

Matching with Contracts: Comment

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The theory of two-sided matching markets has interested researchers for its theoretical appeal and its relevance to the design of real-world institutions. The medical residency matching mechanism in the United States, the National Resident Matching Program (NRMP), and student assignment systems in New York City and Boston are important mechanisms designed by economists using the theory.¹ Hatfield and Paul R. Milgrom (2005) present a unified framework of matching with contracts, which includes the two-sided matching and package auction models as well as a simplified version of a labor market model of Alexander Kelso and Vincent P. Crawford (1982) as special cases. They introduce the substitutes condition of contracts, which is an adaptation of the substitutability condition in the matching literature (Roth and Marilda A. Oliveira Sotomayor 1990) to matching with contracts. They show that there exists a stable allocation if contracts are substitutes for hospitals, and that a number of other results in matching generalize to problems with contracts. They further claim that the substitutes condition on individual hospitals' preferences is necessary to guarantee the existence of a stable allocation for all possible singleton preferences of others: that is, if contracts are not substitutes for a hospital, then there exists a preference profile of other hospitals with single job openings and doctors such that there exists no stable allocation.

We show that this last claim of Hatfield and Milgrom (2005) does not hold in general. The counterexample we present involves a hospital which has two different kinds of positions, and the preferences of the hospital are a combination of preferences of these two positions. While the substitutes condition does not hold for that hospital, we adopt a technique developed by Bettina Klaus and Flip Klijn (2005) and show that there exists a stable allocation.

We then present the weak substitutes condition, which we show is necessary to guarantee existence of stable allocations. The modified result we present, as well as the original contribution of Hatfield and Milgrom (2005), suggest a connection between the substitutes condition and stability.²

I. Model

Our model and notation follow Hatfield and Milgrom (2005). A matching problem with contracts (or simply a problem) is parameterized by a finite set of doctors D , a finite set of hospitals H , a finite set of contracts X , and preferences $[>_a]_{a \in D \cup H}$. Each contract $x \in X$ is bilateral, so that it is associated with a doctor $x_D \in D$ and a hospital $x_H \in H$.³ For any $X' \subset X$, we denote the set of

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¹ For applications to labor markets, see Alvin E. Roth (1984) and Roth and Elliott Peranson (1999). For applications to student assignment, see, for example, Atila Abdulkadiroğlu and Tayfun Sönmez (2003), Abdulkadiroğlu, Parag A. Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2005).

² Further issues related to the current note are investigated by Hatfield and Kojima (2007).

³ Throughout the paper, we assume that, for any $d \in D$ and any $h \in H$, there exists a contract $x \in X$ with $x_D = d$ and $x_H = h$.

corresponding doctors by $x_D(X') = \{d \in D \mid \exists x \in X', x_D = d\}$. We assume that each doctor can sign at most one contract. The null contract, meaning that the doctor has no contract, is denoted by \emptyset . For each $d \in D$, $>_d$ is a strict preference relation on $\{x \in X \mid x_D = d\} \cup \{\emptyset\}$. A contract is *acceptable* if it is preferred to the null contract and *unacceptable* if it is less preferred to the null contract. As notational convention, we write acceptable contracts in order of preferences. For example,

$$P_d : x >_d y >_d z$$

means that, under preference $>_d$ of d , x is the most preferred contract, followed by y and z , and every other contract is unacceptable.

For each $d \in D$ and $X' \subset X$, we define the *chosen set* C_d by

$$C_d(X') = \max_{>_d} (\{x \in X' \mid x_D = d\} \cup \{\emptyset\}),$$

for any $X' \subset X$. Let $C_D(X') = \cup_{d \in D} C_d(X')$ be the set of contracts chosen from X' by some doctor.

We allow each hospital to sign multiple contracts, and assume that each hospital $h \in H$ has a preference relation $>_h$ on subsets of contracts involving it. For any $X' \subset X$, define $C_h(X')$ by

$$C_h(X') = \max_{>_h} \{X'' \subset X' \mid (x \in X'' \Rightarrow x_H = h) \wedge (x, x' \in X'', x \neq x' \Rightarrow x_D \neq x'_D)\}.$$

Let $C_H(X') = \cup_{h \in H} C_h(X')$ be the set of contracts chosen from X' by some hospital. For any $X' \subset X$ and $h \in H$, define the *rejected set* by $R_h(X') = X' \setminus C_h(X')$. $R_h(X')$ is the set of contracts in X' which h is willing to reject.

As with the preferences of doctors, we express the preferences of hospitals by writing all the acceptable sets of contracts in order of preference.

A set of contracts $X' \subset X$ is an *allocation* if $x, x' \in X'$ and $x \neq x'$ imply $x_D \neq x'_D$. In words, a set of contracts is an allocation if each doctor signs at most one contract.

DEFINITION 1: A set of contracts $X' \subset X$ is a *stable allocation* (or a *stable set of contracts*) if

(a) $C_D(X') = C_H(X') = X'$, and

(b) There exists no hospital h and set of contracts $X'' \neq C_h(X')$ such that $X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'')$.

When condition (2) is violated for X' by some h and X'' with corresponding doctors $x_D(X'')$, we say that h and $x_D(X'')$ block X' by X'' .

II. Counterexample

The following condition plays a major role in our analysis.

DEFINITION 2: Contracts are *substitutes for h* if we have $R_h(X') \subset R_h(X'')$ for all $X' \subset X'' \subset X$.

Hatfield and Milgrom (2005) show that there exists a stable allocation when contracts are substitutes for every hospital.

RESULT 1 (Hatfield and Milgrom 2005): *Suppose that contracts are substitutes for every hospital. Then there exists a stable set of contracts.*

Then they investigate further connections between the substitutes condition and stability, and present the following claim.

Claim 1 (Hatfield and Milgrom 2005, Theorem 5): Suppose H contains at least two hospitals, which we denote by h and h' . Further suppose that R_h is not isotone, that is, contracts are not substitutes for h . Then, there exist preference orderings for the doctors in set D , a preference ordering for a hospital h' with a single job opening such that, regardless of the preferences of the other hospitals, no stable set of contracts exists.

The following example shows that Claim 1 does not hold in general. (In the Web Appendix, available at <http://www.aeaweb.org/articles.php?doi=10.1257/aer.98.3.1189>, we point out the part of the proof of Claim 1 that contains an error.)

Example 1: Suppose that h has possible contracts denoted by (d_1, h_r) , (d_2, h_r) , and (d_1, h_c) , where $(d_1, h_r)_D = (d_1, h_c)_D = d_1$ and $(d_2, h_r)_D = d_2$, and $(d_1, h_r)_H = (d_1, h_c)_H = (d_2, h_r)_H = h$.⁴ Preference of h is given by

$$P_h : \{(d_2, h_r), (d_1, h_c)\} \succ_h \{(d_1, h_r)\} \succ_h \{(d_2, h_r)\} \succ_h \{(d_1, h_c)\}.$$

The preference P_h above has the following interpretation. Hospital h has a research position h_r and a clinical position h_c , each with one seat. Its potential candidates are d_1 and d_2 . Although h wants to fill both research and clinical positions, if possible, it prefers to fill a research position if only one doctor is available. Doctor d_1 is eligible both for research and clinical positions, whereas d_2 qualifies only for a research position. Muriel Niederle (2007) and Niederle, Deborah D. Proctor, and Roth (2006) observe that in the gastroenterology fellowship market in the United States, hospitals often wish to hire fellows in research positions and then try to hire fellows in clinical positions in case they fail to hire enough research fellows.

First, note that contracts are not substitutes for h , since $(d_2, h_r) \in R_h(\{(d_2, h_r), (d_1, h_r)\})$, but $(d_2, h_r) \notin R_h(\{(d_2, h_r), (d_1, h_r), (d_1, h_c)\})$.

Next, we show that there exists a stable allocation for any preferences of other hospitals, including h' , such that contracts are substitutes and any preferences of doctors.⁵ To show this, given the original problem, consider an associated problem where h is replaced by two hospitals named h_r and h_c , but hospitals other than h and all the doctors in the original problem are present. The set of contracts in the associated problem is identical to the one in the original problem, but $(d_1, h_r)_H = (d_2, h_r)_H = h_r$ and $(d_1, h_c)_H = h_c$ in the associated problem, that is, contracts involving a research position of h in the original problem belong to hospital h_r and the one involving a clinical position of h belongs to h_c . Preferences of h_r and h_c are given by

$$P_{h_r} : \{(d_1, h_r)\} \succ_{h_r} \{(d_2, h_r)\},$$

$$P_{h_c} : \{(d_1, h_c)\}.$$

⁴ Note that h_r and h_c are not different hospitals. Rather, h_r and h_c refer to different terms of contract with the same hospital h .

⁵ Note that this is a slightly stronger claim than needed to disprove Claim 1, as we allow for preferences of h' with multiple job openings.

Preferences of all other hospitals and doctors are identical to those in the original problem. Contracts are substitutes for h_r and h_c since only singleton sets of contracts are acceptable to them, and also for other hospitals by assumption about the original problem. Hence, by Result 1 there exists a set of contracts X' that is stable in the associated problem. The proof of the following observation, as well as the proofs of other propositions, are given in the Web Appendix.

OBSERVATION 1: *If X' is a stable allocation in the associated problem, then it is a stable allocation in the original problem.*

By Result 1 and Observation 1, there exists a stable set of contracts in the original problem. Since contracts are not substitutes for h , this fact implies that Claim 1 does not hold in the current problem.

Intuitively, the hospital h can be regarded as two different hospitals, h_r and h_c . The decomposition is conducted in such a way that preferences of h_r and h_c satisfy the substitutes condition. The preferences of h are *responsive* (Klaus and Klijn 2005) to those of h_r and h_c .⁶ That is, if the contract pertaining to one position improves according to the preferences of that position, then the set of contracts as a whole improves according to the preferences of the original hospital. Following the idea of Klaus and Klijn (2005), we first show the existence of a stable matching in the associated problem, and then show that the stable matching in the associated problem is stable in the original problem.

The example above is certainly nonpathological and suggests that stable allocations may exist without the substitutes condition in realistic environments. More generally, if one can construct an appropriate mapping from the original problem to an associated problem such that (a) every stable allocation in the associated problem is stable in the original one, and (b) a stable allocation exists in the associated problem, then a stable allocation exists even if contracts are not substitutes.

III. Restoring the Result

We present a weakening of the substitutes condition that is necessary to guarantee existence of a stable allocation.

DEFINITION 3: *Contracts are weak substitutes for h if we have $R_h(X') \subset R_h(X'')$ for all $X' \subset X'' \subset X$ such that $x, y \in X''$ and $x_D = y_D$ imply $x = y$.*

The condition is the same as the substitutes condition, except that we require that no two different contracts with one doctor are in X' or X'' . Clearly, contracts are weak substitutes if they are substitutes. Preferences of h in Section II satisfies the weak substitutes condition but not the substitutes condition, implying that the former is strictly weaker than the latter.

PROPOSITION 1: *Suppose H contains at least two hospitals, which we denote by h and h' . Further suppose that contracts are not weak substitutes for h . Then, there exist preference orderings for the doctors in set D , a preference ordering for a hospital h' with a single job opening such that, regardless of the preferences of the other hospitals, no stable set of contracts exists.*

⁶ Responsiveness as defined by Klaus and Klijn (2005) is different from the standard definition by Roth (1985). Contracts are substitutes if the preference is responsive in the sense of Roth (1985) but not in the sense of Klaus and Klijn (2005). In Example 1, for instance, \succ_h satisfies responsiveness of Klaus and Klijn (2005) but not Roth (1985), and contracts are not substitutes for h .

In the standard matching model in which contracts are specified by the doctor and hospital signing it, contracts are weak substitutes if and only if they are substitutes. Hence, Result 1 and Proposition 1 show that the substitutes condition is sufficient and also necessary for guaranteeing the existence of a stable allocation in that environment. On the other hand, an example in the Web Appendix shows that the weak substitutes condition is not sufficient for the existence of a stable allocation in general. In fact, even a pairwise-stable matching (see Roth and Sotomayor 1990) may fail to exist with the weak substitutes condition.

Preferences of hospital $h \in H$ satisfy the *law of aggregate demand* (Hatfield and Milgrom 2005) if $X' \subset X'' \subset X$ implies $|C_h(X')| \leq |C_h(X'')|$. One conjecture may be that the weak substitutes condition and the law of aggregate demand are sufficient for the existence of a stable allocation. An example in the Web Appendix shows that this is not true.

Complementarity in preferences is analyzed in recent papers by Michael Ostrovsky (2008) and Ning Sun and Zaifu Yang (2006). In a supply chain setup, which is more general than the current model, Ostrovsky (2008) shows that a chain-stable network exists if preferences of each agent satisfy the same-side substitutability and the cross-side complementarity conditions. In an exchange economy setup, Sun and Yang (2006) show that a Walrasian equilibrium exists if preferences of consumers satisfy the condition called gross substitutes and complements. Both of these conditions are weaker than the substitutes condition, and, in fact, some preferences that are not even weak substitutes are allowed.⁷ However, Proposition 1 does not contradict their results. In the models of Ostrovsky (2008) and Sun and Yang (2006), there are restrictions on how violations of substitutes appear across agents, and preferences of agents cannot be chosen freely, as in our Proposition 1.

The job matching problem with adjustable wages has been widely studied since pioneering works of Crawford and Elsie Marie Knoer (1981) and Kelso and Crawford (1982). It recently attracted renewed attention as Crawford (2005) advocated the use of a matching mechanism with wage adjustment in the NRMP.⁸ The current model subsumes a simplified version of Kelso and Crawford (1982), in which only a finite number of wages are allowed.⁹ Although Claim 1 does not hold in the general matching model with contracts, in the Web Appendix we show that the claim holds in the simplified version of Kelso and Crawford (1982) with finite wages.

IV. Concluding Remarks

The matching problem with contracts subsumes a large class of problems, such as the matching model with fixed terms of contract, a version of the job matching model with adjustable wages of Kelso and Crawford (1982), and the package auction model of Lawrence M. Ausubel and Milgrom (2002). On the one hand, Example 1 suggests stable allocations may exist without the substitutes condition in some natural environments. On the other hand, the only slightly weaker condition of weak substitutes is crucial for the existence of stable allocations.

We have observed that a stable allocation exists whenever the problem can be associated in a certain way to another problem in which contracts are substitutes. On the other hand, the weak substitutes condition the current paper introduces does not guarantee existence of a stable

⁷ However, Sun and Yang (2006) consider a slightly different model from the current paper in that a continuum of prices is allowed.

⁸ The proposal of Crawford (2005) is partly motivated by a recently dismissed antitrust lawsuit against the NRMP and certain theoretical support for the plaintiffs' claim people inferred from Jeremy Bulow and Jonathan Levin (2006). See also Kojima (2007), Niederle (2007) and Niederle and Roth (2003), who provide counterarguments to the plaintiffs' claim.

⁹ Kelso and Crawford (1982) consider a model with continuous prices as well as a model with finite prices.

allocation. A condition on preferences that is both sufficient and necessary for guaranteeing existence is still an open question.

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