

Efficient Resource Allocation under Multi-unit Demand

Fuhito Kojima*

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Abstract

We consider a resource allocation problem with multi-unit demand, such as the allocation of courses to students. Unlike the case with single-unit demand, the celebrated (student-proposing) deferred acceptance algorithm fails to achieve desirable properties: It is not strategy-proof and the resulting allocation is not even weakly Pareto efficient under submitted preferences. We characterize the priority structure of courses over students such that the mechanism is strategy-proof or Pareto efficient. We show that either condition holds if and only if the priority structure is essentially homogeneous. This result suggests a sense in which efficient allocation under multi-unit demand is difficult and that the use of the deferred acceptance algorithm (or any stable mechanism) does not necessarily address the problem. *Journal of Economic Literature Classification Numbers: C71, C78, D71, D78, J44.*

Key Words: matching, stability, strategy-proofness, robust stability, acyclicity.

1 Introduction

Suppose that a mechanism designer needs to allocate indivisible goods to a set of agents. Housing allocation in universities and student placement in public schools are important examples in real life.¹ The goal of the mechanism designer is to assign the goods in an efficient and fair way, while eliciting

*Department of Economics, Stanford University, fkojima@stanford.edu. I am grateful to Eric Budish, Yusuke Narita, and Yuki Takagi for comments. I also thank Seung Hoon Lee and Pete Troyan for their very careful proofreading.

¹See Abdulkadiroğlu and Sönmez (1999) and Chen and Sönmez (2002) for application to house allocation, and Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) for student placement.

the true preferences of the agents. In many of these problems each good is endowed with a priority ordering over agents. An allocation is stable if any good that an agent prefers to her assignment is assigned (up to capacity) to others who are granted higher priority for it. Stability is a fairness concept in the sense that there is no justified envy.

In environments where each agent receives only one good, the (agent-proposing) deferred acceptance algorithm (mechanism) of Gale and Shapley (1962) is a prominent solution. There are many advantages justifying the use of the deferred acceptance algorithm. First, it finds a stable matching for any input. In fact, the matching found by the algorithm (weakly) Pareto dominates any other stable matching. Moreover, the deferred acceptance algorithm is strategy-proof, that is, truthful reporting of preferences is a dominant strategy for every agent under this mechanism. Justified by these desirable properties, the deferred acceptance algorithm has been employed in many practical resource allocation problems such as student placement in New York City and Boston (Abdulkadiroğlu, Pathak, and Roth, 2005; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005) and university house allocation in MIT (Guillen and Kesten, 2010).

By contrast, the deferred acceptance algorithm is used less frequently in situations with multi-unit demands, in which each individual may demand more than one good. Drafts in professional sports such as baseball and basketball is a typical example, in which teams take turns to select several players. Another example that recently attracted attention of researchers is course allocation in educational institutions such as business schools.² A pioneering work by Sönmez and Ünver (2010) points out that there are deficiencies in course allocation mechanisms used at business schools in practice. Based on this finding, they propose a new solution building on the deferred acceptance mechanism. In field and laboratory experiments, their mechanism achieves a significant efficiency gain over existing course-bidding mechanisms (Krishna and Ünver, 2008). However, even the deferred acceptance mechanism is neither (even weakly) Pareto efficient nor strategy-proof under multi-unit demand.³ Therefore, not all justifications for the practical use of

²Yet another example is a labor matching market for doctors in the United Kingdom (Roth, 1991). In the U.K., new doctors sought two positions for their postgraduate training, namely one medical position and one surgical position, and the multi-unit demand version of the deferred acceptance mechanism was used in regions such as Cardiff and Edinburgh. This market is not the perfect match to the model of this paper because it is a *two-sided* matching market in which not only doctors but also hospital directors are strategic players. However, the analysis of efficiency and incentives for doctors studied in this paper are applicable to this market as well.

³Sönmez and Ünver (2010) show that students report preferences truthfully under what they call the “price-taking behavior.” We will discuss their analysis in more detail

the deferred acceptance mechanism under single-unit demand carry over to these environments.

While a disappointing result, the lack of efficiency or strategy-proofness of the deferred acceptance mechanism only means that these properties are *not guaranteed for all possible priority structures*. The current paper thus investigates conditions under which the deferred acceptance mechanisms can guarantee Pareto efficiency or strategy-proofness. In other words, we provide a characterization of priority structures such that these desiderata hold. The main result of the paper demonstrates that each of Pareto efficiency and strategy-proofness is equivalent to essential homogeneity of priorities: A priority structure is said to be essentially homogeneous if all goods have the same priority order over students, possibly except for (inconsequential) differences among the very top students who are guaranteed the good if they apply. Thus the result implies that not only is it impossible to guarantee efficiency or incentive compatibility in general, but also that these properties can never be guaranteed except for the trivial case with almost no variation in priority orders. And under such a priority structure, the mechanism is a multi-unit version of the serial dictatorship (Papai, 2001; Ehlers and Klaus, 2003; Hatfield, 2009), under which the first agent receives her most preferred goods up to her quota, the second agent receives his most preferred remaining goods up to his quota, and so forth. As the serial dictatorship suffers from a number of drawbacks (Budish and Cantillon, 2010), our result suggests that some of justifications for the deferred acceptance mechanism under single-unit demand such as student placement may not carry over to situations with multi-unit demand such as course allocation.

Section 2 presents the model. The main result is presented in Section 3. Section 4 provides discussion, and Section 5 concludes. The proof of the main result is in the Appendix.

2 Model

2.1 Preliminary Definitions

A problem is specified by a tuple (S, C, P, \succ, q_C) . S and C are finite and disjoint sets of students and courses (although we frame the model in terms of students and courses, the analysis is applicable to more general resource allocation problems). Each student $s \in S$ has a strict preference relation P_s over the set of subsets of C . The weak preference relation associated with P_s is denoted by R_s , that is, we write $C' R_s C''$ (where $C', C'' \subseteq C$) if either

in Section 5.

$C'P_sC''$ or $C' = C''$. Following a standard assumption in the matching literature, we assume that the preference relation of each student is **responsive** (Roth, 1985), that is, there exists a positive integer q_s , called the **quota** of student s , such that,

- (1) For any $C' \subseteq C$ with $|C'| \leq q_s$, $c \in C \setminus C'$ and $c' \in C'$, $C' \cup c \setminus c'P_sC'$ if and only if cP_sc' ,⁴
- (2) For any $C' \subseteq C$ with $|C'| \leq q_s$ and $c' \in C'$, $C'P_sC' \setminus c'$ if and only if $c'P_s\emptyset$, and
- (3) $\emptyset P_sC'$ for any $C' \subseteq C$ with $|C'| > q_s$.

The set of all responsive preferences for a student is denoted by \mathcal{P} .⁵

Each course $c \in C$ is endowed with a **priority** \succ_c , which is a strict, complete, and transitive binary relation over S .⁶ For each $c \in C$, q_c is the **quota** of c . We write $\succ = (\succ_c)_{c \in C}$ and $q_C = (q_c)_{c \in C}$, and refer to the pair (\succ, q_C) as a **priority structure**.

A **matching** is a vector $\mu = (\mu_s)_{s \in S}$ that assigns a seat at courses $\mu_s \subseteq C$ to each student s , with seats in each course c assigned to at most q_c students. We write $\mu_c = \{s \in S | c \in \mu_s\}$ for the set of students who are assigned seats at course c .

We say that matching μ is **Pareto efficient** if there exists no matching μ' such that $\mu'_s R_s \mu_s$ for all $s \in S$ and $\mu'_s P_s \mu_s$ for at least one $s \in S$.

We say that matching μ is **blocked** by $(s, c) \in S \times C$ if either (1) $cP_s\emptyset$ and $|\mu_s| < q_s$ or (2) cP_sc' for some $c' \in \mu_s$, and either (1) $|\mu_c| < q_c$ or (2) $s \succ_c s'$ for some $s' \in \mu_c$. A matching μ is **individually rational** if $\mu_s R_s C'$ for every $s \in S$ and $C' \subseteq \mu_s$. A matching μ is **stable** if it is individually rational and it is not blocked.

We refer to a tuple (S, C, \succ, q_C) as a **market** and consider a situation where only student preferences are private information while the market (S, C, \succ, q_C) is exogenously given. Given a market, a **mechanism** is a function φ from $\mathcal{P}^{|S|}$ to the set of matchings. Mechanism φ is **Pareto efficient** if $\varphi(P)$ is a Pareto efficient matching for every $P \in \mathcal{P}^{|S|}$. Mechanism φ is **stable** if $\varphi(P)$ is a stable matching for every $P \in \mathcal{P}^{|S|}$. Mechanism φ is **strategy-proof** if $\varphi_s(P) R_s \varphi_s(P'_s, P_{-s})$ for every $P \in \mathcal{P}^{|S|}$, $s \in S$ and

⁴We denote singleton set $\{x\}$ by x when there is no confusion.

⁵The assumption that all students have responsive preferences can be relaxed. In Section 4.2, we show that the results generalizes to the case in which preferences of every student is substitutable.

⁶As we are primarily interested in resource allocation problems such as course assignment, we assume that every student is acceptable to every course.

$P'_s \in \mathcal{P}$. Observe that only students report their preferences while course priorities are publicly known. This assumption is usually satisfied in practical resource allocation problems such as course allocation. Also in the main text we assume that the quota q_s of each student is part of private information about preferences of student s , but in Section 4.3 we show that the conclusion of this paper is robust to different informational structures.

Given P , the (student-proposing) **deferred acceptance algorithm** is defined as follows (Gale and Shapley (1962); see Roth and Sotomayor (1990) for extensions to environments under multi-unit demand).

- Step 1: Each student s applies to her q_s most preferred courses (if any). Each course rejects the lowest-ranking students in excess of its capacity among those who applied to it, keeping the rest of students temporarily (so students not rejected at this step may be rejected in later steps.)

In general, for any $t \geq 2$,

- Step t : Each student s who was not tentatively matched to q_s courses in Step $(t-1)$ applies to her next highest acceptable courses up to quota if any.⁷ Each course considers these students *and* students who are temporarily held from the previous step together, and rejects the lowest-ranking students in excess of its capacity, keeping the rest of students temporarily (so students not rejected at this step may be rejected in later steps.)

The algorithm terminates at the first step when no student applies to a course. Each student tentatively accepted by a course at that step is allocated a seat in that course, resulting in a matching which we denote by $\varphi^S(P)$. The **student-optimal stable mechanism**, or the **deferred acceptance mechanism**, is a mechanism φ^S that produces $\varphi^S(P)$ for every $P \in \mathcal{P}^{|S|}$. It is well known that φ^S is a stable mechanism (Gale and Shapley, 1962).

2.2 Problems under Multi-Unit Demand

In addition to stability, another reason that the deferred acceptance mechanism φ^S is considered very desirable is that it is strategy-proof if each student s has a single-unit demand, that is, $q_s = 1$ (Dubins and Freedman, 1981; Roth, 1982). However, φ^S is not strategy-proof under multi-unit demand as seen in the following example.

⁷An alternative definition of the algorithm would be to have each student apply to only one additional course at each step. All our results are unchanged under this alternative formulation because, for any input, the result of such an algorithm coincides with the one produced by the algorithm employed in this paper.

Example 1. Let $S = \{1, 2\}$, $C = \{a, b\}$, $1 \succ_a 2$, $2 \succ_b 1$, and $q_a = q_b = 1$. Consider the following preferences of students:

$$\begin{aligned} P_1 &: b, a, \emptyset; & q_1 &= 2, \\ P_2 &: a, b, \emptyset; & q_2 &= 1. \end{aligned}$$

In the deferred acceptance algorithm under preference profile P , student 1 applies to both a and b while student 2 applies to a only. At this step course b keeps the only applicant 1 while course a keeps 1 and rejects 2. Rejected from her first choice course a , student 2 applies to her second choice b . Faced with the current match 1 and the new applicant 2, course b keeps 2 while rejecting 1. Student 1 has applied to all her acceptable courses, so the algorithm terminates. Thus the student-optimal stable matching $\varphi^S(P)$ of this market is given by

$$\varphi_1^S(P) = \{a\}, \varphi_2^S(P) = \{b\}.$$

Then consider a reported preference of student i , $P'_1 : b, \emptyset; q_1 = 1$. We write $P' = (P'_1, P_{-1})$. At the first step of the deferred acceptance algorithm under P' , student 1 applies to b only and student 2 applies to a only, and both of them are kept by their first choice courses. Thus the algorithm terminates immediately. Therefore the student optimal stable matching $\varphi^S(P') = \mu$ under P' is given by

$$\varphi_1^S(P') = \{b\}, \varphi_2^S(P') = \{a\}.$$

In particular, $\varphi_1^S(P') = \{b\}$. Since $\varphi_1^S(P') = \{b\} P_1 \{a\} = \varphi_1^S(P)$, φ^S is not strategy-proof. From this example, it is also clear that truthtelling by all agents is not necessarily a Nash equilibrium of the revelation game induced by the deferred acceptance mechanism.

Regarding efficiency, even with single-unit demand, φ^S is not necessarily Pareto efficient. However, Ergin (2002) provides a characterization of priority structures under which φ^S is a Pareto efficient mechanism. The condition is called acyclicity and defined as follows.

Definition 1 (Ergin (2002)). A priority structure (\succ, q_C) is **acyclic** if there exist no $a, b \in C$ and $i, j, k \in S$ such that

- $i \succ_a j \succ_a k$ and $k \succ_b i$, and
- There exist disjoint sets of students $S_a, S_b \subset S \setminus \{i, j, k\}$ such that $|S_a| = q_a - 1$, $|S_b| = q_b - 1$, $s \succ_a j$ for every $s \in S_a$ and $s \succ_b i$ for every $s \in S_b$.

The acyclicity condition requires that there be only little variation in priority orders across different courses. To illustrate the basic intuition, consider a special case in which each course has only one seat (that is, $q_c = 1$ for all $c \in C$). Then the second condition of the definition is vacuously met because $S_a = S_b = \emptyset$ satisfy the condition. Thus in this case, acyclicity requires that there be no courses a and b and students i, j and k such that course a prioritizes i, j , and then k while course b prioritizes k ahead of i . As shown by Theorem 2 of Ergin (2002), this is equivalent to the property that the priority ranking of each student varies at most by one across all courses. Acyclicity in the general case, with course quotas potentially larger than one, is more complicated as can be seen in Definition 1, but the basic idea is unchanged and limits the admissible variations in priorities.

The acyclic priority structure plays an essential role for efficiency under single-unit demand. More specifically, Ergin (2002) shows that φ^S is Pareto efficient if and only if the priority structure is acyclic. Although acyclicity is a strong requirement, some variations in priorities are allowed across different courses. Thus there are some, though limited, cases in which the deferred acceptance mechanism is guaranteed to provide an efficient matching.

Under multi-unit demand, however, inefficiency problems are more serious than under single-unit demand. First, the student-optimal stable matching is not necessarily Pareto efficient even if the priority structure is acyclic. To see this point, note that the priority structure in Example 1 is acyclic, but $\varphi_1^S(P')P_1\varphi_1^S(P)$ and $\varphi_2^S(P')P_2\varphi_2^S(P)$, thus $\varphi^S(P)$ is not Pareto efficient. Second, under single-unit demand, φ^S is weakly Pareto efficient for any priority structure in the sense that there is no matching strictly preferred by every student, but this conclusion also fails under multi-unit demand. To see this point, notice that in Example 1 both students 1 and 2 strictly prefer $\varphi^S(P')$ to $\varphi^S(P)$ according to preference profile P .⁸

Given these negative conclusions under multi-unit demand, a natural question to investigate is whether a mechanism achieves desirable properties in a *specific market*. In other words, we investigate conditions on the priority structure (\succ, q_C) under which a stable mechanism is strategy-proof or Pareto efficient. The following concept will prove to play a central role in our analysis.

Definition 2. A priority structure (\succ, q_C) is **essentially homogeneous** if there exist no $a, b \in C$ and $i, j \in S$ such that

- $i \succ_a j$ and $j \succ_b i$, and

⁸A discussion offered in Section 4.1 provides a further comparison between single-unit demand and multi-unit demand environments.

- There exist sets of students $S_a, S_b \subset S \setminus \{i, j\}$ such that $|S_a| = q_a - 1$, $|S_b| = q_b - 1$, $s \succ_a j$ for every $s \in S_a$ and $s \succ_b i$ for every $s \in S_b$.

As the name suggests, essentially homogeneous priority structures allow for almost no variation in priorities between different courses. To illustrate the basic idea of the condition, consider a special case in which each course has only one seat, that is, $q_c = 1$ for all $c \in C$. Then the second condition of the definition regarding sets S_a and S_b is vacuously satisfied (to see this point, verify that $S_a = S_b = \emptyset$ satisfy the condition). Thus in this case, essential homogeneity requires that there be no courses a and b and students i and j such that i has a higher priority for a than j but j has a higher priority for b than i . This is equivalent to the requirement that the priority orderings for all courses are exactly identical.

Essential homogeneity in the general case, with course quotas potentially larger than one, is more complicated as can be seen in Definition 2. Still, the only variation in the priority ordering allowed involves the top q_c students for course c . Such a student is admitted to course c whenever she applies to it, so how highly she is ordered within those top students does not affect the allocation. In other words, the apparent heterogeneity in priorities among these top q_c students are in fact irrelevant. Under an essentially homogeneous priority structure, the deferred acceptance algorithm coincides with a multi-unit demand version of the serial dictatorship, under which the first student receives her most preferred courses up to her quota, the second student receives his most preferred remaining courses up to his quota, and so forth. In particular, it can easily be seen that essential homogeneity implies acyclicity but not vice versa, thus the former requirement is a strictly stronger restriction than the latter.

3 The Main Result

With the concepts introduced in the last section, we present our main theorem. This result offers a characterization of priority structures such that the deferred acceptance algorithm achieves desirable properties such as Pareto efficiency and strategy-proofness.

Theorem 1. *For market (S, C, \succ, q_C) , the following conditions are equivalent.*

- (1) φ^S is Pareto efficient.
- (2) φ^S is strategy-proof.

(3) *The priority structure (\succ, q_C) is essentially homogeneous.*

Proof. See Appendix. □

It is well known that φ^S weakly Pareto dominates any other stable mechanism (Gale and Shapley, 1962). Also any stable mechanism can be profitably manipulated by a student whenever φ^S can be profitably manipulated by a student (see Pathak and Sönmez (2010)). Thus this theorem implies that, given a market, there exists a stable mechanism that is Pareto efficient or strategy-proof if and only if the priority structure is essentially homogeneous.

While the proof of Theorem 1 is somewhat involved and thus relegated to the Appendix, part of its intuition can be obtained by reviewing Example 1. First, we illustrate how strategy-proofness fails in that example (note that the priority structure violates essential homogeneity in this example). If student 1 reports her preferences truthfully, she applies to both courses a and b as she has quota 2. Her application to a causes student 2 to be rejected from his first choice course a . This rejection in turn induces 2 to apply to b , which rejects student 1. By contrast, if 1 reports that b is her only acceptable choice, then no one is rejected in the first step of the algorithm. This results in an immediate termination, producing a matching in which student 1 is matched to her more preferred course b . Intuitively, by withdrawing an application to her acceptable but less preferred course a , student 1 can eliminate a “rejection chain,” a chain reaction of applications and rejections in the algorithm that displaces 1 from her first choice course b . The main proof of the claim that φ^S is not strategy-proof if the priority structure violates essential homogeneity generalizes this example, showing that one can always find a “rejection chain” similar to the above one for an appropriate preference profile of students.

A main idea behind necessity of essential homogeneity for Pareto efficiency can be seen by Example 1 as well. In that example, a rejection chain caused by student 1’s application to a causes more rejections to students, thus making students worse off. The rejection chain also hurts student 1 because it results in an additional application by student 2 to student 1’s first choice course b , which displaces student 1. The proof for the general case proceeds similarly, by finding an appropriately chosen preference profile of students such that a rejection chain causes efficiency loss.

The proofs of the sufficiency of essential homogeneity are also presented in the Appendix. While these proofs are more complicated than the necessity proofs, the first intuition is fairly simple. An essentially homogeneous priority implies that students are ordered by the common priority, except possibly for an irrelevant ordering among q_c top students at each course c . Hence the mechanism is the multi-unit version of the serial dictatorship. Indeed, it is possible to show that the mechanism is the serial dictatorship first

and then show strategy-proofness and Pareto efficiency as its implications.⁹ However, we chose to prove strategy-proofness and Pareto efficiency directly from essential homogeneity of the priority structure. This is because we believe that our direct proofs illuminate the roles of essential homogeneity for these properties more clearly and intuitively.

Theorem 1 suggests that inefficiency and strategic manipulation may be unavoidable unless the priority structure is so restrictive that the priority is essentially common across all courses. Thus almost no room is left for the match organizers to achieve policy goals by a judicious choice of priority structures. Thus the practical applicability of the deferred acceptance algorithm, or any priority-based stable mechanism, seems to be severely limited in resource allocation under multi-unit demand.

4 Discussion

4.1 Single-unit Demand

While the current paper is the first to, to our knowledge, characterize priority structures for desirable properties under multi-unit demand, similar approaches have been taken under single-unit demand. An influential work by Ergin (2002) shows that the student-optimal stable mechanism is group strategy-proof if and only if the priority structure is acyclic, and this condition is also necessary and sufficient for Pareto efficiency. The condition proved to be also necessary and sufficient for a number of non-manipulability properties of stable mechanisms (Haeringer and Klijn, 2009; Kesten, 2008; Kojima, 2010). Ehlers and Erdil (2010) and Kumano (2009) generalize the concept of acyclicity to coarse priorities and acceptant and substitutable priorities as defined by Kojima and Manea (2010a), respectively.

While sharing the motivation with these studies, the result of the current paper is in a sharp contrast with them. First, the incentive compatibility we require is only strategy-proofness, which is substantially weaker than group

⁹The sketch of the proof that mechanism φ^S under any essentially homogeneous priority structure is a serial dictatorship is as follows. We first construct an ordering over students. Then we verify that, for any preference profile, the allocation under φ^S coincides with the serial dictatorship with the constructed order. A challenge in the proof is that the appropriate order over students is not necessarily obvious from the priority structure. This is because quotas of some courses can be larger than one: Across courses with multiple quotas, some variations in the ordering are allowed even under essential homogeneity. In fact, the ordering over students is not necessarily unique. Thus the proof can proceed by finding one, not necessarily unique, ordering over students and showing that the allocation under φ^S coincides with that of the serial dictatorship with that ordering.

strategy-proofness required by Ergin (2002). In fact, under single-unit demand, the deferred acceptance mechanism φ^S is strategy-proof *regardless of the priority structure*. Second, the priority structure we obtain in the current paper is extremely restrictive. While quite restrictive, acyclic priority structures allow for some variations in priorities across different courses. By contrast, essential homogeneity requires that there be *no variation in priorities at all* across courses except for irrelevant ones. Taken together, the result of this study suggests that designing a satisfactory resource allocation mechanism is much more difficult when agents have multi-unit demand, and desirable properties of certain mechanisms under single-unit demand are unlikely to carry over to situations with multi-unit demand.

4.2 When Preferences Are Not Responsive

In this paper, we have assumed that preferences of all students are responsive. This assumption may not be satisfied in some applications because preferences could exhibit substitutability or complementarity. For instance, in course allocation students may be unable to take multiple courses offered at the same time slot or they may not want to take too many courses in one subject, resulting in substitutable preferences. A complementarity may exist between two courses as well: For instance, if a course in advanced microeconomics builds on concepts and techniques taught in a course in intermediate microeconomics, then these courses may be complements.

When preferences exhibit complementarity, even the existence of stable matching is not guaranteed (Kelso and Crawford, 1982). In fact, Sönmez and Ünver (2010) show that substitutability –the absence of complementarity– is a necessary and sufficient condition for guaranteeing the existence of a stable matching in the sense of a maximal domain.¹⁰ Thus the use of stable mechanisms such as deferred acceptance is difficult and perhaps unrealistic in the first place, even in the absence of efficiency or incentive issues studied in this paper. Thus this paper does not seek to investigate the efficiency or incentive properties under complementarity further.

On the other hand, the result of this paper can be generalized even if we allow students to have and express any substitutable preferences, which may not be responsive. In this sense, the implication of the paper does not depend on responsiveness of student preferences.

To see why our result extends to the case with substitutable preferences, first note that a stable matching exists and can be produced by the gen-

¹⁰Hatfield and Kojima (2008) offer an alternative formulation of the necessary and sufficient condition and show that substitutability is necessary and sufficient for the existence of a stable matching in their sense as well.

eralized deferred acceptance algorithm if preferences of every student are substitutable (see Chapter 6 of Roth and Sotomayor (1990)). Thus the mechanism is well-defined, and one can investigate the question of incentives or efficiency. To show that our theorem generalizes, first observe that the necessity of essential homogeneity extends straightforwardly to the case with substitutability, because the class of responsive preferences is a subset of the class of substitutable preferences. This implies that both Pareto efficiency and strategy-proofness are stronger requirements under substitutability, so necessity is only reinforced. Second, the sufficiency of essential homogeneity also generalizes with little modification. This is because sufficiency is proved by contradiction, and proceeds by finding a sequence of students and colleges that form a cycle. By inspection of the proof, the cycle can be constructed in the same manner as in the proof whether or not student preferences are responsive, and thus the proof goes through.

4.3 When Student Quotas Are Public Information

In this paper, preferences of a student are assumed to be private information of that student, including the quota of the student. The motivation behind this modeling decision is to allow for a full degree of freedom for student preferences within the class of responsive preferences. However, in some applications the quotas of students may be specified exogenously or public information. For instance, a course administrators in business schools often restrict the number of courses that each student can take in one semester.

It turns out that the result of this paper is unchanged if we assume that quotas for students are public information. To see this point, first note that sufficiency of essential homogeneity clearly extends in this case because the set of possible (true or reported) preferences under this assumption is a subset of all possible responsive preferences in which student quotas are private information: In the former, student quotas are known to the mechanism and thus cannot be misreported, while in the latter, students can potentially misreport their quotas. Second, the necessity of essential homogeneity also extends. To see this point, first note that the proof for Pareto efficiency is unchanged because no argument there hinges on informational structure.

The proof of strategy-proofness is less obvious because students cannot misreport their quotas if quotas are publicly known. To consider this case, note that the proof proceeds by contradiction, first assuming that the priority structure is not essentially homogeneous and then finding a profitable deviation. By inspection, the reported preferences used for a successful manipulation in the proof involves a student misreporting only preference ordering over individual courses while reporting her quota truthfully. Thus the

proof goes through without modification even if we assume that the student's quota is publicly known and thus cannot be reported strategically.

5 Conclusion

In resource allocation problems with multi-unit demand, this paper investigated priority structures under which the deferred acceptance mechanism is Pareto efficient or strategy-proof. We demonstrated that each of Pareto efficiency and strategy-proofness is equivalent to the requirement that the priority structure be essentially homogeneous. The result implies that the only priority structures in which Pareto efficiency or strategy-proofness is guaranteed are essentially homogeneous ones, in which almost no variation in priorities is allowed. Our result suggests one reason why the deferred acceptance mechanism (or other stable mechanisms) is rarely used in practical resource allocation problems under multi-unit demand unlike those under single-unit demand. The present paper reinforces the view of some recent studies on resource allocation under multi-unit demand, which suggest that designing a satisfactory mechanism can be very difficult (Budish and Cantillon, 2010).

Before concluding the paper, we discuss possible directions for future research. One possibility would be to study weak priorities (indifferences in priorities).¹¹ In many resource allocation problems such as school choice and course allocation, there are only coarse classes of priorities. That is, more than one student may be granted the same priority at a course. We assumed away this issue in the current study for simplicity, but a similar characterization may exist as long as one focuses on deterministic mechanisms. On the other hand, random mechanisms are often employed if priorities are weak, say in the form of random tie-breaking. Studies such as Hylland and Zeckhauser (1979), Bogomolnaia and Moulin (2001), Abdulkadiroglu, Che, and Yasuda (2009), and Budish and Cantillon (2010) show that randomizations over ex post efficient mechanisms can result in significant ex ante inefficiencies. Extending our analysis to random mechanisms under weak priorities would require a totally different modeling choice and is beyond the scope of this paper, but it would widen the applicability of the analysis.

Another interesting direction of research is to design a practical mechanism in the presence of multi-unit demand. Given the impossibility of exact incentive compatibility or efficiency, achieving approximate versions of these properties may be a fruitful approach. Indeed, such an approach has been

¹¹See Abdulkadiroglu, Pathak, and Roth (2009) and Erdil and Ergin (2008) for issues related to weak priorities in the school choice context.

taken to study asymptotic properties in two-sided matching (Roth and Peranson, 1999; Immorlica and Mahdian, 2005; Kojima and Pathak, 2009) as well as in resource allocation (Manea, 2009; Kojima and Manea, 2010b; Che and Kojima, 2008). In the context of course allocation, Sönmez and Ünver (2010) present a model in which priorities are determined by “bids” in terms of bidding points (mock currency), and show that truthtelling is an optimal decision for a student under the deferred acceptance mechanism under the “price-taking behavior.”¹² As pointed out by Krishna and Ünver (2008), it is in large markets that the assumption of price-taking behavior appears to be the most plausible, thus the deferred acceptance mechanism may be difficult to profitably manipulate in that context. More recently, Budish (2011) proposes a new mechanism motivated by the Walrasian mechanism that provides approximate rather than exact incentive compatibility and efficiency. However, the problem of an “optimal mechanism” is yet to be analyzed. More generally, finding (or perhaps even appropriately defining) a “satisfactory” allocation mechanism under multi-unit demand is an interesting open question.

Appendix: Proof of Theorem 1

We will show $(1) \Rightarrow (3)$, $(3) \Rightarrow (1)$, $(2) \Rightarrow (3)$, and $(3) \Rightarrow (2)$ in the statement of the Theorem in this order.

Proof of “(1) \Rightarrow (3)” We show the claim by contraposition. Suppose that the priority structure is not essentially homogeneous. Then, by definition, there exist $a, b \in C$, $i, j \in S$ such that

- $i \succ_a j$ and $j \succ_b i$, and
- There exist sets of students $S_a, S_b \subset S \setminus \{i, j\}$ such that $|S_a| = q_a - 1$, $|S_b| = q_b - 1$, $s \succ_a j$ for every $s \in S_a$ and $s \succ_b i$ for every $s \in S_b$.

¹²The concept of price taking behavior allows students to fail to recognize the full influence of their own actions on prices, so it does not contradict the fact that the deferred acceptance mechanism is not strategy-proof.

Consider the following preferences of students:

$$\begin{aligned}
P_i &: b, a, \emptyset; & q_i &= 2, \\
P_j &: a, b, \emptyset; & q_j &= 1, \\
P_s &: a, \emptyset; & q_s &= 1 \text{ for every } s \in S_a \setminus S_b, \\
P_s &: b, \emptyset; & q_s &= 1 \text{ for every } s \in S_b \setminus S_a, \\
P_s &: a, b, \emptyset; & q_s &= 2 \text{ for every } s \in S_a \cap S_b, \\
P_s &: \emptyset; & q_s &= 1 \text{ for every } s \in S \setminus [\{i, j\} \cup S_a \cup S_b].
\end{aligned}$$

It is easy to see that

$$\varphi_s^S(P) = \begin{cases} \{a\} & s = i, \\ \{b\} & s = j, \\ \{a\} & s \in S_a \setminus S_b, \\ \{b\} & s \in S_b \setminus S_a, \\ \{a, b\} & s \in S_a \cap S_b, \\ \emptyset & s \in S \setminus [\{i, j\} \cup S_a \cup S_b] \end{cases}. \quad (1)$$

The following matching μ defined as

$$\mu_s = \begin{cases} \{b\} & s = i, \\ \{a\} & s = j, \\ \{a\} & s \in S_a \setminus S_b, \\ \{b\} & s \in S_b \setminus S_a, \\ \{a, b\} & s \in S_a \cap S_b, \\ \emptyset & s \in S \setminus [\{i, j\} \cup S_a \cup S_b] \end{cases}, \quad (2)$$

satisfies that $\mu_s R_s \varphi_s^S(P)$ for all $s \in S$. Moreover, $\mu_i = \{b\} P_i \{a\} = \varphi_i^S(P)$, thus $\varphi^S(P)$ is not Pareto efficient.

Proof of “(3) \Rightarrow (1)” We show the claim by contradiction. To this end, suppose that (\succ, q_C) is essentially homogeneous but φ^S is not Pareto efficient. Then there exist preference profile P and matching μ such that $\mu_s R_s \varphi_s^S(P)$ for all $s \in S$ and $\mu_s P_s \varphi_s^S(P)$ for at least one $s \in S$.

Let s' be a student such that $\mu_{s'} P_{s'} \varphi_{s'}^S(P)$. This implies that there exists $c' \in C$ such that $s' \in \mu_{c'} \setminus \varphi_{c'}^S(P)$ and either $c' P_{s'} \emptyset$ and $|\varphi_{s'}^S(P)| < q_{s'}$ or $c' P_{s'} c'$ for some $c \in (\varphi_{s'}^S(P) \setminus \mu_{s'})$. Because $\varphi^S(P)$ is a stable matching by assumption, this implies that $|\varphi_{c'}^S(P)| = q_{c'}$ and $s \succ_{c'} s'$ for all $s \in \varphi_{c'}^S(P)$ (otherwise (s', c') is a blocking pair, which is a contradiction to stability of

$\varphi^S(P)$). Since $|\varphi_{c'}^S(P)| = q_{c'}$ while $s' \in \mu_{c'} \setminus \varphi_{c'}^S(P)$, there exists a student $s'' \in \varphi_{c'}^S(P) \setminus \mu_{c'}$. This in particular implies $\mu_{s''} \neq \varphi_{s''}^S(P)$. This fact and $\mu_{s''} R_{s''} \varphi_{s''}^S(P)$ (which holds by the assumption that $\mu_s R_s \varphi_s^S(P)$ for all $s \in S$) imply that there exists $c'' \in C$ such that $s'' \in \mu_{c''} \setminus \varphi_{c''}^S(P)$ and $c'' P_{s''} c$ for some $c \in \{\emptyset\} \cup \varphi_{s''}^S(P) \setminus \mu_{s''}$.

Applying the same line of argument as above repeatedly (and noting that the number of students is finite), we conclude that there exist finite sequences of students $s_1, s_2, \dots, s_{n-1}, s_n$ and courses $c_1, c_2, \dots, c_{n-1}, c_n$ such that

$$s_t \in \varphi_{c_{t-1}}^S(P) \setminus \varphi_{c_t}^S(P), \quad (3)$$

$$|\varphi_{c_t}^S(P)| = q_{c_t}, \quad (4)$$

$$s \succ_{c_t} s_t \text{ for all } s \in \varphi_{c_t}^S(P), \quad (5)$$

$$\text{either } c_t P_{s_t} \emptyset \text{ and } |\varphi_{s_t}^S(P)| < q_{s_t} \text{ or } c_t P_{s_t} c \text{ for some } c \in \varphi_{s_t}^S(P), \quad (6)$$

for all $t = 1, 2, \dots, n-1, n$ (where indices are defined modulo n , that is, 0 and n are understood to be equivalent). In particular, relations (3) and (5) imply

$$s_{t+1} \succ_{c_t} s_t, \quad (7)$$

for all $t = 1, 2, \dots, n-1, n$. The following claim plays a key role in the remainder of the proof.

Claim 1. *For any $t = 1, 2, \dots, n-1, n$, we have $s_{t+1} \succ_{c_n} s_t$.*

Proof. We show the claim by induction. Consider first the case for $t = n$ as a base step. In this case, the conclusion holds by relation (7).

Then, as an inductive step, assume that the conclusion of the claim holds for $n, n-1, \dots, t+1$ (where $t < n$) and we shall show that it holds for t . Suppose, for contradiction, that

$$s_t \succ_{c_n} s_{t+1}. \quad (8)$$

First note that

$$s_{t+1} \notin \varphi_{c_n}^S(P), \quad (9)$$

because otherwise the inductive assumption $s_n \succ_{c_n} s_{t+1}$ and the properties (3) and (6) imply that (s_n, c_n) blocks $\varphi^S(P)$, contradicting stability of mechanism φ^S . Define

$$S_{c_t} = \varphi_{c_t}^S(P) \setminus \{s_{t+1}\},$$

$$S_{c_n} = \begin{cases} \varphi_{c_n}^S(P) \setminus \{s_t\} & \text{if } s_t \in \varphi_{c_n}^S(P), \\ \varphi_{c_n}^S(P) \setminus \{s_1\} & \text{otherwise.} \end{cases}$$

Then it follows that

$$S_{c_t}, S_{c_n} \subseteq S \setminus \{s_t, s_{t+1}\}, \quad (10)$$

by construction and properties (3) and (9), and

$$|S_{c_t}| = q_{c_t} - 1, |S_{c_n}| = q_{c_n} - 1 \quad (11)$$

by relations (3) and (4). Moreover,

$$s \succ_{c_t} s_t \text{ for every } s \in S_{c_t} \quad (12)$$

because relation (5) and $S_{c_t} \subseteq \varphi_{c_t}^S(P)$ hold (the latter property follows from the definition of S_{c_t}). Further,

$$s \succ_{c_n} s_{t+1} \text{ for every } s \in S_{c_n} \quad (13)$$

because (i) $s \succ_{c_n} s_n$ by relation (5), and (ii) $s_n \succ_{c_n} s_{n-1} \succ_{c_n} \dots \succ_{c_n} s_{t+1}$ by the inductive assumption. Relations (7), (8), (10), (11), (12), and (13) contradict the assumption that the priority structure is essentially homogeneous. \square

To finish the proof, simply observe that the above Claim 1 implies

$$s_1 \succ_{c_n} s_n \succ_{c_n} \dots \succ_{c_n} s_2 \succ_{c_n} s_1.$$

This is a contradiction, completing the proof.

Proof of “(2) \Rightarrow (3)” We show the claim by contraposition. Suppose that the priority structure is not essentially homogeneous. Consider the market and preference profile P in the proof of “(1) \Rightarrow (3)”. From the previous proof, $\varphi^S(P)$ is given by equation (1). In particular, $\varphi_i^S(P) = \{a\}$. Now consider a false preference of student i , $P'_i : b, \emptyset; q_i = 1$. We write $P' = (P'_i, P_{-i})$. Then it is easy to show that $\varphi^S(P') = \mu$ as defined by (2). In particular, $\varphi_i^S(P') = \{b\}$. Since $\varphi_i^S(P') = \{b\}P_i\{a\} = \varphi_i^S(P)$, we conclude that φ^S is not strategy-proof.

Proof of “(3) \Rightarrow (2)” We show the claim by contradiction. To this end, suppose that (\succ, q_C) is essentially homogeneous but φ^S is not strategy-proof. Then there exist preference profile P , student $s' \in S$ and her reported preference $P'_{s'}$ such that $\mu_{s'}P'_{s'}\varphi_{s'}^S(P)$, where we denote $\mu := \varphi^S(P'_{s'}, P_{-s'})$. This implies that there exists $c' \in C$ such that $s' \in \mu_{c'} \setminus \varphi_{c'}^S(P)$ and either $c'P'_{s'}\emptyset$ and $|\varphi_{s'}^S(P)| < q_{s'}$ or $c'P'_{s'}c$ for some $c \in (\varphi_{s'}^S(P) \setminus \mu_{s'})$. Because $\varphi^S(P)$ is

a stable matching by assumption, this implies that $|\varphi_{c'}^S(P)| = q_{c'}$ and $s \succ_{c'} s'$ for all $s \in \varphi_{c'}^S(P)$ (otherwise (s', c') is a blocking pair, which is a contradiction to stability of $\varphi^S(P)$). Since $|\varphi_{c'}^S(P)| = q_{c'}$ while $s' \in \mu_{c'} \setminus \varphi_{c'}^S(P)$, there exists a student $s'' \in \varphi_{c'}^S(P) \setminus \mu_{c'}$. Since $s'' \succ_{c'} s'$ but $s'' \notin \mu_{c'}$, there exists $c'' \in C$ such that $s'' \in \mu_{c''} \setminus \varphi_{c''}^S(P)$ and $c'' P_{s''} c$ for some $c \in \{\emptyset\} \cup \varphi_{s''}^S(P) \setminus \mu_{s''}$.

Since the number of students is finite, applying the same line of argument as above repeatedly we conclude that there exist sequences of students $s_1, s_2, \dots, s_{n-1}, s_n$ and courses $c_1, c_2, \dots, c_{n-1}, c_n$ such that

$$\begin{aligned} s_t &\in \varphi_{c_{t-1}}^S(P) \setminus \varphi_{c_t}^S(P), \\ |\varphi_{c_t}^S(P)| &= q_{c_t}, \\ s &\succ_{c_t} s_t \text{ for all } s \in \varphi_{c_t}^S(P), \\ \text{either } c_t P_{s_t} \emptyset \text{ and } |\varphi_{s_t}^S(P)| &< q_{s_t} \text{ or } c_t P_{s_t} c \text{ for some } c \in \varphi_{s_t}^S(P), \end{aligned}$$

for all $t = 1, 2, \dots, n-1, n$ (where indices are defined modulo n , that is, 0 and n are understood to be equivalent). These conditions are identical to relations (3)–(6) in the proof of “(3) \Rightarrow (1)”. Thus, following the same argument as in that proof, we can conclude that the priority structure is not essentially homogeneous. This is a contradiction and completes the proof.

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