A Note on Dynamics in Games

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November 24, 2009

Abstract

We introduce the class of dynamics with the best or worst response property. This class generalizes the best response dynamics (Gilboa and Matsui 1991; Matsui 1992) as a special case. We show that a unique Nash equilibrium of any anti-coordination game (Kojima and Takahashi 2007) is globally stable under this class of dynamics.

1 Introduction

Evolutionary game theory has been analyzing the evolution of action distributions in the society with dynamic models. There have been a variety of biologically motivated dynamics, probably the most famous one being the replicator dynamics (Taylor and Jonker 1978). There have been attempts to generalize the replicator dynamics, most notably the class of payoff monotone dynamics (Nachbar 1990; Friedman 1991; Samuelson and Zhang 1992).

Gilboa and Matsui (1991) and Matsui (1992) propose a different class of dynamics motivated by economic analysis, called the best response dynamics. In their setup, there is one large population of agents. At each moment in time, each agent is matched randomly with another in the same population and plays a symmetric two-player game. A fraction of agents are randomly chosen to change their actions at each moment. Each of pure actions chosen by some agents is one of the best responses to the current action distribution. In other words, agents in the best response dynamics maximizes the current average payoff.

Best response dynamics is reasonable in a sense as agents maximize their payoffs, which is consistent with the notion of rational economic agents. On

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the other hand, the specific dynamics may be restrictive in a number of aspects. For instance, revision opportunities arrive to every agent at the same rate irrespective of her current payoff. This may be restrictive since it seems plausible that the higher the current payoff, the less eager an agent is to reconsider her action. Thus it is important to understand how we can relax the assumptions of the best response dynamics while obtaining a strong prediction about the behavior of the population.[1]

We introduce the class of dynamics with best or worst response property. A dynamics is said to satisfy the best response property if the proportion of any currently suboptimal action decreases. A dynamics is said to satisfy the worst response property if the proportion of any action increases if it is not one of the current worst strategies. Weaker versions of these properties, called weak best response property and weak worst response property, require the above conditions only for completely mixed action distributions. The class of dynamics with the best respons property includes the best response dynamics. A dynamics studied by Björnerstedt (1995), in which only worst strategies have negative growth rates, has the weak worst response property.

We consider the dynamic stability of Nash equilibria under this class of dynamics. The class of anti-coordination games (Kojima and Takahashi 2007) is investigated. A symmetric two-player game is said to be an anti-coordination game if any worst response to a mixed strategy is in the support of that strategy. An anti-coordination game is known to have a unique Nash equilibrium. We show that, in a unified approach, the Nash equilibrium in any anti-coordination game is globally stable under any dynamics with best or worst response property. The dynamics of Björnerstedt with the weak worst reponse property exhibits non-convergence to an equilibrium even in a presence of a unique evolutionarily stable strategy (ESS). Actually it can cycle even in a stable game (Hofbauer 2000; Sandholm 2005), in which various dynamics are known to converge to the unique equilibrium. Björnerstedt's dynamics is the first dynamics showen to behave well in anti-coordination games but not in stable games.[2] The result of this paper suggests an important role of anti-coordination games in evolutionary dynamics.

The rest of this paper proceeds as follows. Section 2 introduces dynamics with best or worst response property. Section 3 introduces the class of

[1]The best response dynamics is included in the class of myopic adjustment dynamics (Swinkels 1993). Smoothed best response dynamics (Fudenberg and Levine 1998) is another attempts to relax the extreme assumption of exactly maximizing agents.

[2]Kojima and Takahashi (2007) show that the equilibrium in an anti-coordination game is stable under the perfect foresight dynamics (Matsui and Matsuyama 1995). Dynamic stability of an interior ESS is conjectured by Hofbauer and Sorger (1999), but this claim is neither proved or disproved so far.

anti-coordination games and shows that any dynamics with best or worst response property with any initial state converges to an equilibrium an in anti-coordination game. Section 4 concludes.

2 Dynamics with Best or Worst Response Property

Consider a finite symmetric two-player game. A game is described by an $n \times n$ matrix $A = (a_{ij})$, where n is the cardinality of the set of pure strategies, $\{1, \dots, n\}$. Let x be a generic element of the set of mixed strategies $\Delta := \{x \in \mathbb{R}^n | \forall i, x_i \geq 0, \sum_i x_i = 1\}$. We sometimes refer to a mixed strategy as a state. The set $\Delta^\circ := \{x \in \mathbb{R}^n | \forall i, x_i > 0, \sum_i x_i = 1\}$ denotes the set of completely mixed strategies. We sometimes denote pure-strategy i by $e_i \in \Delta$. For $x \in \Delta$, let $\text{supp}(x) = \{i | x_i > 0\}$. The sets $\text{br}(x) := \arg\max_i e_i \cdot Ax$ and $\text{wr}(x) := \arg\min_i e_i \cdot Ax$ are the sets of pure-strategy best and worst responses, respectively. A state $x \in \Delta$ is a *(symmetric) Nash equilibrium* (NE) if $\text{supp}(x) \subseteq \text{br}(x)$.

Consider a dynamics $x\colon [0, \infty) \to \Delta, x(0) = x$. The state $x \in \Delta$ is called the initial state of the dynamics.

Definition 1. (1) Dynamics $x(\cdot)$ satisfies the *best response property* if $i \notin \text{br}(x(t))$ and $x_i(t) > 0$ imply $d^+x_i(t)/dt < 0$. Dynamics $x(\cdot)$ satisfies the *weak best response property* if $i \notin \text{br}(x(t))$ and $x(t) \in \Delta^\circ$ imply $d^+x_i(t)/dt < 0$.

(2) Dynamics $x(\cdot)$ satisfies the *worst response property* if $i \notin \text{wr}(x(t))$ and $x_i(t) < 1$ imply $d^+x_i(t)/dt > 0$. Dynamics $x(\cdot)$ satisfies the *weak worst response property* if $i \notin \text{wr}(x(t))$ and $x(t) \in \Delta^\circ$ imply $d^+x_i(t)/dt > 0$.

To see what these properties mean, we discuss a number of dynamics in the existing literature. A dynamics $x(\cdot)$ is a myopic adjustment dynamics (Swinkels 1993) if $x(\cdot)$ is right differentiable and satisfies

$$\frac{d^+x(t)}{dt} Ax(t) \geq 0$$

for any $t \in [0, \infty)$. A dynamics $x(\cdot)$ is a payoff monotone dynamics (Nachbar 1990; Friedman 1991; Samuelson and Zhang 1992) if it is right differentiable and

$$\frac{d^+x_i(t)/dt}{x_i(t)} - \frac{d^+x_i(t)/dt}{x_j(t)} > 0 \iff e_i \cdot Ax(t) > e_j \cdot Ax(t).$$

It is clear that any dynamics with either the best response or worst response property is a myopic adjustment dynamics, but it may not be a payoff monotone dynamics. For example, the best response dynamics (Gilboa and Matsui 1991; Matsui 1992) satisfies the best response property and hence is a myopic adjustment dynamics while it violates (strict) payoff monotonicity since every myopically suboptimal strategy has the same negative growth rate. Note that the above class of dynamics with best response property even allows for a better strategy to decrease with a greater rate than a worse strategy, as long as both strategies are suboptimal.

Example 1. Consider the following class of dynamics motivated by human imitation behavior (Björnerstedt and Weibull 1996);

$$\dot{x}_i(t) = \sum_j r_j(x(t))p_j^i(x(t))x_j(t) - r_i(x(t))x_i(t), \tag{1}$$

where $r_i : \Delta \to [0, 1]$ is the revision rate for agents currently taking strategy i (i-strategists), $p_j^i : \Delta \to [0, 1]$ is the proportion of j-strategists who changes their strategies from j to i conditional on obtaining revision opportunity, satisfying $\sum_i p_j^i(x) = 1$ for any $x \in \Delta$ and j. The first term represents the inflow of new i-strategists and the second term represents the outflow of i-strategists to other strategies.[3]

For example, let $r_i(x) = 1$ for any i and $x \in \Delta$ and $p_j^i(x) = 0$ for any $i \notin \text{br}(x)$. This is the best response dynamics (Gilboa and Matsui 1991; Matsui 1992) and satisfies the best response property. Actually, for the dynamics to have the best response property, it is enough to have $r_i(x) > 0$ for any i and $x \in \Delta$ and $p_j^i(x) = 0$ for any $i \notin \text{br}(x)$. The exact specification of the best response dynamics is not needed for a dynamics to have the best response property.

Example 2. Consider the above class of dynamics (1) with $r_i(x) = 1$ and $p_j^i(x) = x_i c_j^i(x) + \delta_j^i(x)$ for any i, where $0 \leq c_j^i(x) < 1$ if $i \notin \text{br}(x)$ and $\delta_j^i(x) = 0$ if $i \notin \text{br}(x)$, $\sum_i \delta_j^i(x) = 1 - \sum_i c_j^i(x)x_i$. In this dynamics, agents revise actions with a constant arrival rate of one. They choose actions by imitation with probability $c_j^i(x)$, but they sometimes choose one of the best

[3]Note that the inflow and the outflow are represented in *gross* terms in (1). That is, i-strategists with revision opportunity who decide to continue taking i are counted both in the inflow and as the outflow in (1).

responses. Now we have

$$\dot{x}_i(t) = \sum_j [x_i(t)c^i_j(x(t)) + \delta^i_j(x(t))]x_j(t) - x_i(t)$$
$$= x_i(t)\left[\sum_j [c^i_j(x(t))x_j(t)] - 1\right] + \sum_j \delta^i_j(x(t))x_j(t),$$

which is strictly negative if $i \notin \mathrm{br}(x(t))$ and $x_i(t) \neq 0$. Thus this dynamics satisfies the best response property. Note that the best response property again does not depend on fine specifications of the dynamics.

Example 3. Now let $r_i(x) = 0$ for $i \notin \mathrm{wr}(x)$, $r_i(x) = 0$ for $i \in \mathrm{wr}(x)$, $r_i(x) = 1$ for any $i \in \mathrm{wr}(x)$ and $p^i_j(x) = x_i$ for any i, j. In this dynamics (Björnerstedt 1995; see also Weibull 1995), only agents with the currently worst strategy review their strategies, and they choose actions by pure imitation. This dynamics has the weak worst response property.

Consider the following payoff matrix provided by Dekel and Scotchmer (1992);

$$\begin{pmatrix} 1 & 2+a & 0 & c \\ 0 & 1 & 2+a & c \\ 2+a & 0 & 1 & c \\ 1+c & 1+c & 1+c & 0 \end{pmatrix}, \qquad 0 < 3c < a.$$

Björnerstedt (1995) shows that under the above dynamics, if the initial state does not have too similar weights on the first three actions, then the fourth action e_4 remains to be taken with positive probability despite the fact that e_4 is strictly dominated by a mixed strategy $x^* = (1/3, 1/3, 1/3, 0)$, which is also the unique equilibrium of this game. Therefore the dynamics does not converge to x^* with some initial states. The reason of the non-convergence is that e_4 does not become the worst strategy in most of the trajectories although it is dominated by a mixed strategy. Given that proportions of only the worst strategies decrease, the dynamics does not eliminate e_4.

Denote $\mathbb{R}^n_0 = \{\zeta \in \mathbb{R}^n | \zeta \neq 0, \sum_{i=1}^n \zeta_i \neq 0\}$. A game A is a *stable game* if $\zeta \cdot A\zeta < 0$ for any $\zeta \in \mathbb{R}^n_0$. The above game is shown to be a stable game with additional assumption $a < 6c$. A Nash equilibrium in a stable game is an ESS and a unique Nash equilibrium. The equilibrium is globally stable under various dynamics, that is, for any initial state, $x(t)$ converges to the equilibrium. Such dynamics include the replicator dynamics, the best response dynamics, smoothed best response dynamics and Brown-von Neumann-Nash (BNN) dynamics.[4] Note that convergence does not hold under the Björnerstedt dynamics in this example, despite the fact the equilibrium in a stable game is known to have dynamic stability under a broad class of dynamics.

3 Dynamic Stability

Consider the following class of anti-coordination games introduced by Kojima and Takahashi (2007).

Definition 2. A symmetric two-player game has the *anti-coordination property* if $\mathrm{wr}(x) \subseteq \mathrm{supp}(x)$ for any $x \in \Delta$.

In words, a symmetric game has the anti-coordination property if, for any mixed strategy, any worst response to the mixed strategy lies in the support of the mixed strategy. In other words, a pure strategy is one of the worst responses only if the strategy is chosen by a positive fraction of the agents in the population.

Proposition 1. *Every anti-coordination game has a unique symmetric Nash equilibrium. The equilibrium is in the interior of* Δ.

Proof. See Kojima and Takahashi (2007). □

The hawk-dove game is a special case of the anti-coordination games. The unique symmetric Nash equilibrium of an anti-coordination game may not be evolutionarily stable, and a game with an interior ESS may not have the anti-coordination property. See Kojima and Takahashi (2007).

Lemma 1. *Suppose that stage game A satisfies the anti-coordination property. Then* $\arg\max(x_i/x_i^*) \cap \mathrm{br}(x) = \emptyset$ *and* $\arg\min_i(x_i/x_i^*) \cap \mathrm{wr}(x) = \emptyset$ *if* $x \neq x^*$.

Proof. Given $x \neq x^*$, define $\alpha = \max_i(x_i/x_i^*) > 1$. We can write $x = \alpha x^* + (1-\alpha)\tilde{x}$, where $\tilde{x} \in \Delta$ and $\arg\max_i(x_i/x_i^*) \cap \mathrm{supp}(\tilde{x}) = \emptyset$. Since A has the anti-coordination property, $1-\alpha < 0$ and x^* is an interior Nash equilibrium, we have $\mathrm{br}(x) = \mathrm{wr}(\tilde{x}) \subseteq \mathrm{supp}(\tilde{x})$. This implies $\arg\max(x_i/x_i^*) \cap \mathrm{br}(x) = \emptyset$. Similarly, we can write $x = \beta x^* + (1-\beta)\bar{x}$, where $\beta = \min_i(x_i/x_i^*) < 1$, $x \in \Delta$ and $\arg\min_i(x_i/x_i^*) \cap \mathrm{supp}(\bar{x}) = \emptyset$. Since A has the anti-coordination property, $1-\beta > 0$ and x^* is an interior Nash equilibrium, we have $\mathrm{wr}(x) = \mathrm{wr}(\bar{x}) \subseteq \mathrm{supp}(\bar{x})$. This implies $\arg\min(x_i/x_i^*) \cap \mathrm{wr}(x) = \emptyset$. □

[4] See Hopkins (1999), Hofbauer (2000), and Sandholm (2005).

Theorem 1. *Let x^* be a Nash equilibrium in an anti-coordination game. Then any dynamics $x(\cdot)$ with either best or worst response property from any initial state $x \in \Delta$ converges to x^*. Moreover, any dynamics $x(\cdot)$ with either weak best or weak worst response property from any initial state $x \in \circ\Delta$ converges to x^*.*

Proof. First assume that dynamics $x(\cdot)$ satisfies the best response property. Then by Lemma 1, the function $U(x) := \max_i(x_i/x_i^*)$ strictly decreases along any solution path and $x(\cdot)$ converges to x^*. If x satisfies the worst response property, then $L(x) := \min_i(x_i/x_i^*)$ increases along any solution path and $x(\cdot)$ converges to x^*. Suppose $x(\cdot)$ satisfies weak best response property and $x \in \Delta^\circ$. Then $\mathrm{supp}(x(t)) = \{1, \ldots, n\}$ for any t by Lemma 1 and the weak best response property, and a similar proof works as in a dynamics with the best response property. A similar proof goes through for a dynamics with weak worst response property. □

Theorem 1 generalizes a result by Kojima and Takahashi (2007) that the Nash equilibrium in any anti-coordination game is globally stable under the best response dynamics. Note that the anti-coordination property assures global stability under a broad class of dynamics including Björnerstedt dynamics, under which even the ESS in a stable game may not be reached.

4 Conclusion

We introduced the class of dynamics with best or worst response property. We showed that a unique Nash equilibrium of any anti-coordination game is globally stable under this class of dynamics.

We conclude with a possible direction of future research. The behavior of dynamics with the best response or the worst response property is largely unknown although this paper proved that they have desirable stability properties in the class of anti-coordination games. Investigating dynamic stability of these dynamics in other classes of games may be interesting.

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