doi: 10.1111/j.1365-2966.2006.0023.x

Stability and instability of the unbeatable strategy in dynamic processes

Fuhito Kojima*

A strategy is *unbeatable* if it is immune to any entrant strategy of any size. This paper investigates static and dynamic properties of unbeatable strategies. We give equivalent conditions for a strategy to be unbeatable and compare it with related equilibrium concepts. An unbeatable strategy is globally stable under replicator dynamics. In contrast, an unbeatable strategy can fail to be globally stable under best response dynamics even if it is also a unique and strict Nash equilibrium.

Key words unbeatable strategy, ESS, replicator dynamics, best response dynamics, smoothed best response dynamics

JEL Classification C72, C73

1 Introduction

Dynamic stability of an action distribution in a large population is of interest both in economics and biology. Hamilton (1967) conceptualizes the notion of unbeatable strategy, without giving a formal definition. A mixed strategy is *unbeatable* if it cannot be successfully invaded by any mutant strategy, no matter how big the mutant population is.

The notion of unbeatable strategy is a stronger concept than that of an evolutionarily stable strategy (ESS) of Maynard Smith and Price (1973) and Maynard Smith (1974), as a strategy is an ESS if it is immune to a small amount of mutants. Although the ESS concept is appropriate for analyzing various situations, the concept of unbeatable strategy might be useful for analyzing some economic situations.

Consider the following example. An economy consists of a continuum of firms. Firms can choose a technology $i = 1, 2$. At each moment of time, a firm is matched with another and plays a symmetric two-player game. Suppose that the stage game has the following payoff matrix:

* Department of Economics, Harvard University, Cambridge, MA 02138, USA. Email: kojima@fas.harvard.edu
I thank Drew Fudenberg for instruction and guidance. I am also grateful to Attila Ambrus, Ulrich Berger, Eric Budish, William H. Sandholm, Satoru Takahashi, an anonymous referee, and seminar participants at Harvard University and Tokyo University for helpful comments and suggestions, to Satoshi Takahashi for his help with the simulations presented in Section 3.3, and to Josef Hofbauer for, among other things, introducing me to his results on best response dynamics discussed in Section 3.2

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Technologies are strategic complements in the sense that the more firms use technology i, the more desirable technology i becomes for each firm. Denote the fraction of firms using i by x_i, $x = (x_1, x_2)$ and a point mass on i by e_i. There are two ESS, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. That is, once every firm adopts technology i, it is established as the de facto standard in the sense that no small amount of firms can upset it by adopting an alternative technology $j \neq i$.

However, should we necessarily think that e_1 and e_2 are stable once it is established? Suppose, for instance, that there is a government in this economy, who wants to establish the first technology although every firm currently uses the second technology. Suppose that the government changes technologies of more than one-third of the population to the first technology temporarily through government purchases. Then firms would find it more profitable to use the first technology, which suggests that all the firms might eventually switch to the first technology. Although e_2 is an ESS, it can be destabilized by a large amount of agents switching to the other technology (initiated by the government in this case).

Now consider a different technology expressed by the following payoff matrix:

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

Technologies are strategic substitutes: the more firms use technology i, the less desirable technology i becomes for each firm. For payoff matrix B there is a unique ESS, $x^* = (1/2, 1/2)$. x^* is also unbeatable. Suppose the government tries to change the current state x^* by influencing technologies of some fraction of the firms temporarily. This time such an attempt is likely to fail, because the average payoff of the firm that changed the technology is lower than that of x^* in the new mixed population.[1]

As the above argument clarifies, unbeatability (robustness to both a small and large amount of mutants) is a useful concept in some cases, where a large amount of mutants are expected. This point is succinctly expressed by Sigmund (2001) as follows:

> The notion of ESS is particularly appropriate for large, well-mixed populations, for in that case any invading mutant is obviously rare. In highly structured populations, it is not necessarily the case, and it could well be that Hamilton conceived the stronger condition of unbeatability because he was used to thinking of viscous populations.

Moreover, the unbeatability concept is so weak that it is applicable to a wide range of situations. For instance, every global superior strategy (GSS) (defined below) is unbeatable, but some unbeatable strategies are not GSS.[2] In light of the above observations, it is important to understand the nature of unbeatable strategies to understand evolution of behavior in a highly structured human society, where some mutants might be large.

[1] x^* is not only an unbeatable strategy, but also an interior ESS, which is a much stronger condition. However, unbeatability is enough for the above argument that x^* is immune to a large population of alternative strategies.

[2] The relationship between an unbeatable strategy and other equilibrium concepts will be investigated in detail in the present paper.

This paper investigates static and dynamic properties of unbeatable strategies. First, we investigate normal-form properties of an unbeatable strategy. We obtain equivalent conditions for a strategy to be unbeatable. We also compare an unbeatable strategy with related concepts. A mixed strategy x^* is a GSS if, for any mixed strategy $x \neq x^*$, the average payoff of x^* against x is strictly better than the average payoff of x against itself. A game with a negative definite payoff matrix and a game with an interior ESS are also considered. It has been shown that a game with an interior ESS has a negative definite payoff matrix, and a game with a negative definite payoff matrix has a unique Nash equilibrium (NE), which is also a GSS. We show that a GSS is unbeatable. Multiple equilibria are possible in a game with an unbeatable strategy, in contrast to a game with a GSS. However, we show that there exists no other interior NE or ESS.

Next we investigate the dynamic stability of an unbeatable strategy under three dynamic processes, replicator dynamics, best response dynamics (BRD) and smoothed BRD. For the replicator dynamics, we show that any solution originating at an interior state converges to an unbeatable strategy, slightly generalizing an analogous result for a GSS. The main result concerns BRD. We show that the solution to BRD might not converge to an unbeatable strategy. This is shown by an example of a four-strategy variant of the rock–scissors–paper game. The non-convergence of BRD to an unbeatable strategy is in sharp contrast to a GSS, which is known to be globally stable under BRD. Smoothed BRD is investigated numerically. Although we cannot obtain an analytical solution, numerical solutions to the smoothed BRD approach equilibrium in our simulation. This result suggests that the instability of an unbeatable strategy under BRD might not be robust to some perturbations described by smoothed BRD.

The non-convergence in BRD is particularly interesting. Many existing results in dynamic processes suggest that dynamic stabilities of an ESS coincide in a wide range of dynamics. For example, an interior ESS is globally stable under replicator dynamics, BRD, smoothed BRD and Brown von Neumann–Nash dynamics.[3] An unbeatable strategy, by contrast, exhibits quite different behavior in the replicator dynamics, BRD and smoothed BRD. Investigating an unbeatable strategy might help in understanding differences among various dynamic processes.

The rest of this paper is organized as follows. Section 2 defines an unbeatable strategy and investigates its normal-form properties. Section 3 investigates dynamic stability and shows that an unbeatable strategy is not necessarily globally stable under BRD. Section 4 concludes.

2 Definition and normal-form properties

Consider a finite symmetric two-player game. It is specified by an $n \times n$ payoff matrix $A = (a_{ij})$, where n is the number of pure strategies. Denote the set of mixed strategies by $\Delta := \{x \in \mathbb{R}^n \mid \forall i,\ x_i \geq 0,\ \sum_i x_i = 1\}$. $\Delta^\circ := \{x \in \Delta \mid \forall i,\ x_i > 0\}$ is the set of interior (totally mixed) strategies. Let e_i be a point mass on strategy i, which corresponds to pure strategy i.

[3] See Hofbauer and Sigmund (1998), Hofbauer (1995, 2000) and Sandholm (2004). Sandholm (2004) considers the class of potential dynamics that generalizes Brown–von Neumann Nash dynamics and obtains global stability.

For $x \in \Delta$, let $\mathrm{supp}(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i > 0\}$ and $\mathrm{br}(x) := \arg\max_i e_i \cdot Ax$. A state $x^* \in \Delta$ is a (*symmetric*) NE if $\mathrm{supp}(x^*) \subseteq \mathrm{br}(x^*)$. x^* is an *evolutionarily stable strategy* (ESS) if it is a NE and $x^* \cdot Ax > x \cdot Ax$ for any $x \neq x^*$ with $\mathrm{supp}(x) \subseteq \mathrm{br}(x^*)$. x^* is an ESS if and only if there exists $\bar{t} > 0$ such that $x^* \cdot A[(1-t)x^* + tx] > x \cdot A[(1-t)x^* + tx]$ for any $x \neq x^*$ and any $t \in (0, \bar{t})$.

Definition 1 $x^* \in \Delta$ is an *unbeatable strategy* if

$$x^* \cdot A[(1-t)x^* + tx] > x \cdot A[(1-t)x^* + tx] \tag{3}$$

for any $x \neq x^*$ and any $t \in (0, 1)$.

A mixed strategy is unbeatable if it cannot be successfully invaded by any mutant strategy of any size. An unbeatable strategy is an ESS, because the ESS concept requires the above inequality only for sufficiently small t. The following example shows that an ESS is not unbeatable in general (see also Example 3), and also highlights differences of these concepts with respect to economic implication.

Example 1 Consider the example of technology choice discussed in the Introduction. The payoff matrix (1) has two ESS, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. e_1 is not unbeatable, because for $x = e_2$ and $t \geq 2/3$, we have $e_1 \cdot A[(1-t)e_1 + te_2] = 2(1-t) \leq t = e_2 \cdot A[(1-t)e_1 + te_2]$, violating (3). By a similar argument, e_2 is not unbeatable. Next, consider payoff matrix (2). There is a unique NE, $x^* = (1/2, 1/2)$. It is easy to verify that x^* is unbeatable.

Economic interpretation is as follows. Firms are faced with a choice from two available technologies. Payoff matrix (1) is a case in which technologies are complements where there are two states that the economy might settle in. Neither of these states is, however, immune to large amounts of deviators. By contrast, payoff matrix (2) is a case of substitutes, where there is a unique steady state. The state is immune not only to small amounts of deviators but also to a large one.

The following proposition gives characterizations of the unbeatable strategy.

Proposition 1 *The following conditions are equivalent.*

(i) x^* *is an unbeatable strategy.*
(ii) x^* *is an ESS and satisfies* $x^* \cdot Ax \geq x \cdot Ax$ *for any* $x \in \Delta$.
(iii) *For any* $x \neq x^*$, *we have* $x^* \cdot Ax^* \geq x \cdot Ax^*$ *and* $x^* \cdot Ax \geq x \cdot Ax$. *Moreover, at least one of the above inequalities is strict.*
(iv) $x^* \cdot Ax > x \cdot Ax$ *for any* $x \neq x^*$ *with* $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(x)$.

PROOF: (i) $\Rightarrow$ (ii). Suppose that x^* is unbeatable. Then it is clear that x^* is an ESS. By continuity as t approaches 1, we obtain $x^* \cdot Ax \geq x \cdot Ax$ as desired.

(ii) $\Rightarrow$ (iii). Suppose that x^* satisfies (ii). Then for any $x \neq x^*$, it is obvious that weak inequalities hold. If both inequalities hold with equality, then the strict inequality in the definition of ESS does not hold. So (ii) implies (iii).

(iii) $\Rightarrow$ (iv). Suppose $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(x)$ and $x \neq x^*$. Then there exists $t \in (0, 1)$ and $y \neq x^*$ such that $x = (1-t)x^* + ty$, or equivalently, $x^* - x = t(x^* - y)$. Now we have

$$\begin{aligned} x^* \cdot Ax - x \cdot Ax &= t(x^* - y) \cdot A[(1-t)x^* + ty] \\ &= t(1-t)(x^* - y) \cdot Ax^* + t^2(x^* - y)Ay \\ &> 0. \end{aligned}$$

The strict inequality results from (iii). Therefore, (iv) holds.

(iv) $\Rightarrow$ (i). For any $x \neq x^*$ and $t \in (0, 1)$, let $y = (1-t)x^* + tx$.

$$x^* \cdot Ay - y \cdot Ay = t(x^* - x) \cdot Ay,$$

which is positive because $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(y)$. Therefore, we have $(x^* - x) \cdot Ay > 0$, which implies that x^* is unbeatable. □

We consider a slightly stronger concept than the unbeatable strategy: $x^* \in \Delta$ is a GSS if $x^* \cdot Ax > x \cdot Ax$ for any $x \neq x^*$ (Cressman and Hofbauer 2005).[4] Proposition 1 (iv) implies that any GSS is unbeatable.

The following example shows that an unbeatable strategy need not be a GSS or a unique NE.

Example 2 Consider the following payoff matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

e_1 and e_2 are Nash equilibria and e_1 is unbeatable. However, e_1 is not a GSS.

As Example 2 illustrates, an unbeatable strategy need not be a unique NE and another equilibrium can even be undominated. The following proposition shows, however, that the existence of an unbeatable strategy does exclude the existence of particular types of equilibria.

Proposition 2 *Suppose that x^* is an unbeatable strategy. Then*

(i) *There exists no interior NE different from x^*.*
(ii) *x^* is a unique ESS. In particular, x^* is a unique unbeatable strategy.*

PROOF:
(i) For any interior strategy $x \neq x^*$, we have $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(x)$. By Proposition 1, we have $x^* \cdot Ax > x \cdot Ax$; hence, x is not an NE.

(ii) Let x^* be an unbeatable strategy and $x \neq x^*$ be a NE. By Proposition 1 (iii), $x^* \cdot Ax \geq x \cdot Ax$ holds. Because x is an NE, this impies $x^* \cdot Ax = x \cdot Ax$. Again by Proposition 1 (iii), we have $x^* \cdot Ax^* \geq x \cdot Ax^*$. Therefore, x is not an ESS. Because an unbeatable strategy is an ESS (Proposition 1 (ii)), x^* is a unique unbeatable strategy.

[4] Cressman and Hofbauer (2005) define a GSS for possibly infinite strategy spaces.

Although any unbeatable strategy is an ESS, some ESS might not be unbeatable. For instance, e_1 and e_2 of payoff matrix (1) in Example 1 are both ESS, and neither of them is unbeatable by Proposition 2 (ii). The following example shows that, more strongly, even an ESS which is also a unique NE might fail to be unbeatable. □

Example 3 Consider the following payoff matrix:

$$A = \begin{pmatrix} 0 & 2 & -2 \\ 2 & 0 & 1 \\ \epsilon & \epsilon & 0 \end{pmatrix}, \quad \epsilon \in (0, 1/2).$$

There exists a unique NE $x^* = (1/2, 1/2, 0)$. It can be shown by computation that x^* is an ESS. However, x^* is not unbeatable, because $-1/2 = x^* \cdot Ae_3 < e_3 \cdot Ae_3 = 0$, violationg Proposition 1 (ii). This example shows, in particular, that Proposition 2 does not characterize an unbeatable strategy.

If a game admits an interior NE, then any unbeatable strategy in such a game is an interior ESS by Proposition 2 (i) and, hence, a GSS. By contrast, a partially mixed unbeatable strategy might not be a GSS even if it is a unique NE.

Example 4 Consider the following payoff matrix:

$$A = \begin{pmatrix} 0 & -b & a & 0 & 0 \\ a & 0 & -b & 0 & 0 \\ -b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad 0 < a < b.$$

There exists a unique NE $x^* = (0, 0, 0, 1/2, 1/2)$, which is also unbeatable. x^* is not a GSS because $x^* \cdot Ae_1 = 0 = e_1 \cdot Ae_1$.

Let $\mathbb{R}^n_0 = \{x \in \mathbb{R}^n \mid x \neq 0, \sum_i x_i = 0\}$. A game A is said to be *negative definite* if $\zeta \cdot A\zeta < 0$ for any $\zeta \in \mathbb{R}^n_0$. Intuitively, a game with a negative definite payoff matrix has negative externality. Payoff matrix (2) is a simple example of a negative definite payoff matrix. If A has an interior ESS, then A is negative definite. A negative definite game has a unique NE. Let x^* be the NE of a negative definite game and let $x \neq x^*$. Because $x^* - x \in \mathbb{R}^n_0$, we have $(x^* - x) \cdot A(x^* - x) < 0$, which is equivalent to $(x^* - x) \cdot Ax > (x^* - x) \cdot Ax^*$. The right-hand side is nonnegative because x^* is an NE. Therefore, if A is negative definite, then the unique NE of the game is a GSS and, hence, unbeatable (Hofbauer 2000). In summary, if the game is negative definite, then concepts of NE, ESS, an unbeatable strategy and GSS coincide.

A game with a GSS might not be negative definite, as the following example shows. The example also highlights differences among unbeatability, GSS and negative definiteness with a one-parameter family of mayoff matrices.

Example 5 Consider the following payoff matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & a & 1 \\ 0 & -1 & -1 \end{pmatrix}, a \in (-\infty, 0].$$

For $a = 0$ (the case of Example 2), e_1 and e_2 are Nash equilibria and e_1 is unbeatable but not a GSS. For $a < 0$, e_1 is a unique Nash equilibrium and also a GSS. By computation, A is not negative definite if $a \geq -5/4$.

3 Dynamic stability

3.1 Replicator dynamics

Consider the replicator dynamics (Taylor and Jonker 1978):

$$x : [0, \infty) \to \Delta,$$

$$x(0) = x,$$

$$\dot{x}_i(t) = x_i(t)[e_i \cdot Ax(t) - x(t) \cdot Ax(t)],$$

with $x \in \Delta$. x is called an initial state.

Proposition 3 *Assume that the stage game has an unbeatable strategy x^*. For any $x \in \Delta^\circ$, a solution $x(t)$ of the replicator dynamics converges to x^*.*

Proof: Define $V : \Delta \to \mathbb{R}$ by $V(x) = \prod_{i \in \mathrm{supp}(x^*)} x_i^{x_i^*}$. V attains its unique maximum at $x = x^*$. Along the solution $x(t)$ of the replicator dynamics, we have

$$\frac{d \log V(x(t))}{dt} = x^* \cdot Ax(t) - x(t) \cdot Ax(t),$$

which is positive by Proposition 1 (iv) as long as $x(t) \neq x^*$, for $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(x(t))$ for any interior strategy $x(t) \in \Delta^\circ$. Therefore, V is a global Lyapunov function, completing the proof. □

Proposition 3 shows that an unbeatable strategy is globally stable in the replicator dynamics. Actually, we can show a slightly stronger result using the same method: the solution converges to x^* for any $x \in \Delta$ with $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(x)$. Note that the replicator dynamics cannot converge to x^* if $\mathrm{supp}(x^*) \subseteq \mathrm{supp}(x)$ does not hold by definition. Another remark is that Proposition 3 states that, starting at an interior strategy, a solution converges to x^* *even if there is another NE.*

Proposition 3 follows a standard Lyapunov method (see Hofbauer and Sigmund 1998). Cressman and Hofbauer (2005) show that the measure dynamics, which generalizes the standard replicator dynamics for finite strategies, converges to a GSS in games with infinite as well as finite strategies. Proposition 3 slightly generalizes existing results on a GSS such as theirs, showing that the global stability under the replicator dynamics is not lost by a slight weakening of a GSS to an unbeatable strategy.

Any ESS is locally asymptotically stable under the replicator dynamics, but it might not be globally stable in general in contrast to an unbeatable strategy. Consider the following payoff matrix (Zeeman 1980):

$$A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}.$$

There are two NE: $x = (1/3, 1/3, 1/3)$ and e_1. e_1 is a unique ESS because it is a strict NE (and, hence, an ESS) and the other NE is interior (note that if an interior strategy were an ESS, then it would be a unique NE). e_1 is not unbeatable by Proposition 2 (i) and the fact $x \in \Delta^\circ$. x is locally asymptotically stable because the Jacobian at x has eigenvalues $(-1 \pm i\sqrt{2})/3$ with negative real parts. This implies that some paths fail to converge to e_1 under the replicator dynamics, even though it is a unique ESS.

3.2 Best response dynamics

Consider BRD (Gilboa and Matsui 1991; Matsui 1992):

$$x : [0, \infty) \to \Delta,$$

$$x(0) = x,$$

$$\tfrac{d^+x}{dt}(t) = \alpha(t) - x(t),$$

$$\mathrm{supp}(\alpha(t)) \subseteq \mathrm{br}(x(t)).$$

A microfoundation of BRD is as follows. There is one large population of agents, with action distribution at time t represented by $x(t)$. At each moment of time, agents are randomly matched and play the stage game. A fraction of the agents change their strategies at each moment. The distribution of strategies chosen at time t is proportional to $\alpha(t)$, and every pure strategy chosen by some agents maximizes average payoff against the current action distribution.

The following proposition shows that even if a game admits an unbeatable strategy, BRD might fail to converge to any equilibrium. This is a striking result, because any GSS is the limit of BRD, as discussed later in this section.

Proposition 4 *A solution to BRD might fail to converge to an unbeatable strategy.*

The proof is based on an example. More precisely, in Example 6 below we show that there exist many solutions that fail to converge to a unique, strict and unbeatable NE.

Example 6 Consider the following payoff matrix:

$$A = \begin{pmatrix} 0 & -b & a & 0 \\ a & 0 & -b & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad 0 < a < b.$$

This game has a unique NE, e_4, which is unbeatable and also strict. We will show that there is a solution to BRD that does not converge to e_4. We construct such a path using the technique of Gaunersdorfer and Hofbauer (1995).

For any $y \in \Delta$, define $V(y) := \max_{i \neq 4} e_i \cdot Ay$. Suppose that we start at an initial state $x(0) = x$ that satisfies $V(x) > x_4 \geq 0$. We show inductively that the solution converges to the so-called "Shapley triangle", $S := \{y \in \Delta \,|\, V(y) = 0,\ y_4 = 0\}$. Suppose that $V(x(t)) > x_4(t)$ for some $t \geq 0$ (the inequality is satisfied for $t = 0$). Then $\alpha(t) = e_i$ for some $i \neq 4$. Set $t' = \inf\{s > t \,|\, \alpha(s) \neq e_i\}$. Clearly, t' is finite (if t' were infinite, then $x(s) \rightarrow e_i$. This contradicts the fact that i is a best response to $x(s)$ for a large value of s given the payoff matrix). For any $s \in (t, t')$, we have $V(x(s)) = e_i \cdot Ax(s)$ and $\dot{x}(s) = e_i - x(s)$. It follows that

$$\begin{aligned} \dot{V}(x(s)) &= e_i \cdot A\dot{x}(s) \\ &= e_i \cdot A(e_i - x(s)) \\ &= -e_i \cdot Ax(s) \\ &= -V(s). \end{aligned}$$

It follows that $V(x(s)) = V(x(t))e^{t-s}$ and $x(s) = x(t)e^{t-s} + e_i(1 - e^{t-s})$ for any $s \in [t, t']$. This implies that $V(x(t')) = V(x(t))e^{t-t'} > x_4(t)e^{t-t'} = x_4(t')$. Inductively we have that $V(x(t)) = V(x)e^{-t} > 0$ and $x_4(t) = x_4 e^{-t}$ for any $t \geq 0$, with $V(x(t)) \rightarrow 0$ and $x_4(t) \rightarrow 0$ as $t \rightarrow \infty$. This implies that $x(t)$ converges not to e_4 but to limit cycle S. If initial state x satisfies $V(x) < x_4$, then the solution is easily shown to be $x(t) = xe^{-t} + e_4(1 - e^{-t})$, which converges to e_4. Finally, if the initial state satisfies $V(x) = x_4$, then there is a continuum of solutions, some converging to e_4 and others approaching S.

Note that the non-convergence of BRD to the unbeatable strategy is not nongeneric with respect to the initial state. In particular, $x(t)$ does not converge to e_4 but exhibits a limit cycle on S even if the initial state is in the interior Δ°, in contrast to the replicator dynamics.

Similarly to the above payoff matrix, a solution to BRD might not converge to the unbeatable strategy x^* in Example 4.

As the above example shows, an unbeatable strategy might not be globally stable under BRD. This is in a sharp contrast to the global stability of a GSS shown by Hofbauer (1995). He defines

$$W(x) = \max_i e_i \cdot Ax - w_{\bar{B}}(x),$$

where $\bar{\mathcal{B}}$ is the set of mixed strategies b such that every pure strategy in supp(b) is indifferent against b, and

$$w_{\bar{\mathcal{B}}}(x) = \max \left\{ \sum_{b \in \bar{\mathcal{B}}} b \cdot Ab\lambda^b \,\middle|\, \lambda^b \geq 0,\ \sum_{b \in \bar{\mathcal{B}}} \lambda^b = 1,\ \sum_{b \in \bar{\mathcal{B}}} b\lambda^b = x \right\}.$$

Suppose that x^* is a GSS. Then we have

$$\begin{aligned} w_{\bar{\mathcal{B}}}(x) &= \max \left\{ \sum_{b \in \bar{\mathcal{B}}} b \cdot Ab\lambda^b \,\middle|\, \lambda^b \geq 0,\ \sum_{b \in \bar{\mathcal{B}}} \lambda^b = 1,\ \sum_{b \in \bar{\mathcal{B}}} b\lambda^b = x \right\} \\ &\leq \max \left\{ \sum_{b \in \bar{\mathcal{B}}} x^* \cdot Ab\lambda^b \,\middle|\, \lambda^b \geq 0,\ \sum_{b \in \bar{\mathcal{B}}} \lambda^b = 1,\ \sum_{b \in \bar{\mathcal{B}}} b\lambda^b = x \right\} \\ &= x^* \cdot Ax. \end{aligned}$$

The above inequality holds with equality if and only if $x = x^*$. Clearly we have

$$\begin{aligned} W(x) &= \max_i e_i \cdot Ax - w_{\bar{\mathcal{B}}}(x) \\ &\geq x^* \cdot Ax - w_{\bar{\mathcal{B}}}(x) \\ &\geq 0. \end{aligned}$$

$W(x) = 0$ if and only if $x = x^*$. Hofbauer (1995) shows that $W(x(t))$ is decreasing along any path $x(t)$ of BRD except for $x(t) = x^*$. It follows that any solution to BRD converges to x^*.

Although an unbeatable strategy might not be globally stable, it is locally stable. Hofbauer (1995) uses the above function to show that any ESS and, hence, any unbeatable strategy in particular, is locally asymptotically stable.

3.3 Smoothed best response dynamics

Consider the smoothed BRD (Fudenberg and Kreps 1993; Fudenberg and Levine 1998):

$$\begin{aligned} x &: [0, \infty) \to \Delta, \\ x(0) &= x, \\ \dot{x}(t) &= \sigma(x(t)) - x(t), \end{aligned}$$

where $\sigma : \Delta \to \Delta$ is defined by $\sigma_i(y) = \Pr(i \in arg\, max_j e_j \cdot Ay + \epsilon_j)$ and ϵ_j are i.i.d. random variables with a strictly positive density function $f : \mathbb{R} \to \mathbb{R}$.

Smoothed BRD is different from BRD in one respect. Unlike in BRD, each agent receives a stochastic payoff. The current payoff of each agent is given by $e_i \cdot Ax(t) + \epsilon_i$ for action i, where ϵ_i is a stochastic component of payoff distributed i.i.d. with the density function f.

Agents choose actions to maximize these perturbed current payoffs. Given the distribution of ϵ_i, the proportion of agents choosing i is proportional to $\sigma_i(x(t))$ at t.[5]

Consider the following special case where the stochastic components of payoff are distributed according to cumulative distribution function $F(x) = \exp(-\exp(-\lambda x - \gamma))$, where γ is Euler's constant and $\lambda \geq 0$. It is well known that the corresponding choice function, which we denote by σ^λ, is given by the logistic quantal response function with parameter λ:

$$\sigma_i^\lambda(x) = \frac{\exp(\lambda e_i \cdot Ax)}{\sum_j \exp(\lambda e_j \cdot Ax)},$$

for each i (see Anderson *et al.* 1992 for derivation). $x \in \Delta$ is a symmetric logit equilibrium with parameter λ if $x = \sigma^\lambda(x)$. As $\lambda \to \infty$, σ^λ converges to the best response correspondence in the sense that $\lim_{\lambda\to\infty} \sigma_i^\lambda(x) = 0$ for any $i \notin \mathrm{br}(x)$. By definition, a state is a rest point of the smoothed BRD associated with the logistic quantal response function if and only if it is a symmetric logit equilibrium.

Because σ^λ approximates the best-response correspondence for a large value of λ, one might expect that this dynamics exhibits a cycle similar to BRD in Example 6. Simulation suggests otherwise, however. Numerical solutions with initial state $x = (0.7, 0.15, 0.10, 0.05)$ are carried out for six sets of parameters: $(a, b) = (1, 2), (1, 1.1)$ as payoff parameters, and parameters for the dynamics $\lambda = 10, 100, 500$ for each set of payoff parameters. For each set of parameters, $x(t)$ remains close to the solution of exact BRD for small t, approaching the Shapley triangle. However, $x_4(t)$ begins to increase when $x(t)$ has come close enough to the Shapley triangle. Then $x_4(t)$ continues to increase, approaching a logit equilibrium corresponding to e_4. The results are presented graphically in Figures 1 and 2 for $(a, b) = (1, 2), (1, 1.1)$ and $\lambda = 500$. For each set of parameters, the figure on the left represents $x_i(t)$ as a function of t for each i, whereas the trajectory of $x(t)$ is represented in the figure on the right.

The above simulation suggests that when BRD is perturbed by introducing stochastic payoffs, then the solution might converge to the logit equilibrium. In this sense, the non-convergence in BRD might not be robust to payoff perturbations of the dynamics modeled by smoothed BRD.

4 Conclusion

We investigated static and dynamic properties of unbeatable strategies. Equivalent conditions for a strategy to be unbeatable were established. We investigated relationships among an unbeatable strategy, an ESS and a GSS. Three dynamic processes were then considered.

[5] Choice function σ can be derived from a different model where an agent directly chooses a mixed strategy. Specifically, Hofbauer and Sandholm (2002, Theorem 2.1) show that, for any σ derived by the above model of perturbed payoffs with strictly density function, if σ is continuously differentiable, then there is a function V such that $D^2 V(y)$ is positive definite with respect to $\mathbb{R}^n_0$, $\|V(y)\|$ approaches infinity as y approaches the boundary of Δ, and $\sigma(y) = arg\ \max_{z\in\Delta^\circ}\{z \cdot Ay - V(z)\}$.

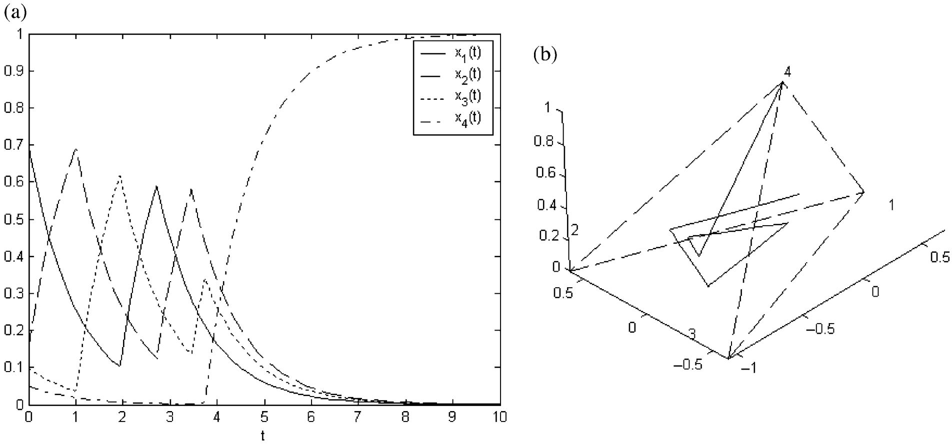

Figure 1 Simulation for smoothed best response dynamics: $(a, b) = (1, 2)$, $\lambda = 500$.

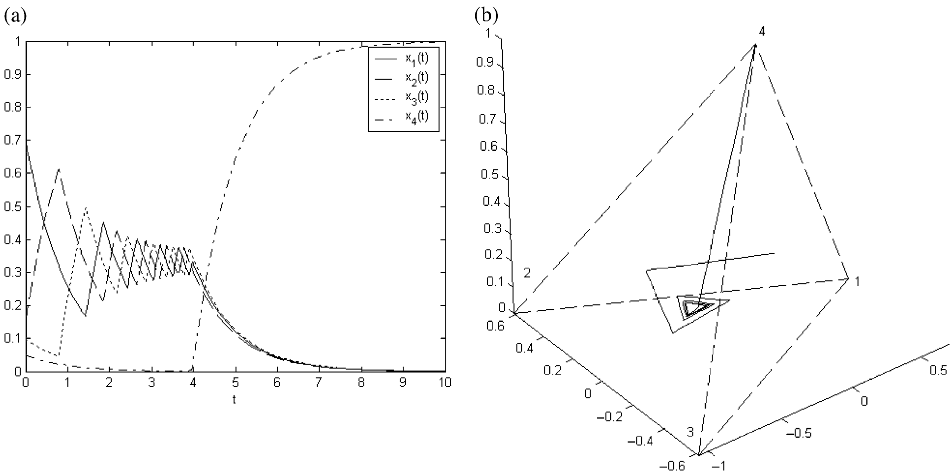

Figure 2 Simulation for smoothed best response dynamics: $(a, b) = (1, 1.1)$, $\lambda = 500$.

We showed the global stability under replicator dynamics. For BRD, in contrast, an example is given in which an unbeatable strategy fails to be globally stable. This is in contrast to a GSS, for which the global stability holds. Also, our example is in sharp contrast to equilibria in games with negative definite payoff matrices (and, hence, interior ESS), which are globally stable under various dynamics including replicator dynamics, BRD, smoothed BRD and Brown–von Neumann Nash dynamics (Brown and von Neumann 1950; Nash 1951). Finally, we investigated smoothed BRD. Numerical solutions suggest that the global stability of an unbeatable strategy might hold in smoothed BRD.

Dynamic stability of unbeatable strategies under other dynamics is an open question. Simulation suggests that global stability might hold in Example 6 under smoothed BRD, unlike for exact BRD, but neither a proof nor a counterexample is available yet. Stability under the Brown–von Neumann Nash dynamics is also unknown.

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