The Equivalence Between Costly and Probabilistic Voting Models

Yuichiro Kamada and Fuhito Kojima*

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Abstract

In a costly voting model, voters abstain when a stochastic cost of voting exceeds the benefit from voting. In a probabilistic voting model, they always vote for a candidate who generates the highest utility, which is subject to random shocks. We prove an equivalence result: In two-candidate elections, given any costly voting model, there exists a probabilistic voting model that generates winning probabilities identical to those in the former model for any policy announcements, and vice versa. Thus predictions established in one of the models hold in the other as well, providing robustness of the conclusions to model specifications.

*Kamada: Department of Economics, Harvard University, Cambridge, MA 02138, e-mail: ykamada@fas.harvard.edu; Kojima: Department of Economics, Stanford University, Stanford, CA 94305, e-mail: fkojima@stanford.edu. We are grateful to John William Hatfield, Matt Jackson, Martin Osborne, and Yuki Takagi.

1 Introduction

Electoral competitions have been analyzed with game theoretic tools, beginning with seminal studies by Hotelling (1929), Downs (1957) and Black (1958). Their models are useful but highly stylized benchmarks, and many subsequent works modified their models to better fit theoretical predictions to the real world. There are at least two assumptions that these works have relaxed. One is the assumption that no voter abstains from voting, and the other is that it is with certainty that each voter votes for a candidate whose policy is strictly closer to her bliss point.

Costly voting is a model that addresses the first point. In that model, each voter randomly draws a cost of voting from a certain distribution, and votes for the candidate with a closer policy if and only if the voting cost is smaller than her utility gain from doing so. Otherwise, a voter abstains from voting. *Probabilistic voting* tackles the second issue. In that model, voters cannot abstain. Each voter obtains a random utility shock for the candidates in addition to the deterministic utility from announced policies, and votes for the candidate whose overall utility is higher.[1]

These two models have proved to be useful for predictions, and properties of each model have been investigated extensively in the literature.[2] However, these models have modified different assumptions of the basic Hotelling-Downs model, and the relationship between them has been unclear in the literature.

In this paper, we show that these two models are equivalent in the context of two-candidate election: For any pair of the two candidates' announced policies, the winning probability of each candidate is identical under these two models. More specifically, our main result demonstrates that (1) given any cost distribution, there exists a distribution

[1]Standard interpretations of the random utility shocks include ideology or personal characteristics of the party leadership (Persson and Tabellini, 2000).

[2]See Banks and Duggan (2005) and references therein. For example, in probabilistic voting models it is known that with certain regularity conditions the existence of a Nash equilibrium is established. In addition, a Nash equilibrium is welfare-maximizing if voters have convex utility over policies, but not necessarily otherwise (Kamada and Kojima, 2011). On the costly voting model, see an extensive survey by Osborne (1995) as well as more recent studies such as Borgers (2004) for instance.

of the random utility shocks such that the costly voting model with abstention associated with the former distribution is equivalent to the probabilistic voting model with the latter, and (2) given any random utility shock distribution, there exists a cost distribution such that the probabilistic voting model associated with the former distribution is equivalent to the costly voting model with abstention associated with the latter. Therefore, predictions established in one of the models carry over to the other, providing robustness of the conclusions to model specifications. The sets of equilibrium policy profiles announced by the two candidates who try to maximize the winning probabilities, for instance, are identical under these two models.

We also analyze a more general hybrid model with both voting cost and random utility shocks. Generalizing our main result we show that, given any such hybrid model, there exist a costly voting model and a probabilistic voting model each of which is equivalent to the given hybrid model. This result implies that there is no loss of generality in assuming away either the cost or random utility shocks from the model. This result can facilitate analysis of voting by giving justification for studying relatively simple models with only either cost or utility shocks but not both.

The rest of this paper proceeds as follows. Section 2 introduces the model. Section 3 presents the main theorem. In Section 4, we consider a generalization to hybrid models. Section 5 concludes. Some proofs are relegated to the Appendix.

2 The Model

There is a policy space X, endowed with a Borel probability measure μ. A continuum of voters are distributed over X according to μ. There are two political candidates, A and B, who simultaneously choose their policies on X, x_A and x_B, respectively. A voter with position $x \in X$ obtains a deterministic utility $u(x, x')$ from the policy x'. A standard example for this is that the utility depends (only) on the "distance" between x and x', which is defined over the set X.

A **voting model** M is characterized by a corresponding random function $w^M(x_A, x_B) \in \{A, B\}$ that assigns a winner of the election given the policy profile (x_A, x_B). We say that two voting models M and M' are **equivalent** if $\Pr(w^M(x_A, x_B) = A) = \Pr(w^{M'}(x_A, x_B) = A)$ for any given policy profile (x_A, x_B).

Two voting models are defined below. In each voting model M, $w^M(x_A, x_B) = i$ if the measure of voters that vote for candidate i is greater than that for the other candidate. If they are exactly equal then $w^M(x_A, x_B)$ takes A and B with equal probabilities.

Costly Voting Model

Each voter experiences a cost $c \geq 0$ from voting. The costs are independent of voters' positions and across voters, and each cost follows a continuous cumulative distribution function $F : \mathbb{R} \to [0, 1]$ such that $F(t) = 0$ for all $t \leq 0$.[3] Let $\mathcal{F}$ be the set of all possible cumulative distribution functions with such a restriction.

In this model, a voter with position x votes for candidate A with probability 1 if

$$\phi\left(u(x, x_A) - u(x, x_B)\right) > c,$$

and she votes for candidate B with probability 1 if

$$\phi\left(u(x, x_B)) - u(x, x_A)\right) > c,$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Otherwise, the voter abstains from voting.

This model subsumes a setup in which the motivation for each voter is to influence

[3]To be formal, a continuum of voters are distributed over the space $X \times \mathbb{R}_+$ according to a product measure of μ and a measure induced by F, and a voter associated with a pair (x, c) is interpreted as positioning at a point x in the policy space and experiencing a cost c from voting. Note that we assume that the distribution of voters' positions and realized costs is given by μ and F, rather than modeling a continuum of random position-cost pairs. This is a standard modeling approach in the literature, based on the intuitive idea of the "law of large numbers." See however Judd (1985) and Uhlig (1996) for technical issues associated with invoking the law of large numbers for a continuum of random variables. Analogous remarks also apply to the probabilistic voting and hybrid models below.

the electoral outcome and a voter estimates that, potentially incorrectly, the probability that she is pivotal is a fixed number $p > 0$ (thus $\phi(t) = pt$ for any $t \in \mathbb{R}$), but many other possibilities are subsumed as well. For example, the voter may vote because of "expressive" motives (see Schuessler (2000) and Glaeser et al. (2005)), in which case function ϕ specifies the salience of the political issue in the voter's mind. Denote by $M^C(F)$ the costly voting model with cost distribution $F \in \mathcal{F}$.

Probabilistic Voting Model

Each voter does not have a choice to abstain, while their decision on who to vote for depends on two elements: The first is the deterministic utility of electing the respective candidates defined earlier, and the second is a random shock in the evaluation of the two candidates. Specifically, each voter experiences a random shock in utility that favors candidate A. The random shocks are independent of voters' positions and across voters, and each random shock follows a continuous cumulative distribution function $G : \mathbb{R} \to [0, 1]$ which is symmetric around 0. Let $\mathcal{G}$ be the set of all cumulative distribution functions with such a restriction.

In this model, More specifically, a voter with position x votes for candidate A with probability 1 if

$$u(x, x_A) + \xi > u(x, x_B),$$

and for candidate B with probability 1 if

$$u(x, x_B) > u(x, x_A) + \xi.$$

If $u(x, x_A) + \xi = u(x, x_B)$, the voter votes for the two candidates with probability $\frac{1}{2}$ for each. Denote by $M^P(G)$ the probabilistic voting model with shock distribution $G \in \mathcal{G}$.

3 The Equivalence Theorem

Theorem 1. *Fix ϕ. For any cost distribution $F \in \mathcal{F}$, there exists a utility shock distribution $G \in \mathcal{G}$ such that the costly voting model $M^C(F)$ is equivalent to the probabilistic voting model $M^P(G)$. Conversely, for any $G \in \mathcal{G}$, there exists $F \in \mathcal{F}$ such that $M^C(F)$ is equivalent to $M^P(G)$.*

This is the main theorem of this paper, which establishes the equivalence between a costly voting model and a probabilistic voting model. As we have discussed in the Introduction, the theorem implies that predictions established in one of the models carry over to the other, which provides robustness of the conclusions to model specifications. The sets of equilibrium policy profiles announced by the two candidates who try to maximize the winning probabilities, for instance, are identical under these two models.

The proof is simple: We take the difference of expected vote shares of the two candidates in each model, which is a sufficient statistic of the winning probability. Then we construct a probability distribution of random utility shocks (resp. costs of voting) that generates the same expected vote share difference given any policy profile and a probability distribution of cost of voting (resp. random utility shock). It turns out that the restrictions on these probability distributions described in each model are sufficient to guarantee that the construction works.

Proof. Fix a policy profile (x_A, x_B). Given a costly voting model or a probabilistic voting model M, let $R(M, x_A, x_B)$ be the "relative vote share," the measure of voters who vote for candidate A minus that for candidate B under M under policy profile (x_A, x_B). To prove the theorem, it suffices to show that (i) for any $F \in \mathcal{F}$ there exists $G \in \mathcal{G}$ such that $R(M^C(F), x_A, x_B) = R(M^P(G), x_A, x_B)$ for every (x_A, x_B), and (ii) for any $G \in \mathcal{G}$ there exists $F \in \mathcal{F}$ such that $R(M^C(F), x_A, x_B) = R(M^P(G), x_A, x_B)$ for every (x_A, x_B).

By definition, for any (x_A, x_B) we have

$$R(M^C(F), x_A, x_B) = \int_X \left[F\left(\phi(u(x, x_A) - u(x, x_B))\right) - F\left(\phi(u(x, x_B) - u(x, x_A))\right)\right] d\mu(x)$$

for any $F \in \mathcal{F}$, and

$$R(M^P(G), x_A, x_B) = \int_X \left[G\left(u(x, x_A) - u(x, x_B)\right) - G\left(u(x, x_B) - u(x, x_A)\right)\right] d\mu(x)$$

for any $G \in \mathcal{G}$.

Denoting $t = u(x, x_A) - u(x, x_B)$ for notational simplicity, we have:

$$R(M^C(F), x_A, x_B) = \int_X \left[F\left(\phi(t)\right) - F\left(\phi(-t)\right)\right] d\mu(x)$$

and

$$R(M^P(G), x_A, x_B) = \int_X \left[G\left(t\right) - G\left(-t\right)\right] d\mu(x).$$

This means that it suffices to show that (i') for any $F \in \mathcal{F}$ there exists $G \in \mathcal{G}$ such that the following condition (1) holds, and (ii') for any $G \in \mathcal{G}$ there exists $F \in \mathcal{F}$ such that condition (1) holds,[4] where

$$F\left(\phi(t)\right) - F\left(\phi(-t)\right) = G\left(t\right) - G\left(-t\right) \text{ for all } t \in \mathbb{R}. \tag{1}$$

Note that the left-hand side is $F(\phi(t))$ if $t \geq 0$ and $-F(\phi(-t))$ if $t < 0$. Also, note that the right hand side is $2(G(t) - \frac{1}{2})$ if $t \geq 0$ and $-2(G(-t) - \frac{1}{2})$ if $t < 0$. Now we prove (i') and (ii').

[4]Note that (i') and (ii') imply (i) and (ii) above, respectively.

Part (i'): Let $F \in \mathcal{F}$ be given. Define function G by

$$G(t) = \begin{cases} \frac{1}{2}F(\phi(t)) + \frac{1}{2}, & t \geq 0, \\ -\frac{1}{2}F(\phi(-t)) + \frac{1}{2}, & t < 0. \end{cases}$$

Since G is nondecreasing and satisfies $\lim_{t\to-\infty} G(t) = -\frac{1}{2}\lim_{t\to-\infty} F(\phi(-t)) + \frac{1}{2} = 0$ and $\lim_{t\to\infty} G(t) = \frac{1}{2}\lim_{t\to\infty} F(\phi(t)) + \frac{1}{2} = 1$, G is a cumulative distribution function over $\mathbb{R}$. Moreover, G is symmetric around zero since, for any $t \geq 0$,

$$\begin{aligned} G(-t) &= -\frac{1}{2}F(\phi(t)) + \frac{1}{2} \\ &= 1 - G(t). \end{aligned}$$

Hence $G \in \mathcal{G}$. It is straightforward by inspection that the above definition of G satisfies condition (1).[5] This completes the proof.

Part (ii'): Let $G \in \mathcal{G}$ be given. Define F by

$$F(t) = \begin{cases} 2(G(\phi^{-1}(t)) - \frac{1}{2}), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Since F is nondecreasing and satisfies $F(t) = 0$ for all $t \leq 0$, and $\lim_{t\to\infty} F(t) = 2(\lim_{t\to\infty} G(\phi^{-1}(t)) - \frac{1}{2}) = 1$, F is a cumulative distribution function over $\mathbb{R}$ and $F \in \mathcal{F}$. It is straightforward by inspection that the above definition of F satisfies condition (1). This completes the proof. $\square$

[5]For $t \geq 0$, $G(t) - G(-t) = F(\phi(t)) = F(\phi(t)) - F(\phi(-t))$. For $t < 0$, $G(t) - G(-t) = -F(\phi(-t)) = F(\phi(t)) - F(\phi(-t))$. Thus condition (1) is satisfied for any t.

4 Hybrid Models

In this section we analyze a general hybrid model with both voting cost and random utility shocks. We show that, given any such hybrid model, there exist a costly voting model and a probabilistic voting model each of which is equivalent to the given hybrid model. This result implies that there is no loss of generality in assuming away either the cost or random utility shocks from the model. This result can facilitate analysis of voting by giving justification for studying relatively simple models with only either cost or utility shocks but not both.

In a hybrid model, each voter experiences cost of voting, denoted c, distributed according to $F \in \mathcal{F}$, and a random shock in the evaluation of the two candidates, denoted ξ, distributed according to $G \in \mathcal{G}$. Hence, a voter with position x votes for candidate A with probability 1 if

$$\phi\left((u(x, x_A) + \xi - u(x, x_B)\right) > c$$

and she votes for candidate B with probability 1 if

$$\phi\left(u(x, x_B) - (u(x, x_A) + \xi)\right) > c.$$

Otherwise, the voter abstains from voting. Denote by $M^H(F, G)$ the voting model with cost distribution $F \in \mathcal{F}$ and utility shock distribution $G \in \mathcal{G}$. $w^{M^H(F,G)}$ is defined in the same way as in the costly and probabilistic voting models.

Theorem 2. *Fix ϕ. For any cost distribution $F \in \mathcal{F}$ and utility shock distribution $G \in \mathcal{G}$, there exist $\bar{F} \in \mathcal{F}$ and $\bar{G} \in \mathcal{G}$ such that each of the costly voting model $M^C(\bar{F})$ and the probabilistic voting model $M^P(\bar{G})$ is equivalent to the hybrid model $M^H(F, G)$.*

Proof. See the Appendix. □

5 Conclusion

In this paper, we showed that costly and probabilistic voting models are equivalent in the two-candidate election: For any pair of the two candidates' announced policies, the winning probability of each candidate is identical under these two models. This result implies that predictions established in one of the models carry over to the other, providing robustness of the conclusions to model specifications. The sets of equilibrium policy profiles announced by the two candidates who try to maximize the winning probabilities, for instance, are identical under these two models. We also showed that, given any hybrid model with both voting cost and random utility shocks, there exist a costly voting model and a probabilistic voting model each of which is equivalent to the given hybrid model. This result implies that there is no loss of generality in assuming away either the cost or random utility shocks from the model.

Appendix: Proof of Theorem 2

Proof. Extend the definition of $R(M, x_A, x_B)$ in the proof of Theorem 1 to include hybrid models. Again this is well-defined. Let H be the cumulative distribution function of $\phi^{-1}(c) - \xi$.

By definition, we have

$$R(M^C(\bar{F}), x_A, x_B) = \int_X \left[\bar{F}\left(u(x, x_A) - u(x, x_B)\right) - \bar{F}\left(u(x, x_B) - u(x, x_A)\right)\right] d\mu(x)$$

for any $\bar{F} \in \mathcal{F}$ as before, and

$$R(M^H(F, G, x_A, x_B)) = \int_X \left[H\left(u(x, x_A) - u(x, x_B)\right) - H\left(u(x, x_B) - u(x, x_A)\right))\right] d\mu(x)$$

for any $H \in \mathcal{H}$. Denoting $t = u(x, x_A) - u(x, x_B)$, we have:

$$R(M^C(\bar{F}), x_A, x_B) = \int_X \left[\bar{F}(t) - \bar{F}(-t)\right] d\mu(x)$$

and

$$R(M^H(F, G), x_A, x_B) = \int_X \left[H(t) - H(-t)\right] d\mu(x).$$

This means it suffices to show that for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ there exists $\bar{F} \in \mathcal{F}$ such that condition (2) holds, where

$$\bar{F}(t) - \bar{F}(-t) = H(t) - H(-t). \tag{2}$$

Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ be given. Define $\bar{F}$ by

$$\bar{F}(t) = \begin{cases} H(t) - H(-t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Since H is nondecreasing by construction, $\bar{F}$ is nondecreasing. Moreover, $\bar{F}$ satisfies $\bar{F}(t) = 0$ for all $t \leq 0$ by definition, and $\lim_{t\to\infty} \bar{F}(t) = \lim_{t\to\infty} H(t) - H(-t) = 1$. Therefore $\bar{F}$ is a cumulative distribution function over $\mathbb{R}$ and $\bar{F} \in \mathcal{F}$. It is straightforward by inspection that the above definition of $\bar{F}$ satisfies condition (2).

To complete the proof note that, by Theorem 1, there exists $\bar{G} \in \mathcal{G}$ such that $M^P(\bar{G})$ is equivalent to $M^C(\bar{F})$. Since $M^C(\bar{F})$ is equivalent to $M^H(F, G)$ by the preceding argument, this implies that $M^P(\bar{G})$ is equivalent to $M^H(F, G)$. This completes the proof.

□

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