under this mechanism, students submit their preference lists to the central clearing house. Given the reported preferences, the clearinghouse follows an algorithm that tries to match as many students to their stated preferred schools as possible subject to prespecified priorities of students at each school: seats of each school are allocated to students who rank that school first, then to those who rank it second *if there is any remaining seat*, and so forth.

Abdulkadiroğlu and Sönmez (2003) find various shortcomings of the Boston mechanism: the mechanism is not stable, that is, it can cause an unfair outcome where a student is not admitted to a school she likes while a student with a lower priority than her is admitted to that school; the Boston mechanism is not strategy-proof and, even worse, easy to manipulate. Abdulkadiroğlu and Sönmez (2003) point out that the (student-proposing) deferred acceptance mechanism (Gale and Shapley, 1962) solves both problems because it is both stable and strategy-proof. Subsequent studies find additional pitfalls of the Boston mechanism; its (complete information) Nash equilibrium outcomes are Pareto-dominated by the outcome of the deferred acceptance mechanism (Ergin and Sönmez, 2006); a higher fraction of individuals misreport their preferences to manipulate the Boston mechanism than the deferred acceptance mechanism in an experimental environment (Chen and Sönmez, 2006);[1] if some students sincerely report their preferences truthfully while others are sophisticated in the sense of taking best responses, then sophisticated students can be better off under the Boston mechanism than in the deferred acceptance mechanism at the expense of the sincere students (Pathak and Sönmez, 2008). Consistently with recommendations by these studies, the Boston mechanism was recently replaced by the deferred acceptance mechanism in Boston.

Despite these suggested shortcomings, the Boston mechanism continues to be a very popular student placement mechanism. School districts in the U.S. mentioned by Abdulkadiroğlu and Sönmez (2003), such as Minneapolis and Lee County of Florida, are just a few of the examples. The mech-

[1]Related experiments are conducted by Pais and Pinter (2007) and Calsamiglia, Haeringer, and Klijn (2009).

anism is used in many school choice systems around the globe. Even outside the school choice context, mechanisms similar to the Boston mechanisms have been tried in various times and contexts although they have sometimes failed: Examples include university housing assignment (Hylland and Zeckhauser, 1979) and labor market clearinghouses for doctors (Roth, 1991). Another interesting case is the school choice program in Seattle. The district recently started using a mechanism inspired by deferred acceptance instead of the Boston mechanism, but the system will transit back to a version of the Boston mechanism starting with the 2011 enrollment season.[2]

Even in Boston, where the deferred acceptance mechanism is adopted and still being used instead of the Boston mechanism, some parents raised concerns about the change as follows:

> I'm troubled that you're considering a system that takes away the little power that parents have to prioritize... what you call this strategizing as if strategizing is a dirty word... (Recording from Public Hearing by the School Committee, 05/11/2004).

Furthermore, there is a recent surge in research that finds advantages of the Boston mechanism over the deferred acceptance mechanism. Abdulkadiroğlu, Che, and Yasuda (2009) find an (incomplete information) environment in which students are better off at any symmetric Bayesian equilibrium under the Boston mechanism than under the deferred acceptance mechanism. Moreover, some sincere players may be placed better in highly ranked schools under the Boston mechanism than under the deferred acceptance mechanism.[3] In a similar model, Miralles (2008) shows that the Boston mechanism outperforms the deferred acceptance mechanism with respect to certain ex-ante efficiency criteria. Featherstone and Niederle (2008) offer a setting where truth-telling is an equilibrium under the Boston mechanism and efficiency under the mechanism is higher than under the deferred acceptance mechanism.[4] They also conduct laboratory experiments whose outcomes confirm their predictions.

Almost all research and popular opinions on the Boston mechanism focus on an important welfare property: Under the mechanism, each school admits all qualified students who rank it higher before

[2]Seattle Public Schools, New Student Assignment Plan Transition Plan for 2010-11, January 2010 draft, retrieved from http://www.seattleschools.org/area/board/09-10agendas/010610agenda/nsaptransitionattachment.pdf on 01/27/2010.

[3]Based on their findings, Abdulkadiroğlu, Che, and Yasuda (2008) propose a mechanism that could be regarded as a hybrid of the deferred acceptance and the Boston mechanisms.

[4]Ehlers (2008) shows that Boston mechanism is difficult to manipulate when there is limited symmetric information.

admitting any student who ranks it lower. This property is intuitive as a welfare criterion, and also enables students to express strength of their preferences for a particular school by ranking that school higher. However, the same property causes many of the Boston mechanism's shortcomings, since it promotes preference manipulations in a very obvious manner: A student may put a school higher than it actually is in the hope of being admitted to that school.

In this paper we aim to provide a basic understanding of the good and bad properties of the Boston mechanism using axiomatic tools. First, we formalize the aforementioned welfare property of the Boston mechanism: We say that a mechanism respects preference rankings if whenever a qualified student prefers a school to the school assigned by the mechanism, all the seats of the former are allocated to students who rank it at least as high as the initial student. The Boston mechanism satisfies this property. We further show that all mechanisms respecting preference rankings have certain efficiency property (Proposition 1). The Boston mechanism shares this welfare property because it respects preference rankings.

Then we note that respect of preference rankings is in a fundamental conflict with strategy-proofness, let alone group strategy-proofness: As mentioned before, a student may benefit by ranking a certain school higher than it actually is. This fact also means that the Boston mechanism violates the famous Maskin monotonicity, because group strategy-proofness is known to be equivalent to Maskin monotonicity in our environment. Faced with this conflict, we introduce a new property that relaxes Maskin monotonicity so that it can be consistent with respect of preference rankings. We call our new concept rank-respecting Maskin monotonicity.

We show that these two new properties and mild fairness axioms fully characterize the class of Boston mechanisms (since each priority profile induces a Boston mechanism, we refer to the class of such mechanisms as Boston mechanisms in the plural form). We provide two independent characterizations of the Boston mechanisms using different axioms.

Our first result, Theorem 1, states that a mechanism respects preference rankings and satisfies rank-respecting Maskin monotonicity, consistency, and resource monotonicity if and only if it is in the class of Boston mechanisms. A mechanism is consistent if, whenever we fix the school assignment of a student at the mechanism's outcome and re-run the mechanism for the remaining group of students with one less seat at the fixed school of the initial student, all remaining students are assigned the same schools. Consistency can be seen as a robustness property of a school choice mechanism. A mechanism is resource-monotonic if increasing the capacity of a school makes all students weakly better off. These axioms are commonly considered reasonable.

The other result is motivated by general resource allocation problems. In problems such as housing and office allocation, it is natural to assume that there is a single copy of each object unlike in the school choice problem where each school has multiple seats to fill. Theorem 2 shows that when each object to be distributed has a single copy, a mechanism respects preference rankings and satisfies individual rationality, rank-respecting Maskin monotonicity, and population monotonicity if and only if it is in the class of Boston mechanisms. A mechanism is population-monotonic if whenever a student is removed from the problem, the mechanism's outcome makes every remaining student weakly better off than its original outcome. We also show that our properties are independent in each of our characterizations, confirming that each axiom is indispensable.

Both of our results use respect of preference rankings and rank-respecting Maskin monotonicity, while employing different combinations of standard axioms for the two characterizations. Moreover, rank-respecting Maskin monotonicity is a weakening of the famous Maskin monotonicity property to accommodate respect of preference rankings. Thus respect of preference rankings introduced in this paper is the main cause of the repsesentative features of the Boston mechanism, and once this property is accepted, then a Boston mechanism is to be adopted. The results also suggest that the weakening of Maskin monotonicity, which is necessitated by the respect of preference rankings, causes Boston mechanisms to be strategically manipulable (recall that Maskin monotonicity is equivalent to group strategy-proofness in our setup).

Related Literature

While the current paper is the first to characterize the Boston mechanisms, other allocation mechanisms have been previously characterized. The closest study to ours is Kojima and Manea (2009), who axiomatize the class of the deferred acceptance mechanisms with substitutable (and acceptant) priorities. Their paper and the current paper complement each other, as these two studies together provide characterizations of the main competing mechanisms in school choice. The analysis by Kojima and Manea (2009) is followed by Ehlers and Klaus (2009), who axiomatize deferred acceptance mechanisms with responsive priorities.[5] Papai (2000) characterizes the hierarchical exchange mech-

[5]Ehlers and Klaus (2003, 2006) characterize the class of deferred acceptance mechanisms with acyclic priority structures (Ergin, 2002).

anisms, which generalize the priority-based top trading cycle mechanisms of Abdulkadiroğlu and Sönmez (2003). Pycia and Ünver (2009) characterize a more general class of mechanisms by group strategy-proofness and Pareto efficiency. **??**) and Thomson (2007) give comprehensive surveys on allocation mechanisms and axiomatic studies on the subject.

More broadly, this study is part of the rapidly growing literature on school choice mechanisms.[6] Practical considerations in designing school choice mechanisms in Boston and New York City are discussed by Abdulkadiroğlu, Pathak, and Roth (2005, 2009) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005, 2006). Erdil and Ergin (2006) and Kesten (2009) propose alternative mechanisms that may produce more efficient student placements than those used in New York City and Boston. When there are no priorities, as in the supplementary round of New York City's student placement process, the probabilistic serial mechanism (Bogomolnaia and Moulin, 2001) is more efficient than the current random priority mechanism, but the former mechanism is not strategy-proof. Kojima and Manea (2008) show that the probabilistic serial mechanism is incentive compatible when the number of seats in each school is sufficiently large. Che and Kojima (2009) subsequently show that these two mechanisms are asymptotically equivalent as the market becomes infinitely large.[7] Kesten and Ünver (2009) propose a generalization of the probabilistic serial mechanism for school choice systems with priorities. Ergin (2002) shows that the deferred acceptance mechanism is Pareto efficient if and only if the priority structure is acyclic. The acyclic priority structure has proved crucial for the deferred acceptance mechanisms to have a number of desirable properties (Haeringer and Klijn, 2009; Kesten, 2006a,b; Ehlers and Erdil, 2009).

Finally, this paper is part of an extensive field of matching theory initiated by Gale and Shapley (1962). The field is too large to summarize in this paper: Roth and Sotomayor (1990) provide a comprehensive survey of the early literature, while more recent advances are discussed by Roth (2008) and Sönmez and Ünver (2008).

[6]See also an earlier contribution by Balinski and Sönmez (1999).

[7]Manea (2006) provides sufficient conditions for asymptotic efficiency and inefficiency of the random priority mechanism.

2 The Model

Let I and C be nonempty and finite sets of **students** and **schools.** Each student shall be assigned to a school or remain unmatched. Each school $c \in C$ has a maximum capacity of students to admit, referred to as its **quota** and denoted by q_c. Let $q = (q_c)_{c \in C}$ be the quota vector associated with the schools. We refer to being unmatched as being matched to a **null school** $\emptyset$. The null school is interpreted as an outside option, such as a school in a different district or a private school. We set the quota of the null school as $q_\emptyset = \infty$. Each student i has a **strict preference relation** (denoted by P_i) over $C \cup \{\emptyset\}$. Denote the set of strict preference relations by $\mathcal{P}$. Let $P = (P_i)_{i \in I} \in \mathcal{P}^{|I|}$ be a **preference profile**. Denote by R_i the weak preference relation associated with P_i, i.e., cR_id if and only if $c = d$ or cP_id. Let $P_i(c)$ be the ranking of school c at P_i, i.e., if school c is the ℓ^{th} choice of student i under P_i, then $P_i(c) = \ell$. Thus, for all $c, d \in C \cup \{\emptyset\}$, $P_i(c) < P_i(d)$ if and only if cP_id. A (**school choice) problem** is specified by I, C, P, and q.

A **matching** is a function $\mu : I \to C \cup \{\emptyset\}$ where μ_i is the school assigned to student i for all $i \in I$. For each $c \in C \cup \{\emptyset\}$, we write $\mu_c = \{i \in I : \mu_i = c\}$ for the set of students assigned to school c. We require that any matching μ satisfy $|\mu_c| \leq q_c$ for all $c \in C \cup \{\emptyset\}$, i.e., no school is assigned to more students than its quota.

A (**school choice) mechanism** is a systematic procedure that assigns a matching for each problem. Throughout the paper we fix I and C and denote a problem simply by its preference profile and quota vector $[P; q]$. For any problem $[P; q]$, let $\mathcal{M}[q]$ be the set of matchings and let $\varphi[P; q] \in \mathcal{M}[q]$ be the matching generated by mechanism φ at $[P; q]$.

3 The Boston Mechanism

The most commonly used school choice mechanism is the so-called Boston mechanism (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005, 2006). The mechanism was used by the Boston Public Schools until 2005 and is currently in use in Lee County of Florida and Minneapolis among other districts.

To define the mechanism, we introduce some additional terminology. For each school $c \in C$, a **priority order** $\succ_c$ is a linear order over the set of students and a vacant position denoted $\emptyset$. We interpret that if $i \succ_c j$ then student i has a higher priority at school c than student j, and if $\emptyset \succ_c i$ then student i is unacceptable at school c. Let Π be the set of priority orderings. Let $\succ = (\succ_c)_{c \in C} \in \Pi^{|C|}$ be a priority order profile. Formally, the **Boston (school choice) mechanism** at a priority order profile $\succ$ is defined through the following iterative algorithm for each problem $[P; q]$:

Algorithm 1 The Boston mechanism:

Step 1: Only the top choices of the students at P are considered. For each school c, consider the students who have listed it as their top choice and assign seats of the school c to these students one at a time following their priority order at $\succ_c$ until either there are no seats left at c (i.e., q_c students have been assigned) or there is no student left who has listed it as her top choice and is acceptable to c.

$\vdots$

Step ℓ: Consider the remaining students. Only the ℓ^{th} choices of these students at P are considered. For each school c still with available seats, consider the students who have listed it as their ℓ^{th} choice and assign the remaining seats of c to these students one at a time following their priority order at $\succ_c$ until either there are no seats left at c (i.e., all q_c seats have been assigned in the current and previous steps) or there is no student left who has listed it as her ℓ^{th} choice and is acceptable to c.

Let $\psi^{\succ}[P; q]$ denote the resulting matching of the Boston mechanism induced by priority profile $\succ$ in problem $[P; q]$.

Example 1 How does the Boston mechanism assign students to schools? Consider a problem with $I = \{1, 2, ..., 5\}$ and $C = \{a, b, c, d, e\}$ where the quota of each school is one. Let P be given as

P_1	P_2	P_3	P_4	P_5
d	b	d	a	b
a	d	$\vdots$	$\vdots$	$\vdots$
e	e			
$\vdots$	$\emptyset$			
	$\vdots$			

Consider the Boston mechanism induced by the following priority profile $\succ$:

$\succ_a = \succ_b$	$\succ_c$	$\succ_d$	$\succ_e$
1	$\vdots$	2	1
5		3	2
2		1	$\vdots$
4		$\vdots$	
$\vdots$			

Matching $\psi^{\succ}[P;q]$ is found as follows.

Step 1:

- 1 and 3 apply to d. Since $3 \succ_d 1$ and $q_d = 1$, only 3 is admitted to d.
- 2 and 5 apply to b. Since $5 \succ_b 2$ and $q_b = 1$, only 5 is admitted to b.
- 4 applies to a and is admitted.

Step 2:

- 1 applies to a. Since quota of a is already full, she is not admitted.
- 2 applies to d. Since quota of d is already full, she is not admitted.

Step 3:

- 1 and 2 apply to e. Since $1 \succ_e 2$ and $q_e = 1$, only 1 is admitted to e.

Step 4:

- 2 applies to $\emptyset$ and is admitted.

The algorithm terminates with the resulting matching

$$\psi^{\succ}[P;q] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ e & \emptyset & d & a & b \end{pmatrix}.$$

An important observation about the mechanics of the algorithm is that a student may have justified envy in the resulting matching–a school that a student prefers to her assigned school may be attended by students with lower priority for it. For example, although student 1 prefers a to $\psi_1^{\succ}[P;q] = e$ and has higher priority at school a than student $\psi_a^{\succ}[P,q] = 4$, 4 is admitted while 1 is not. This is because the Boston mechanism matches seats at each school to students who rank it high. In the current problem, for instance, 4 is matched to a while 1 is not since 4 ranks a higher than 1 does. Similarly, student 2 is not assigned to a more preferred school d than her assignment $\emptyset$ while student 3 is assigned to d, even though she has lower priority at d, because 3 ranks d higher than 2 does.

Remark 1 *It is easy to see that no two distinct priority profiles induce the same Boston mechanism (except for irrelevant parts of orders among unacceptable students). Thus the characterization results we offer in the sequel lead to unique representations.*

The Boston mechanism has an intuitive welfare maximization property, subject to some feasibility constraints. More specifically, one interpretation of the mechanism is that it assigns artificial utility indices to the alternatives consistent with each individual's preferences and finds a matching that maximizes total welfare of the economy.[8] In Appendix E, we present a particular welfare maximization problem using linear programming whose solution is equivalent to the outcome of the Boston mechanism for a given priority profile.

The popularity of the Boston mechanism may be due to this seemingly appealing welfare maximization property. In this regard, note that other welfare-maximizing mechanisms such as the Walrasian mechanism are widely applauded as desirable solutions for many economic problems. However, this promising property of the mechanism comes at a cost. Like many other welfare-based solutions, the Boston mechanism is vulnerable to manipulation of preference revelations by students. For example, in Example 1, student 1 can successfully manipulate the mechanism by submitting

P_1'
a
e
$\vdots$

instead of P_1 and be matched with a instead of e.

[8]Note that no information about the utility functions of individuals are provided in the description of the problem: Rather, only ordinal preferences are primitive in the problem.

4 Axioms

In this section, we introduce several axioms regarding school choice mechanisms. First, we introduce new axioms that play central roles in the remainder of the paper. To do this, for any $c \in C \cup \{\emptyset\}$, let $I_c = \{i \in I | \varphi_i[P; q] = c \text{ for some } P, q\}$ be the set of students who are matched to c for at least one preference-quota profile under φ. The set I_c is the set of "qualified" students at c in the sense that there is an instance in which they are matched to c. Given mechanism φ, and hence I_c for each $c \in C \cup \{\emptyset\}$ induced by φ, matching μ **respects preference rankings** for φ if, for all $c \in C \cup \{\emptyset\}$ and $i \in I_c$, $cP_i\mu_i$ implies $|\mu_c| = q_c$ and $P_j(c) \leq P_i(c)$ for all $j \in \mu_c$. That is, a matching respects preference rankings if whenever a student $i \in I_c$ is assigned a worse school than a school c, then c's quota is filled with students who rank c at least as high as student i. A mechanism **respects preference rankings** if for every problem, it finds a matching that respects preference rankings. Respecting preference rankings is a representative feature of the Boston mechanism, as illustrated in Example 1. Naturally, properties similar to respect of preference rankings have been extensively studied in the literature on the Boston mechanism. However, the property has not been formalized in the form presented here.

One motivation of respecting preference rankings is based on welfare considerations. Intuitively, the social planner respecting preference rankings tries to assign school seats to students who value them as highly as possible. Of course, Pareto efficiency is not necessarily guaranteed even if respect of preference rankings holds: This is because we allow some student i to be simply unacceptable to some school c (i.e., $i \notin I_c$). However, there is a sense in which respect of preference rankings implies a milder concept of efficiency. Given mechanism φ, and hence I_c for each $c \in C \cup \{\emptyset\}$ induced by φ, a matching $\mu \in \mathcal{M}[q]$ is **constrained Pareto-efficient** for φ at a problem $[P; q]$ if there exists no matching $\nu \in \mathcal{M}[q]$ such that $\nu_i R_i \mu_i$ for all $i \in I$ and $\nu_i P_i \mu_i$ for some $i \in I$ as well as $i \in I_{\nu_i}$ for every $i \in I$. In words, a matching is constrained Pareto-efficient if there is no matching that makes every student weakly better off and at least one student strictly better off while no unqualified student is admitted to a school. A mechanism is **constrained Pareto-efficient** if $\varphi[P; q]$ is constrained Pareto-efficient for φ at every problem $[P; q]$. The following proposition establishes a relationship between respecting preference rankings and constrained Pareto efficiency.

Proposition 1 *If matching μ respects preference rankings, then it is constrained Pareto-efficient.*

In some school districts, the law may require that all students are qualified for all schools. Under this presumption, Proposition 1 asserts that respect of preference rankings implies Pareto efficiency.

Before introducing the next axiom, we review a standard axiom in the implementation literature. We say that P_i' is a **monotonic transformation** of P_i at $c \in C \cup \{\emptyset\}$ (P_i' *m.t.* P_i at c) if every school that is ranked above c under P_i' is ranked above c under P_i, i.e.,

$$b\, P_i'\, c \;\; \Rightarrow \; b\, P_i\, c, \; \forall b \in C \cup \{\emptyset\}.$$

P' is a **monotonic transformation** of P at a matching μ (P' *m.t.* P *at* μ) if P_i' *m.t.* P_i at μ_i for all $i \in I$. A mechanism φ satisfies **Maskin monotonicity** (Maskin, 1999) if

$$P' \; m.t. \; P \; at \; \varphi[P; q] \Rightarrow \varphi[P', q] \;\; = \;\; \varphi[P; q].$$

Maskin monotonicity is regarded as a reasonable property and is equivalent to group strategy-proofness in the current setting (**?**). The property is indeed satisfied by a number of promising mechanisms. Examples include the priority-based top-trading cycles mechanisms (Abdulkadiroğlu and Sönmez, 2003) and their generalizations such as the hierarchical exchange mechanisms (Papai, 2000) and the trading cycles with brokers and owners (Pycia and Ünver, 2009).

However, the Boston mechanism may violate Maskin monotonicity. To see this point consider, for instance, a change in student 1's reported preferences in Example 1. Although $(P_1', P_{-1})\, m.t.\, P$ at $\psi^{\succ}[P; q]$, we have $\psi_1^{\succ}[P_1', P_{-1}; q] = a \neq e = \psi_1^{\succ}[P; q]$.

This observation motivates a certain weakening of Maskin monotonicity. More specifically, we seek a condition that obtains the same conclusion as Maskin monotonicity does, but for which the hypothesis is restricted in such a way that the condition is not in conflict with respecting preference rankings. To proceed, define

$$U_i(P, \mu) = \{j \in I \;:\; P_j(\mu_i) \leq P_i(\mu_i), \; P_j(\mu_i) \leq P_j(\mu_j)\},$$
$$V_i(P, \mu) = \{j \in I \;:\; P_j(\mu_i) < P_i(\mu_i), \; P_j(\mu_i) \leq P_j(\mu_j)\}.$$

In words, the set $U_i(P, \mu)$ (resp. $V_i(P, \mu)$) is the group of students who, at preference profile P, rank μ_i weakly (resp. strictly) higher than i does and weakly higher than their assignments at μ. Intuitively, they are the sets of students who are potentially in competition with i for a seat in school μ_i. We say that P' is a **rank-respecting monotonic transformation** of P at a matching μ (P' *r.r.m.t.* P *at* μ) if P' *m.t.* P *at* μ and, for all i with $\mu_i \in C$, $U_i(P', \mu) \subseteq U_i(P, \mu)$ and $V_i(P', \mu) \subseteq V_i(P, \mu)$. A mechanism φ satisfies **rank-respecting (r.r.) Maskin monotonicity** if

$$P' \; r.r.m.t. \; P \; at \; \varphi[P; q] \Rightarrow \varphi[P'; q] \;\; = \;\; \varphi[P; q].$$

In words, r.r. Maskin monotonicity requires that a matching is unchanged when students promote the rankings of their original assignments, as long as doing so does not increase the competition for schools assigned to others. Thus r.r. Maskin monotonicity does not conflict with respecting preference rankings and is satisfied by the Boston mechanism.

Next we introduce axioms that are standard in the literature.

A mechanism φ is **resource-monotonic** if for all $P \in \mathcal{P}^{|I|}$, if $q_c \geq q'_c$ for all $c \in C$, then $\varphi_i[P;q]\, R_i \varphi_i[P;q']$ for all $i \in I$ (Thomson, 1978). In words, a mechanism is resource-monotonic if whenever the supply of school seats increases, the mechanism's outcome makes each student weakly better off than its original outcome.

A mechanism φ is **individually rational** if $\varphi_i[P;q] R_i \emptyset$ for all P, q and $i \in I$. Note that any resource-monotonic mechanism φ satisfies

$$\varphi_i[P;q]\, R_i\, \varphi_i[P;(0,\dots,0)] = \emptyset,$$

for any P, q, and $i \in I$, so any mechanism that is resource monotonic is individually rational in our context. Also, since $I_\emptyset = I$ (because $\varphi_i[P;(0,\dots,0)] = \emptyset$) and $q_\emptyset = \infty$, any mechanism that respects preference rankings is individually rational.[9]

Next, we define two properties of mechanisms regarding variable populations. We introduce one additional notation. For any $i \in I$, let $\mathcal{P}^\emptyset$ be the set of preference relations that rank $\emptyset$ as the first choice. A mechanism φ is **population-monotonic** if, for all $P \in \mathcal{P}^{|I|}$ and all $P_i^\emptyset \in \mathcal{P}^\emptyset$, we have $\varphi_j\left[P_i^\emptyset, P_{-i};q\right] R_j \varphi_j[P;q]$ for all $j \neq i$ and $\varphi_i[P_i^\emptyset, P_{-i};q] = \emptyset$ (Thomson, 1983a,b).[10] That is, a mechanism is population-monotonic if whenever a student is removed from the problem, the mechanism's outcome makes every remaining student weakly better off than its original outcome.

Population monotonicity is usually defined for variable populations. To avoid further notational complexity, we use a fixed population representation of population monotonicity. We interpret a change in preferences of student i from P_i to $P_i^\emptyset$ as a situation where student i leaves the school choice problem.

A mechanism φ is **consistent** if $\varphi_j\left[P_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \varphi_j[P;q]$ for all $j \neq i$ and

[9] Respect of preference rankings does not imply individual rationality when the quota is fixed at $q_c = 1$ for each $c \in C$ as in Section 5.2.

[10] This definition implies that for all $P_i^\emptyset, \tilde{P}_i^\emptyset \in \mathcal{P}^\emptyset, \varphi\left[P_i^\emptyset, P_{-i};q\right] = \varphi\left[\tilde{P}_i^\emptyset, P_{-i};q\right]$.

$\varphi_i\left[P_i^{\emptyset}, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \emptyset$ for all $P \in \mathcal{P}^{|I|}$ and all $P_i^{\emptyset} \in \mathcal{P}^{\emptyset}$ (Thomson, 1988).[11,12] In words, a mechanism is consistent if whenever a student is removed from the problem with her assigned school seat, the assignment for each remaining student is unchanged.

Consistency is usually defined for variable populations like population monotonicity. We use a fixed population representation of consistency, as we do for population monotonicity. We express a situation where a student leaves the problem with her assigned school by decreasing the quota of this school by one and having this student rank the null school as her top choice.

The following Lemma offers a relationship between resource monotonicity, consistency, and population monotonicity.

Lemma 1 *If a mechanism φ satisfies resource monotonicity and consistency, then it satisfies population monotonicity.*[13]

5 Characterizing the Boston Mechanism

This section presents the main results of this paper. It turns out that respect of preference rankings is a crucial property of the Boston mechanisms in the sense that this axiom, together with more standard ones, characterizes the mechanisms. We provide alternative characterizations. Respect of preference rankings is one of the axioms for each of the characterizations while different sets of other axioms appear in alternative characterizations, which suggests that this property is the main content of the Boston mechanism.

5.1 Main Result: Characterization

Theorem 1 *A mechanism φ is the Boston mechanism induced by some priority profile $\succ$, i.e., $\varphi = \psi^{\succ}$, if and only if φ respects preference rankings and satisfies consistency, resource monotonicity, and r.r. Maskin monotonicity.*

[11] It follows that, for all $P_i^{\emptyset}, \tilde{P}_i^{\emptyset} \in \mathcal{P}^{\emptyset}$, $\varphi\left[P_i^{\emptyset}, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \varphi\left[\tilde{P}_i^{\emptyset}, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right]$.

[12] We stipulate $\infty - 1 = \infty$, so consistency implies that when an unmatched student is removed from the problem, assignments for all remaining students are unchanged.

[13] Although we suspect that this lemma is well known, we were unable to find a reference.

As pointed out in Remark 1, no two distinct priority profiles induce the same Boston mechanism. Therefore, the representation of the mechanism in Theorem 1 in terms of the Boston mechanism is unique.

The following examples show that the axioms in Theorem 1 are independent when $|I| \geq 3$ and $|C| \geq 2$ (in Appendix F we study all other cases and show that the axioms are not independent when $|I| \leq 2$ or $|C| = 1$).

Example 2 A mechanism violating only respect of preference rankings: Fix $i \in I$ and $c \in C$. Consider the mechanism such that

1. if $q_c \geq 2$ and $cP_i\emptyset$, then assign c to i and $\emptyset$ to every other student, and

2. otherwise, assign $\emptyset$ to every student.

This mechanism satisfies consistency, resource monotonicity, and r.r. Maskin monotonicity, but does not respect preference rankings. To show that the mechanism violates respect of preference rankings, first note that i is assigned c in Case (1) above by assumption, so $i \in I_c$. Next, suppose that $q_c = 1$ and $cP_i\emptyset$. Then $\varphi_i[P; q] = \emptyset$ by assumption of Case (2) and, moreover, no student different from i is matched to c even though $cP_i\emptyset$. Thus respect of preference rankings is violated.

Example 3 A mechanism violating only consistency: Suppose that $|I| \geq 3$: Let $I = \{1, \ldots, n\}$ with $n \geq 3$. Fix a school $c \in C$ and define priority orders $\succ_c$ and $\succ'_c$ by

$\succ_c$	$\succ'_c$
1	1
2	3
3	2
4	4
$\vdots$	$\vdots$
n	n
$\emptyset$	$\emptyset$

For all $d \neq c$, let

$\succ_d$
1
2
3
4
$\vdots$
n
$\emptyset$

Consider the mechanism φ defined by

$$\varphi[P;q] = \begin{cases} \psi^{\succ'_c, \succ_{-c}}[P;q] & \text{if } q_c = 1, \\ \psi^{\succ}[P;q] & \text{otherwise.} \end{cases}$$

It can be verified that φ respects preference rankings and satisfies resource monotonicity and r.r. Maskin monotonicity. To see that consistency is violated, consider a quota profile q with $q_c = 2$ and preference profile P where c is top-ranked by every student. Students 1 and 2 are matched to c at $\varphi[P;q]$, but student 3 is matched to c at $\varphi[P_1^{\emptyset}, P_{-1}; q_c - 1, q_{-c}]$. Therefore φ does not satisfy consistency.

Example 4 A mechanism violating only resource monotonicity: Suppose that $|C| \geq 2$ and $|I| \geq 2$: Let $I = \{1, \ldots, n\}$. For each $d \in C$, define priority orders $\succ_d, \succ'_d, \succ''_d \in \Pi$ by

$\succ_d$	$\succ'_d$	$\succ''_d$
1	2	$\emptyset$
2	1	$\vdots$
3	3	
$\vdots$	$\vdots$	
n	n	
$\emptyset$	$\emptyset$	

Fix some school $c \in C$ and define mechanism φ by, for any problem $[P;q]$,

$$\varphi[P;q] = \begin{cases} \psi^{\succ''_c, \succ_{-c}}[P;q] & \text{if } q_c \geq 2, \\ \psi^{\succ''_c, \succ'_{-c}}[P;q] & \text{otherwise.} \end{cases}$$

It can be shown that φ respects preference rankings and satisfies consistency and r.r. Maskin monotonicity. On the other hand, it violates resource monotonicity. To see this point consider preference and quota profiles P and q where there is a school $c' \in C \setminus c$ (such a school exists since $|C| \geq 2$) that is top-ranked by every student at P and $q_c = 2, q_{c'} = 1$. At this profile, student 2 is not matched to her most preferred school c'. If q_c is reduced to 1, student 2 is matched to her most preferred school c', violating resource monotonicity.

Example 5 A mechanism violating only r.r. Maskin monotonicity: Suppose that $|C| \geq 2$ and $|I| \geq 2$: Let $I = \{1, ..., n\}$ for some $n \geq 2$. Fix $c \in C$. Define priority orders $\succ_c, \succ'_c \in \Pi$ such that

$\succ_c$	$\succ'_c$
1	2
2	3
3	4
4	$\vdots$
$\vdots$	n
n	1
$\emptyset$	$\emptyset$

Fix some priority order profile $\succ_{-c} \in \Pi^{|C|-1}$ for other schools. Fix a school $d \neq c$ (such a school exists since $|C| \geq 2$). Define mechanism φ such that for any problem $[P; q]$,

$$\varphi [P; q] = \begin{cases} \psi^{\succ_c, \succ_{-c}} [P; q] & \text{if } P_1 (d) < P_1 (\emptyset), \\ \psi^{\succ'_c, \succ_{-c}} [P; q] & \text{otherwise.} \end{cases}$$

It can be shown that φ respects preference rankings and satisfies consistency and resource monotonicity. On the other hand, it violates r.r. Maskin monotonicity. To see this, consider $P_1, P'_1 \in \mathcal{P}$ satisfying

P_1	P'_1
c	c
d	$\emptyset$
$\emptyset$	d
$\vdots$	$\vdots$

Let $P_{-1} \in \mathcal{P}^{|I|-1}$ be such that $P_2(c) = 1$. Let q be a quota vector with $q_c = 1$. Then, $\varphi_1[P;q] = \psi_1^{\succ_c, \succ_{-c}}[P;q] = c$. Thus, (P_1', P_{-1}) is a rank-respecting monotonic transformation of P at $\varphi[P;q]$, yet $\varphi_1[P_1', P_{-1};q] = \psi_1^{\succ_c', \succ_{-c}}[P_1', P_{-1};q] = \emptyset$, violating r.r. Maskin monotonicity.

5.2 Single-Unit Supply

We consider an environment in which q_c is fixed at 1 for all $c \in C$. In this setup, in the statement of the axioms, we do not allow q to vary. Hence, we denote a problem by its preference profile P and the matching produced under a mechanism φ for this problem as $\varphi[P]$. For example, the set I_c is defined as $I_c = \{i \in I | \varphi_i[P] = c \text{ for some } P\}$ for the fixed quota profile $q = (1, 1, ..., 1)$. As a consequence, respect of preference rankings does not imply individual rationality in this environment unlike in the case in which quotas are allowed to vary.

While the assumption that each school has quota one may not hold in the context of student placement in schools, it may be satisfied in problems such as housing or office allocation where each object is arguably unique. Moreover, this restriction enables a simpler characterization of the Boston mechanism as stated below.

Theorem 2 *Fix the quota profile at q with $q_c = 1$ for all $c \in C$. A mechanism φ is the Boston mechanism induced by some priority profile $\succ$, i.e., $\varphi = \psi^{\succ}$, if and only if φ respects preference rankings and satisfies individual rationality, population monotonicity, and r.r. Maskin monotonicity.*

The following examples show that the axioms in Theorem 2 are independent when $|I| \geq 3$ and $|C| \geq 2$ (in Appendix F we study all other cases and show that the axioms are not independent when $|I| \leq 2$ or $|C| = 1$).

Example 6 A mechanism violating only respect of preference rankings: Suppose that $|C| \geq 2$ and $|I| \geq 2$. Fix a school $c \in C$ and students $1, 2 \in I$ (such a school and students exist by assumption). Consider a mechanism such that

$$\varphi_i[P] = \begin{cases} c & \text{if } i = 1 \text{ and } cP_1\emptyset \text{ or} \\ & \quad i = 2,\ \emptyset P_1 c \text{ and } cP_2\emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

It can be verified that φ satisfies individual rationality, population monotonicity and r.r. Maskin monotonicity. On the other hand, the mechanism violates respect of preference rankings. To see this point first observe that $2 \in I_c$ by the construction of the mechanism. Consider a preference profile P such that

P_1	P_2	P_i $(i \neq 1,2)$
d	c	$\emptyset$
c	$\vdots$	$\vdots$
$\vdots$		

where d is a school different from c (such a school exists since $|C| \geq 2$). By the definition of the mechanism φ, it follows that $c\,P_2\,\emptyset = \varphi_2[P]$ although $2 \in I_c$ and there is no student $i \neq 2$ with $P_i(c) \leq P_2(c)$, violating respect of preference rankings.

Example 7 A mechanism violating only individual rationality: Fix $c \in C$ and $i \in I$. Consider a mechanism φ such that $\varphi_i[P] = c$ and $\varphi_j[P] = \emptyset$ for all $j \neq i$ at every preference profile P. The mechanism trivially respects preference rankings and satisfies population monotonicity and r.r. Maskin monotonicity. Mechanism φ violates individual rationality since for any preference profile P such that $\emptyset\,P_i\,c$, we have $\emptyset\,P_i\,c = \varphi_i[P]$.

Example 8 A mechanism violating only population monotonicity: Suppose that $|I| \geq 3$ and $|C| \geq 2$: Let $I = \{1, 2, 3, \ldots, n\}$. For each $c \in C$, define priority orders $\succ_c, \succ'_c \in \Pi$ such that

$\succ_c$	$\succ'_c$
1	1
2	3
3	2
$\emptyset$	$\emptyset$
$\vdots$	$\vdots$

Let $\succ = (\succ_c)_{c \in C}$ and $\succ' = (\succ'_c)_{c \in C}$. Define mechanism φ by,

$$\varphi\,[P] = \begin{cases} \psi^{\succ}\,[P] & \text{if } P_1(\emptyset) = 1, \\ \psi^{\succ'}[P] & \text{otherwise.} \end{cases}$$

Mechanism φ respects preference rankings and satisfies individual rationality and r.r. Maskin monotonicity. On the other hand, φ violates population monotonicity. To see this point consider a preference profile P such that student 1 top-ranks a school $c \in C$ and all other students top-rank $d \in C \setminus c$ (such schools c and d exist by the assumption $|C| \geq 2$). Then $\varphi_3[P] = d$ by definition of the mechanism φ. However, by definition of φ it follows that $\varphi_3[P_1^{\emptyset}, P_{-1}] \neq d$, thus violating population monotonicity.

Example 9 A mechanism violating only r.r. Maskin monotonicity: Suppose that $|I| \geq 2$ and $|C| \geq 2$: The mechanism presented in Example 5 satisfies respect of preference rankings, individual rationality, and population monotonicity, while violating r.r. Maskin monotonicity.

A Proof of Proposition 1

For contradiction, assume that matching μ respects preference rankings but is not constrained Pareto-efficient. Since μ respects preference rankings, there exists no school c and student i such that $i \in I_c$, $|\mu_c| < q_c$ and $cP_i\mu_i$. This and the assumption that μ is not constrained Pareto-efficient imply that there exists a sequence of students $i_1, i_2, \ldots, i_n$ such that $i_k \in I_{\mu_{i_{k+1}}}$, and $\mu_{i_{k+1}} P_{i_k} \mu_{i_k}$ or equivalently $P_{i_k}(\mu_{i_{k+1}}) < P_{i_k}(\mu_{i_k})$ for all $k = 1, \ldots, n$ (with the convention that $n + 1 = 1$). Since μ respects preference rankings, $P_{i_{k+1}}(\mu_{i_{k+1}}) \leq P_{i_k}(\mu_{i_{k+1}})$ for all $k = 1, \ldots, n$. Combining these inequalities, we obtain

$$P_{i_1}(\mu_{i_1}) \leq P_{i_n}(\mu_{i_1}) < P_{i_n}(\mu_{i_n}) \leq P_{i_{n-1}}(\mu_{i_n}) < P_{i_{n-1}}(\mu_{i_{n-1}}) \leq \cdots \leq P_{i_1}(\mu_{i_2}) < P_{i_1}(\mu_{i_1}),$$

a contradiction.

B Proof of Lemma 1

Suppose that φ satisfies resource monotonicity and consistency, and fix P, q, and i arbitrarily. If $\varphi_i[P;q] = \emptyset$, then consistency implies

$$\varphi_j[P_i^{\emptyset}, P_{-i}; q] = \varphi_j[P;q] \text{ for all } j \neq i \text{ and all } P_i^{\emptyset} \in \mathcal{P}^{\emptyset}.$$

Suppose $\varphi_i[P;q] \neq \emptyset$. Then, by consistency, $\varphi_j\left[P_i^{\emptyset}, P_{-i}; q_{\varphi_i[P;q]} - 1; q_{-\varphi_i[P;q]}\right] = \varphi_j[P;q]$ for all $j \neq i$ and all $P_i^{\emptyset} \in \mathcal{P}^{\emptyset}$. Hence, resource monotonicity implies that for all $j \neq i$

$$\varphi_j\left[P_i^{\emptyset}, P_{-i}; q\right] R_j \varphi_j\left[P_i^{\emptyset}, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \varphi_j[P;q].$$

The above displayed relations show that φ satisfies population monotonicity.

C Proof of Theorem 1

It is straightforward to see that the Boston mechanism for an arbitrary priority profile $\succ$ satisfies all the axioms in the statement. Thus, we show the converse.

Based on Lemma 1, we invoke population monotonicity of φ in various parts of the proof. For any $c \in C$ and $I' \subseteq I$, let $\mathcal{P}^{|I|}\langle c; I' \rangle$ be the set of preference profiles such that all students in I' rank c as the first choice and all other students rank $\emptyset$ as the first choice. We prove the theorem in two parts.

C.1 Part 1: Construction of Priority Profile $\succ$

For each $c \in C$, construct $\succ_c$ as follows: Let $q_c = 1$ and $q_{c'} = 0$ for all $c' \in C \setminus \{c\}$. Fix some $P^{(1)} \in \mathcal{P}^{|I|}\langle c, I \rangle$ and let the top priority student under $\succ_c$, denoted by i_c^1, be the student in $\varphi_c[P^1; q]$ if any (who is unique, if existent, since $q_c = 1$). Iteratively continue, such that: For each $\ell \geq 2$, fix some $P^{(\ell)} \in \mathcal{P}^{|I|}\left\langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}\right\rangle$ and let the ℓ^{th} priority student $\succ_c$, denoted by i_c^ℓ, be the student in $\varphi_c[P^{(\ell)}; q]$ if any (who is unique, if existent, since $q_c = 1$). If $\varphi_c[P^{(\ell)}; q] = \emptyset$ for any $\ell \geq 1$, then order all students in $I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$ so that $\emptyset \succ_c i$ for all $i \in I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$. This procedure defines a priority order $\succ_c$. It is an implication of the following claim that the construction of $\succ_c$ is independent of the choice of preference $P^{(\ell)} \in \mathcal{P}^{|I|}\left\langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}\right\rangle$ in each step ℓ:

Claim 1 *Let $I' \subseteq I$. Suppose $\varphi_c[P; q] = i$ where $P \in \mathcal{P}^{|I|}\langle c, I' \rangle$ and $q_c = 1, q_{c'} = 0$ for all $c' \neq c$. Then $\varphi_c[P'; q'] = i$ for all q' with $q'_c = 1$ and $P' \in \mathcal{P}^{|I|}\langle c, I'' \rangle$ with $\{i\} \subseteq I'' \subseteq I'$.*

Proof. We will prove the claim in three steps:

Proof Step 1:

First, we show the claim when $q' = q$ and $I'' = I'$. Let $P \in \mathcal{P}^{|I|}\langle c, I' \rangle$ and $\varphi_i[P; q] = c$. Assume $P'_j \neq P_j$ for some $j \in I$ and $P'_k = P_k$ for all $k \neq j$ without loss of generality. Two cases are possible for the identity of j, where Case 2 has also two subcases:

Case 1 $j = i$:

Then, since $\varphi_i[P; q] = c$ and c is top-ranked both at P_i and P'_i, P' *m.t.* P at $\varphi[P; q]$ and

$$U_k(P, \varphi[P; q]) = U_k(P', \varphi[P; q]),$$

$$V_k(P, \varphi[P; q]) = V_k([P', \varphi[P; q])$$

for all $k \in I$. Thus $P'\, r.r.m.t.\, P$ at $\varphi[P;q]$. Since φ satisfies r.r. Maskin monotonicity, we conclude that $\varphi[P';q] = \varphi[P;q]$ and hence $\varphi_i[P';q] = c$.

Case 2-(i) $j \neq i$ and $\varphi_j[P';q] \neq c$:

For any $P_j^{\emptyset} \in \mathcal{P}^{\emptyset}$, since φ satisfies population monotonicity, $\varphi_i[P_j^{\emptyset}, P_{-j};q] R_i \varphi_i[P;q] = c$. Since P_i top-ranks c, we obtain

$$\varphi_i[P_j^{\emptyset}, P_{-j};q] = c. \tag{1}$$

Suppose for contradiction that $\varphi_i[P';q] \neq c$. Then, since φ respects preference rankings, $i \in I_c$, and $\varphi_j[P;q] \neq c$, there exists $i' \neq i, j$ such that $\varphi_{i'}[P';q] = c$. Thus $P_{i'} = P'_{i'}$ top-ranks c as φ respects preference rankings (otherwise i' receives $\emptyset$ by individual rationality) and, by population monotonicity,

$$\varphi_{i'}[P_j^{\emptyset}, P'_{-j};q] = c \tag{2}$$

Relations (1) and (2) contradict each other since $i' \neq i$, $q_c = 1$, and $P'_{-j} = P_{-j}$.

Case 2-(ii) $j \neq i$ and $\varphi_j[P';q] = c$:

Then, since both P_j and P'_j top-rank c (otherwise j receives $\emptyset$ by individual rationality), $P\, m.t.\, P'$ at $\varphi[P';q]$ and

$$U_k\left(P, \varphi[P';q]\right) = U_k\left(P', \varphi[P';q]\right),$$
$$V_k\left(P, \varphi[P';q]\right) = V_k\left(P', \varphi[P';q]\right)$$

for all $k \in I$. Thus $P\, r.r.m.t.\, P'$ at $\varphi[P';q]$. Since φ satisfies r.r. Maskin monotonicity we conclude $\varphi_j[P;q] = c$, a contradiction to $j \neq i$, $\varphi_i[P;q] = c$ and $q_c = 1$.

Proof Step 2:

Given the preceding argument, population monotonicity of φ implies that the claim holds for all cases when $q' = q$ and $\{i\} \subseteq I'' \subseteq I'$, noting that c is the top-ranked school at P_i.

Proof Step 3: Finally, we will show that the claim holds for all q' with $q'_c = 1$ and $P' \in \mathcal{P}^{|I|}\langle c, I''\rangle$ with $\{i\} \subseteq I'' \subseteq I'$. Thus assume that q' satisfies $q'_c = 1$ and $P' \in \mathcal{P}^{|I|}\langle c, I''\rangle$ with $\{i\} \subseteq I'' \subseteq I'$. By Proof Step 2,

$$\varphi_i[P';q] = c. \tag{3}$$

Since φ satisfies resource monotonicity and $q'_c \geq q_c$ for all $c \in C$ (as $q'_c = q_c = 1$ and $q'_{c'} \geq 0 = q_{c'}$ for all $c' \neq c$),

$$\varphi_i[P'; q'] R'_i \varphi_i[P'; q]. \tag{4}$$

Relations (3) and (4) imply

$$\varphi_i[P'; q'] R'_i c. \tag{5}$$

Since c is top-ranked under P'_i by assumption, relation (5) implies

$$\varphi_i[P'; q'] = c,$$

completing the proof. ■

C.2 Part 2: Proof that $\varphi = \psi^{\succ}$

Let $\succ = (\succ_c)_{c \in C}$ be the priority order profile constructed in Part 1. For any given preference profile P and quota profile q, we will show that $\varphi[P; q] = \psi^{\succ}[P; q]$. Construct the following student sets and quotas:

For any $c \in C \cup \{\emptyset\}$, define

$$I_c(0) = \emptyset,$$
$$J_c(0) = \emptyset,$$
$$q_c(0) = q_c.$$

For any $\ell \geq 1$, given $(I_c(0), J_c(0), q_c(0))_{c \in C \cup \{\emptyset\}}, \ldots, (I_c(\ell - 1), J_c(\ell - 1), q_c(\ell - 1))_{c \in C \cup \{\emptyset\}}$, recursively define

$$I_c(\ell) = \left\{ i \in I_c \setminus \left(\bigcup_{d \in C \cup \{\emptyset\}} \bigcup_{\ell'=1}^{\ell-1} J_d(\ell') \right) \;:\; P_i(c) = \ell \right\},$$
$$J_c(\ell) = \{ i \in I_c(\ell) \;:\; \varphi_i[P; q] = c \},$$
$$q_c(\ell) = q_c(\ell - 1) - |J_c(\ell - 1)|,$$

$q_c(\ell)$ denotes the number of seats remaining after assigning seats to students who rank c as the $(\ell - 1)^{\text{st}}$ choice or higher. $I_c(\ell)$ is the set of qualified students who rank c as their ℓ^{th} choice and have not received any higher-ranked school. $J_c(\ell)$ is the set of students in $I_c(\ell)$ who receive seats at c under $\varphi\,[P; q]$. Let $\succ_\emptyset \in \Pi$ be an arbitrary priority order. Individual rationality of φ (following from resource monotonicity of φ) implies $I_\emptyset = I$, and it suffices to show the following claim.

Claim 2 *For all* $\ell \geq 1$, $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ *for all* $c \in C \cup \{\emptyset\}$, *where*

$$\arg\max_{I_c(\ell), q_c(\ell)} \succ_c := \{i \in I_c(\ell) \ : \ |\{j \in I_c(\ell) \ : j \succeq_c i\}| \leq q_c(\ell)\},$$

is the set of (at most) $q_c(\ell)$ *students who have the highest priorities at* c *among those in* $I_c(\ell)$.

Proof. Let $c \in C \cup \{\emptyset\}$. Fix $\ell \geq 1$. If $|I_c(\ell)| \leq q_c(\ell)$, then $\arg\max_{I_c(\ell), q_c(\ell)} \succ_c = I_c(\ell)$. Then $J_c(\ell) = I_c(\ell)$ because φ respects preference rankings. Hence the conclusion $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ holds.

Thus, assume $|I_c(\ell)| > q_c(\ell)$. Suppose for contradiction that the conclusion $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ does not hold. Since φ respects preference rankings, it follows that $|J_c(\ell)| = q_c(\ell)$ and hence, there exist $i \in \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ such that $\varphi_i[P;q] \neq c$ and $j \in I_c(\ell) \setminus \left(\arg\max_{I_c(\ell), q_c(\ell)} \succ_c\right)$ such that $\varphi_j[P;q] = c$.

Let $P' = (P_i, P_j, P^{\emptyset}_{-i,j})$ for some $P^{\emptyset}_{-i,j} \in \left(\mathcal{P}^{\emptyset}\right)^{|I|-2}$. Let $q' = (q'_{c'})_{c' \in C}$ be defined by

$$q'_{c'} = q_{c'} - |\{k \in I \setminus \{i,j\} : \varphi_k[P;q] = c'\}|$$

for each $c' \in C$. By construction of $I_c(\ell)$, since $i \in I_c(\ell)$, she has not been matched to a higher choice school than c. Thus, $cP_i\varphi_i[P;q]$. Since φ is consistent,

$$\varphi_j[P';q'] = \varphi_j[P;q] = c,$$
$$\varphi_i[P';q'] = \varphi_i[P;q] \text{ and hence, } cP_i\varphi_i[P';q'].$$

Consider preference P''_i, P''_j such that c is top-ranked at P''_i and P''_j and relative rankings of all other schools are unchanged from P'_i and P'_j, respectively. Then $P'' := (P''_i, P''_j, P'_{-i,j})$ *r.r.m.t.* P' at $\varphi[P';q']$. Thus, since φ satisfies r.r. Maskin monotonicity, $\varphi_i[P'';q'] \neq c$. This is a contradiction since Claim 1 and the assumption that i has a higher priority than j at c imply $\varphi_i[P'';q'] = c$. ■

Claim 2 completes the proof of the Theorem.

D Proof of Theorem 2

The Boston mechanism for an arbitrary priority profile clearly satisfies all the axioms in the statement. Let q be the quota vector with $q_c = 1$ for all $c \in C$, which is fixed throughout the current analysis.

D.1 Part 1: Construction of Priority Profile $\succ$

For each $c \in C$, construct $\succ_c$ as follows: Fix some $P^{(1)} \in \mathcal{P}^{|I|}\langle c, I\rangle$ and let the top priority student under $\succ_c$, denoted by i_c^1, be the student in $\varphi_c[P^{(1)}]$ (who is unique, if existent, since $q_c = 1$). Iteratively continue, such that: For each $\ell \geq 2$, fix some $P^{(\ell)} \in \mathcal{P}^{|I|}\left\langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}\right\rangle$ and let the ℓ^{th} priority student $\succ_c$, denoted by i_c^ℓ, be the student in $\varphi_c[P^{(\ell)}]$ (who is unique, if existent, since $q_c = 1$). If $\varphi_c[P^{(\ell)}] = \emptyset$ for any $\ell \geq 1$, then order all students in $I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$ so that $\emptyset \succ_c i$ for all $i \in I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$. This procedure defines a priority order $\succ_c$.[14] It is an implication of the following claim that the construction of $\succ_c$ is independent of the choice of preference $P^{(\ell)} \in \mathcal{P}^{|I|}\left\langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}\right\rangle$ in each step ℓ:

Claim 3 *Let $I' \subseteq I$. Suppose $\varphi_c[P] = i$ where $P \in \mathcal{P}^{|I|}\langle c, I'\rangle$. Then $\varphi_c[P'] = i$ for all $P' \in \mathcal{P}^{|I|}\langle c, I''\rangle$ with $\{i\} \subseteq I'' \subseteq I'$.*

Proof. The proof is omitted since it is identical to Steps 1 and 2 of the proof of Claim 1. ■

D.2 Part 2: Proof that $\varphi = \psi^{\succ}$

Let $\succ = (\succ_c)_{c \in C}$ be the priority order profile constructed in Part 1. For any given preference profile P, we will show that $\varphi[P] = \psi^{\succ}[P]$. For each $c \in C \cup \{\emptyset\}$ and $\ell = 1, 2, \ldots,$ define $I_c(\ell)$, $J_c(\ell)$, and $q_c(\ell)$ as in Part 2 of the Proof of Theorem 1. Recall that $I_c(\ell)$ is the set of students who rank c as their ℓ^{th} choice and have not received any higher-ranked school (except those who cannot be matched to c under any preference profile), $J_c(\ell)$ is the set of students in $I_c(\ell)$ who receive seats at c under $\varphi[P]$, and $q_c(\ell) \in \{0, 1\}$ denotes the number of seats in c remaining after assigning seats to students who rank c as the $(\ell-1)^{\text{st}}$ choice or higher. Let $\succ_\emptyset \in \Pi$ be an arbitrary priority order. Individual rationality of φ implies $I_\emptyset = I$, and it suffices to show the following claim.

Claim 4 *For all $\ell \geq 1$, $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ for all $c \in C \cup \{\emptyset\}$, where*

$$\arg \max_{I_c(\ell), q_c(\ell)} \succ_c := \{i \in I_c(\ell) \ : \ |\{j \in I_c(\ell) \ : j \succeq_c i\}| \leq q_c(\ell)\},$$

is the set of (at most) $q_c(\ell)$ students who have the highest priorities at c among those in $I_c(\ell)$.

[14] Note that the construction of $\succ_c$ is similar to the one in the proof of Theorem 1. The only difference is that $q_{c'} = 1$ for every $c' \in C$ here, while $q_{c'} = 0$ for every $c' \neq c$ in the proof of Theorem 1.

Proof. Let $c \in C \cup \{\emptyset\}$. Fix $\ell \geq 1$. If $|I_c(\ell)| \leq q_c(\ell)$, then $\arg\max_{I_c(\ell), q_c(\ell)} \succ_c = I_c(\ell)$. Then $J_c(\ell) = I_c(\ell)$ because φ respects preference rankings. Hence the conclusion $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ holds.

Thus, assume $|I_c(\ell)| > q_c(\ell)$. Suppose for contradiction that the conclusion does not hold. Since φ respects preference rankings, it follows that $|J_c(\ell)| = 1$ and hence, there exist $i \in \arg\max_{I_c(\ell)} \succ_c$ such that $\varphi_i[P] \neq c$ and $j \in I_c(\ell) \setminus \left(\arg\max_{I_c(\ell), q_c(\ell)} \succ_c\right)$ such that $\varphi_j[P] = c$.

Consider preference profile P' such that (1) at P'_k for every student $k \in I_c(\ell)$, c is top-ranked and relative rankings of all other schools are unchanged from P_k, and (2) preferences of all other students are unchanged. Since $q_c = 1$, it follows that P' *r.r.m.t.* P at $\varphi[P]$. Thus, since φ satisfies r.r. Maskin monotonicity, $\varphi_j[P'] = c$.

Let $P'' = (P'_i, P'_j, P^{\emptyset}_{-i,j})$ for some $P^{\emptyset}_{-i,j} \in \left(\mathcal{P}^{\emptyset}\right)^{|I|-2}$. Since φ satisfies population monotonicity,

$$\varphi_j[P''] R'_j \varphi_j[P'] = c.$$

This and the assumption that c is top-ranked at P'_j imply

$$\varphi_j[P''] = c.$$

Since $q_c = 1$, the above relation implies

$$\varphi_i[P''] \neq c,$$

which is a contradiction since Claim 3 and the construction of $\succ$ that gives i a higher priority than j at c imply $\varphi_i[P''] = c$. ■

Claim 4 completes the proof of the Theorem.

E A Welfare Maximization Interpretation of the Boston Mechanism

For each problem $[P; q]$, the solution of the following linear assignment program, which maximizes the sum of the induced utilities, is equivalent to the solution of the Boston mechanism induced by a priority profile $\succ$: First, define for each $c \in C$, $I_c = \{i \in I | i \succ_c \emptyset\}$ as the set of acceptable students for school c.

$$\max_{[z_{i,c}]} \sum_{i\in I, c\in C\cup\{\emptyset\}} z_{i,c} u_i(c)$$

subject to

$$i \notin I_c \Rightarrow z_{i,c} = 0 \qquad \forall i \in I, c \in C.$$

$$0 \leq z_{i,c} \qquad \forall i \in I,\ c \in C \cup \{\emptyset\}$$

$$\sum_{c\in C\cup\{\emptyset\}} z_{i,c} = 1 \qquad \forall i \in I$$

$$\sum_{i\in I} z_{i,c} \leq q_c \qquad \forall c \in C \cup \{\emptyset\},$$

where $q_\emptyset = \infty$ and $(u_i)_{i\in I} \subseteq (\mathbb{R}_+^{|C|\cup\{\emptyset\}})^{|I|}$ are utility functions consistent with the given preferences P (i.e., for any $i \in I$ and $c, d \in C \cup \{\emptyset\}$, we have $u_i(c) > u_i(d)$ if and only if $c\, P_i\, d$) satisfying

$$|I|\, u_i(c) < u_j(d) \qquad \forall i, j \in I \text{ and } c, d \in C \cup \{\emptyset\} \text{ with } P_j(c) < P_i(d)$$

$$|I|\, u_i(c) < u_j(c) \qquad \forall c \in C \cup \{\emptyset\} \text{ with } P_i(c) = P_j(c) \text{ and } \succ_c(j) < \succ_c(i)$$

In the solution $[z_{i,c}]_{i\in I, c\in C\cup\{\emptyset\}}$, which is unique and where $z_{i,c} \in \{0, 1\}$ for all $i \in I$ and $c \in C \cup \{\emptyset\}$, school i is matched with school c if and only if $z_{i,c} = 1$.

F Independence of Axioms for Remaining Cases

The main text has presented examples showing that the axioms in the characterizations are independent for all but a few values of $|I|$ and $|C|$. This section completes the investigation by considering all other cases.

F.1 Axioms for Theorem 1

A mechanism violating only consistency: Suppose $|I| \leq 2$: If $|I| = 1$, then consistency is vacuously satisfied by any mechanism. If $|I| = 2$, then respect of preference rankings implies consistency. To see this first note that, for any $i \in I$ and $j \neq i$, $\varphi_i[P; q]$ is the most preferred school in $\{c \in C \cup \{\emptyset\} | i \in I_c, q_c - \mathbf{1}_{\varphi_j[P;q]=c} \geq 1\}$ by respect of preference rankings, where $\mathbf{1}_{\varphi_j[P;q]=c} = 1$ if $\varphi_j[P; q] = c$ and 0 otherwise. By inspection, respect of preference rankings implies that

$\varphi_i[P_i, P_j^{\emptyset}; (q_c - \mathbf{1}_{\varphi_j[P;q]=c})_{c\in C}]$ is the most preferred school in $\{c \in C \cup \{\emptyset\} | i \in I_c, q_c - \mathbf{1}_{\varphi_j[P;q]=c} \geq 1\}$, showing consistency. Thus, there is no mechanism that violates consistency while respecting preference rankings.

A mechanism violating only resource monotonicity: Suppose that $|I| = 1$ or $|C| = 1$: If $|I| = 1$, then respect of preference rankings implies resource monotonicity. To see this point observe that the unique agent, denoted i, receives her most preferred school in $\{c \in C | i \in I_c, q_c \geq 1\}$. Since this set is increasing in each q_c in the set inclusion sense, resource monotonicity follows.

If $|C| = 1$, then respect of preference rankings, consistency, and r.r. Maskin monotonicity imply resource monotonicity. To show this first recall that respect of preference rankings implies individual rationality (since $I_{\emptyset} = I$ and $q_{\emptyset} = \infty$). Now suppose, for contradiction, a mechanism φ satisfies respect of preference rankings, consistency and r.r. Maskin monotonicity, while violating resource monotonicity. Then there exists a student $i \in I$, preference profile P, and a quota q_c of the unique school c such that

$$\varphi_i[P; q_c - 1] = c, \tag{6}$$

$$\varphi_i[P; q_c] = \emptyset, \tag{7}$$

$$c \, P_i \, \emptyset. \tag{8}$$

By relationships (6) – (8) and respect of preference rankings, $|\varphi_c[P; q_c]| = q_c$ and hence there exists a student $j \neq i$ such that

$$\varphi_j[P; q_c] = c, \tag{9}$$

$$\varphi_j[P; q_c - 1] = \emptyset. \tag{10}$$

By consistency of φ and relation (9), it follows

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = \varphi_i[P; q_c]. \tag{11}$$

Relations (7) and (11) imply

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = \emptyset. \tag{12}$$

Meanwhile relation (10) implies that $(P_j^{\emptyset}, P_{-j}) \, r.r.m.t. \, P$ at $\varphi[P; q_c - 1]$. Thus by r.r. Maskin monotonicity of φ,

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = \varphi_i[P; q_c - 1]. \tag{13}$$

Then, by relations (6) and (13), we obtain

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = c. \tag{14}$$

Relations (12) and (14) contradict each other, showing that φ is resource monotonic.

A mechanism violating only r.r. Maskin monotonicity: Suppose that $|C| = 1$ or $|I| = 1$: If $|C| = 1$, then consistency, resource monotonicity, and respect of preference rankings imply r.r. Maskin monotonicity. To see this point first recall that these properties imply individual rationality and population monotonicity. Let φ be a mechanism that these axioms and for each $i \in I$ let

P'_i	P''_i
c	$\emptyset$
$\emptyset$	c

Fix a preference profile P arbitrarily. Since φ satisfies individual rationality,

$$\varphi_i[P; q] = c \Rightarrow P_i = P'_i \tag{15}$$

Moreover, if $\varphi_i[P; q]$ is the top-ranked school for i at P_i, then the only monotonic transformation of P_i at $\varphi_i[P; q]$ is P_i itself. By this fact and (15), if $\tilde{P}$ is a monotonic transformation of P at $\varphi[P; q]$, then for any $i \in I$, either

1. $\tilde{P}_i = P_i$, or
2. $P_i = P'_i$, $\varphi_i[P; q] = \emptyset$, and $\tilde{P}_i = P''_i$.

For any i such that Case 1 above applies, population monotonicity of φ and Cases 1 and 2 imply $\varphi_i[\tilde{P}; q] \, R_i \varphi_i[P; q]$. So, if $\varphi_i[P; q] = c$, then $\varphi_i[\tilde{P}; q] = c$. Since φ respects preference rankings, it follows that $\varphi_i[P; q] = \emptyset$ implies $\varphi_i[\tilde{P}; q] = \emptyset$. Thus we conclude $\varphi_i[P; q] = \varphi_i[\tilde{P}; q]$. For any i such that Case 2 above applies, individual rationality of φ implies $\varphi_i[\tilde{P}; q] = \emptyset = \varphi_i[P; q]$. Therefore we conclude $\varphi[\tilde{P}; q] = \varphi[P; q]$, showing r.r. Maskin monotonicity.

If $|I| = 1$, then respect of preference rankings implies r.r. Maskin monotonicity. To show this, let i be the unique student in I and $C_i = \{c \in C \cup \{\emptyset\} | i \in I_c, q_c \geq 1\}$. Since i is the unique student, respect of preference rankings imply that $\varphi_i[P_i; q]$ is the school that is top-ranked by P_i within C_i. Any monotonic transformation P'_i of P_i at $\varphi_i[P_i; q]$ leaves the top-ranked school in C_i unchanged (namely $\varphi_i[P_i; q]$), so $\varphi_i[P'_i; q] = \varphi_i[P_i; q]$ by respect of preference rankings. This shows that φ satisfies r.r. Maskin monotonicity.

F.2 Axioms for Theorem 2

A mechanism violating only respect of preference rankings: Suppose that $|C| = 1$ or $|I| = 1$: If $|C| = 1$, then any mechanism φ that satisfies individual rationality and r.r. Maskin monotonicity respects preference rankings. To see this, let $C = \{c\}$ and assume for contradiction that φ does not respect preference rankings. Then there exists $i \in I_c$ (so $\varphi_i[P] = c$ for some P) such that $cP'_i\varphi_i[P']$ for some P' where either $\varphi_c[P'] = \emptyset$ or $\varphi_c[P'] = j$ with $P'_i(c) < P'_j(c)$. The latter is a contradiction to individual rationality of φ, because $P'_i(c) < P'_j(c)$ and $|C| = 1$ imply $\emptyset P'_j c$. Thus assume $\varphi_c[P'] = \emptyset$. Since $q_c = 1$ and $C = \{c\}$, it follows that $\varphi_j[P] = \varphi_j[P'] = \emptyset$ for all $j \neq i$. Also note that $P_i = P'_i$ because $\varphi_i[P] = c$ implies $cP_i\emptyset$ by individual rationality of φ, $cP'_i\emptyset$ from before, and $|C| = 1$. Hence $(P_i, P^{\emptyset}_{-i}) r.r.m.t. P$ at $\varphi[P]$ and $(P_i, P^{\emptyset}_{-i}) r.r.m.t. P'$ at $\varphi[P']$ for any $P^{\emptyset}_{-i} \in (P^{\emptyset})^{|I|-1}$. Thus r.r. Maskin monotonicity implies $c = \varphi_i[P] = \varphi_i[P_i, P^{\emptyset}_{-i}] = \varphi_i[P'] = \emptyset$, a contradiction.

If $|I| = 1$, then r.r. Maskin monotonicity implies respect of preference rankings. To see this point suppose for contradiction that a mechanism φ violates respect of preference rankings while satisfying r.r. Maskin monotonicity. Then, for the unique student i, there exist P_i, P'_i such that

$$\varphi_i[P'_i] P_i \varphi_i[P_i]. \tag{16}$$

Then a preference P''_i such that

$$\begin{array}{c} P''_i \\ \hline \varphi_i[P'_i] \\ \varphi_i[P_i] \\ \vdots \end{array}$$

satisfies $P''_i\, r.r.m.t.\, P_i$ at $\varphi[P_i]$ and $P''_i\, r.r.m.t.\, P'_i$ at $\varphi[P'_i]$. By r.r. Maskin monotonicity we obtain

$$\varphi[P_i] = \varphi[P''_i] = \varphi[P'_i],$$

a contradiction to relation (16).

A mechanism violating only population monotonicity: Suppose that $|C| = 1$ or $|I| \leq 2$: If $|I| = 1$, then population monotonicity is vacuously satisfied. If $|I| = 2$, then individual rationality and respect of preference rankings imply population monotonicity. To see this point let $I = \{1, 2\}$ and $P^{\emptyset}_2 \in \mathcal{P}^{\emptyset}$. Consider an arbitrary preference profile P. Since φ is individually rational,

$\varphi_1[P]\, R_1\, \emptyset$. If $\varphi_1[P] = \emptyset$, then individual rationality of φ implies that $\varphi_1[P_1, P_2^{\emptyset}]\, R_1\, \emptyset = \varphi_1[P]$, satisfying the conclusion of population monotonicity. Thus suppose $\varphi_1[P] \neq \emptyset$ and, for contradiction, that $\varphi_1[P]\, P_1\, \varphi_1[P_1, P_2^{\emptyset}]$. Then, by respect of preference rankings, $\varphi_2[P_1, P_2^{\emptyset}] = \varphi_1[P]$. However, $\emptyset\, P_2^{\emptyset}\, \varphi_1[P]$ since $P_2^{\emptyset}$ top-ranks $\emptyset$, which contradicts individual rationality of φ. By a symmetric argument the conclusion of population monotonicity holds for matchings of student 2, showing that population monotonicity holds.

If $|C| = 1$, then individual rationality and r.r. Maskin monotonicity imply population monotonicity. To see this point first note that $\varphi_i[P]\, R_i\, \emptyset$ for all i and P since φ is individually rational. Hence if $\varphi_i[P] = \emptyset$, then $\varphi_i[P_j^{\emptyset}, P_{-j}] R_i \emptyset = \varphi_i[P]$ for any $i \in I, j \neq i$ and $P_j^{\emptyset} \in \mathcal{P}^{\emptyset}$, thus the conclusion of population monotonicity holds. So suppose $\varphi_i[P] = c \neq \emptyset$. Then, since $|C| = 1$ and $q_c = 1$, $\varphi_k[P] = \emptyset$ for all $k \neq i$. Thus, for any $j \neq i$ and $P_j^{\emptyset} \in \mathcal{P}^{\emptyset}$, we have $(P_j^{\emptyset}, P_{-j})\, r.r.m.t.\, P$ at $\varphi[P]$. By r.r. Maskin monotonicity of φ, we obtain $\varphi_i[P_j^{\emptyset}, P_{-j}] = \varphi_i[P]$, showing that the conclusion of population monotonicity holds. These arguments show the claim.

A mechanism violating only r.r. Maskin monotonicity: Suppose that $|I| = 1$ or $|C| = 1$: If $|I| = 1$ or $|C| = 1$, then the paragraph on r.r. Maskin monotonicity in Section F.1 shows that there exists no mechanism that satisfies respect of preference rankings, individual rationality, population monotonicity and, yet, violates r.r. Maskin monotonicity.[15]

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[15]The paragraph in Section F.1 supposes, in addition to respect of preference rankings, resource monotonicity and consistency instead of individual rationality and population monotonicity. However, consistency and resource monotonicity are used only to obtain individual rationality and population monotonicity. Therefore the example is valid in the current context.

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