A velocity-pressure Navier-Stokes solver using a B-spline collocation method

By O. Botella

1. Motivation and objectives

B-spline functions are bases for piecewise polynomials that possess attractive properties for turbulent flow simulations; they have compact support and yield numerical schemes with a high resolving power, with an order of accuracy k that is a mere input parameter. In a sense, a B-spline basis can be viewed as a finite-element basis of arbitrary order k. Its main restriction is that multidimensional approximations are constructed as tensor products of one-dimensional B-splines, which limits approximations to structured meshes. Recently however, Shariff & Moser (1998) circumvented the tensor product limitation by constructing B-spline bases on zonal embedded grids, allowing a treatment of the zonal interfaces consistent with the high continuity of the B-splines.

The approximation of differential problems with B-splines is obtained by the method of weighted residual, of which the Galerkin and collocation methods are particular cases. The Galerkin method is the most widely used method for B-spline approximations. Not only does this method provide an optimal order of accuracy, but it also guarantees the conservation of quadratic invariants such as kinetic energy. Galerkin B-spline schemes have been developed in Kravchenko *et al.* (1996, 1999) and Kim (1998) for solving the Navier-Stokes equations. However, a high order Galerkin method is burdened by the cost of evaluating nonlinear terms; as observed by Kravchenko *et al.* (1999), 50% of computational time is spent on their evaluation. A collocation method, on the other hand, represents an economical alternative since it only requires the evaluation of these terms at grid points. This method has been most successfully applied in the past to orthogonal polynomial approximations (see *e.g.* Quarteroni & Valli, 1994), and applications to B-spline approximations for solving fluid flow problems are still in an early stage (Fairweather & Meade, 1989).

The aim of this study is to develop an efficient collocation scheme for solving the incompressible Navier-Stokes equations with a fractional step method. To our knowledge, this work represents the first attempt in developing such a B-spline scheme and, in particular, in dealing with compatible B-spline bases for the approximation of the velocity and the pressure. For this purpose, the geometry is restricted to a rectangular domain, and the dependent variables are represented by tensor-product B-splines. The extension to complex geometries and the handling of embedded meshes are left for future work. The first part of this report is devoted to the evaluation of B-spline collocation methods for one-dimensional problems. In the second part, the collocation scheme using "staggered" spline bases for preventing pressure oscillations is introduced. Finally, some results for benchmark Navier-Stokes problems are presented.

2. A comparison of some B-spline approximation methods

2.1 Construction of B-spline bases

A spline function is a piecewise polynomial of order k (the polynomial degree is $k-1$ at most) defined on the interval $\Lambda =]a, b[$, whose high order derivatives possess jump-discontinuities at some breakpoints $\boldsymbol{\xi} = \{\xi_i,\ i = 1, \ldots, l+1\}$ defined by

$$a = \xi_1 < \xi_2 < \ldots < \xi_i < \ldots < \xi_l < \xi_{l+1} = b. \tag{1}$$

In the following, we will restrict our characterization to splines having jump-discontinuities at their $m+1$ derivative at each $\xi_i \in \Lambda$, *i.e.* splines belonging to the space $C^m(\Lambda)$.

The spline $u(x)$ is commonly described in its B-representation

$$u(x) = \sum_{i=1}^{N} \alpha_i B_i^k(x), \tag{2}$$

where $B_i^k(x)$ is a special spline function of order k called a B-spline which has, in particular, the property of having compact support. The number N of the B-splines, depending on the order k and the index of regularity m, will be defined later. Most properties of B-splines can be found in de Boor (1978), and Fortran software for efficient computations with B-splines is presented in de Boor (1977).

The B-splines of order 1 are step functions defined by

$$B_i^1(x) = \begin{cases} 1 & \text{if } x \in [\xi_i, \xi_{i+1}], \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

and an efficient construction of the B-splines of order $k > 1$ is given by the recurrence relation of Curry and Schoenberg (see *e.g.* de Boor (1978)):

$$B_i^k(x) = \frac{x - t_i}{t_{i+k-1} - t_i} B_i^{k-1}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} B_{i+1}^{k-1}(x). \tag{4}$$

This formula introduces the knots $\{t_i,\ i = 1, \cdots, N+k\}$, where the number N of the B-splines is

$$N = l\,(k - m - 1) + m + 1. \tag{5}$$

The regularity of the B-spline basis is imposed through the definition of the knots by requiring

$$t_{k+(i-2)(k-m-1)+1} = \cdots = t_{k+(i-1)(k-m-1)} = \xi_i \quad \text{for } i = 2, \cdots, l. \tag{6}$$

The construction of the basis given by Eqs. (3)-(6) leaves freedom in the first k and last k of the knots. A convenient choice for the approximation of boundary value problems is to set

$$t_1 = \cdots = t_k = a, \quad t_{N+1} = \cdots = t_{N+k} = b. \tag{7}$$

In that case, by using the properties that $B_i^k(x)$ have compact support in $[t_i, t_{i+k}]$ (de Boor, 1978), the spline function (2) verifies

$$u(a) = \alpha_1 \quad \text{and} \quad u(b) = \alpha_N,$$

so that Dirichlet boundary conditions are imposed strongly. This choice of the remaining knots is to be used throughout this report. Note that a choice analogous to (7) for the first k and last k knots that is suitable for imposing periodic boundary conditions is discussed in *e.g.* de Boor (1978) and Kravchenko *et al.* (1996).

Finally, we mention that the B-splines bases considered in this report are characterized by the triplet $(k, m, \boldsymbol{\xi})$, *i.e.* the polynomial order k, the index of continuity m, and the distribution of breakpoints $\boldsymbol{\xi}$. The bases are then constructed by application of relations (3)-(7).

2.2 Approximation of differential problems

In this part, we give a brief overview of the properties of B-spline approximation methods that lead to the development of the collocation scheme described in the subsequent sections. The efficiency of a B-spline method for solving differential problems depends crucially on the choice of:

- The approximation methods (*i.e.* Galerkin or Collocation),
- The B-spline basis.

The Galerkin method, which satisfies the equations in an average sense, has been the preferred method for spline approximations of fluid flow problems (see *e.g.* Kravchenko *et al.* 1996, Kim 1998). Not only does this method provide an optimal order of accuracy, namely $O(N^{-k})$ with B-splines of order k, but also guarantees the conservation of quadratic invariant such as the kinetic energy. In spite of the compact support of the B-splines, when the order k is raised this high quality method requires enormous work in term of storage cost of the matrices and calculation of nonlinear terms.

On the other hand, the collocation method consists in simply satisfying the equations at a discrete set of points $\{x_j,\, j = 1, \ldots, N\}$, the collocation points. The mathematical properties of this method are less well established for the main reason that the order of accuracy depends on the properties of the B-spline bases. Moreover, the choice of the collocation points may appear arbitrary in some cases.

Two collocation methods have been compared. The first one uses the "smoothest" splines $(k, m = k-2, \boldsymbol{\xi})$, *i.e.* splines of order k of maximum continuity $k-2$. The collocation points are in this work chosen as the maximum of the B-splines. It is important to note that the order of accuracy obtained is suboptimal, typically in $O(N^{d-k})$ for a differential problem of order d (Prenter, 1975). For this reason, this method has been overlooked in favor of the Gauss-collocation method of de Boor and Swartz (1973). This method belongs to the class of orthogonal collocation methods (Prenter, 1975) meaning that, when the collocation grid is defined as the quadrature points of the Legrendre-Gauss rule, the optimal $O(N^{-k})$ accuracy of the Galerkin method is recovered. However, note that this method uses B-splines defined by $(k, m = d-1, \boldsymbol{\xi})$ with $k \geq 3$ so that the continuity of the basis is imposed only by the order d of the problem to be solved, irrespective of k. As an example, a second-order problem would only be solved by using C^1 splines.

A comparison of these methods is now performed on the boundary value problem

$$Lu(x) = f(x), \quad x \in \Lambda \tag{8a}$$

$$u(a) = g_0, \quad u(b) = g_1, \tag{8b}$$

where L is a second order elliptic operator. This problem is to be solved with the different spline methods on a uniform distribution of breakpoints whose main properties are given in Table 1. In this Table, the methods using smooth splines are designated *SCo* and *SCe* for the collocation method using k odd and even respectively, and *SG* for the Galerkin method; *GC* refers to the Gauss-collocation method. The average bandwidth of a scheme refers to the ratio of the non-zero entries of the associated linear system divided by its size; this quantity gives a rough quantification of the amount of work required for solving this system. A typical example concerns the numerical solution of (8) in $\Lambda =]-1,1[$ with $L = \lambda^2 - d^2/dx^2$, $f = \lambda^2$, $g_0 = 1$, $g_1 = 0$, whose smooth solution displays a boundary layer of thickness $O(1/\lambda)$ near $x = 1$. For the case $\lambda = 10$, Fig. 1 displays the maximal error for the various methods of average bandwidth 7. It is observed that, with smoothest splines, collocation methods are more accurate than the Galerkin method. The striking result is that even though *GC* displays the highest convergence rate, it does not give the best accuracy for moderate values of N. This tendency has also been observed for methods of bandwidth 5 as well as for steeper solutions.

An interesting hint of this behavior is given by comparing in Fig. 2 the resolving power of the first derivative yielded by the various approximations (Lele, 1992). First, let us mention that a well known result of Swartz & Wendroff (1974) shows that the resolving power of *SCe* and *SG* are identical. The Gauss-collocation method gives by far the worst behavior in the finest scales. To show that this phenomenon is not mainly related to the collocation approximation, we have also included in this figure the modified wavenumber of the Galerkin method using the same B-splines basis that displays similar wiggles. This phenomenon is most likely related to the low C^1 continuity at the breakpoints that is enforced on the basis; note that a similar behavior is found for the Fourier analysis of C^1 piecewise cubic (*i.e.* $k = 4$) Hermite Galerkin schemes.

2.4 Concluding remarks

The conclusions that can be drawn form this study are twofold. Firstly, a spline method has better approximation properties when the bases display high continuity even if it would result in dropping some order of accuracy as is the case for the smoothest collocation method. Secondly, a collocation method requires less work for the evaluation of nonlinear terms (Botella, 1999) and is more cost-effective for solving linear problems even though it should be reminded that collocation matrices are non-symmetric.

For these reasons, efforts are now engaged towards solving fluid flow problems with the smoothest collocation method.

	SCo	SCe	SG	GC
Average bandwidth	k	$k-1$	$2k-1$	k
Order of accuracy	$k-1$	$k-2$	k	k
Continuity at the breakpoints	C^{k-2}	C^{k-2}	C^{k-2}	C^1

Table 1. Properties of the different methods using B-splines of order k for solving problem (8).

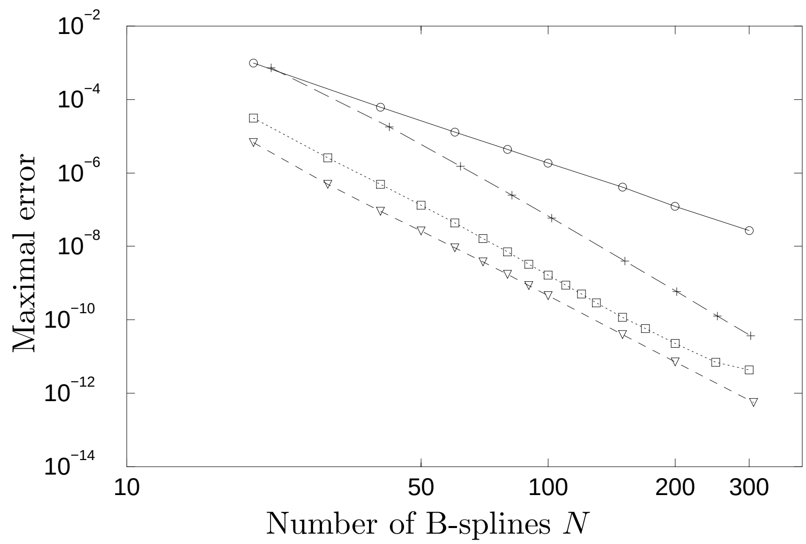

FIGURE 1. Maximal value of the error, taken on 1001 equidistant points, vs. the number of B-splines for the various methods of average bandwidth 7: *SCo*, $k = 7$ (□⋯⋯□); *SCe*, $k = 8$ (▽- - -▽); *SG*, $k = 4$ (○——○); *GC*, $k = 7$ (+— — —+)

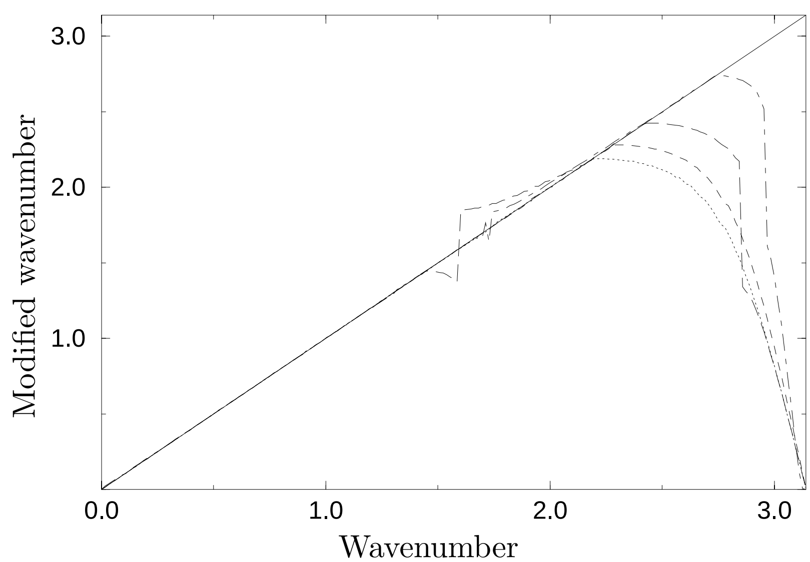

FIGURE 2. The modified wavenumber of the first derivative, for the collocation methods of bandwidth 7: ——— (exact); ⋯⋯ (*SCo*, $k = 7$); - - - - (*SCe*, $k = 8$); — — — (*GC*, $k = 7$); and Galerkin method with the Gauss-splines basis —·— (*GG*, $k = 7$).

3. Navier-Stokes solver

3.1 Fractional-step scheme

In this part, we consider solving in the domain $\Omega =]a,b[^2$ the unsteady Navier-Stokes equations for an incompressible fluid in the velocity-pressure formulation with the following Adams-Bashforth/Backward Euler fractional step scheme

$$\frac{3\,\widetilde{\mathbf{v}} - 4\,\mathbf{v}^n + \mathbf{v}^{n-1}}{2\Delta t} - \nu\nabla^2\widetilde{\mathbf{v}} + \nabla p^n + 2\,\mathbf{v}^n\cdot\nabla\mathbf{v}^n - \mathbf{v}^{n-1}\cdot\nabla\mathbf{v}^{n-1} = \mathbf{f}^{n+1}, \tag{9a}$$

$$\widetilde{\mathbf{v}}_{|\partial\Omega} = \mathbf{g}^{n+1}, \tag{9b}$$

where ν denotes the inverse of the Reynolds number Re, and

$$\frac{3}{2}\,\frac{\mathbf{v}^{n+1} - \widetilde{\mathbf{v}}}{\Delta t} + \nabla\left(p^{n+1} - p^n\right) = 0, \tag{10a}$$

$$\nabla\cdot\mathbf{v}^{n+1} = 0, \tag{10b}$$

$$\mathbf{v}^{n+1}\cdot\mathbf{n}_{|\partial\Omega} = \mathbf{g}^{n+1}\cdot\mathbf{n}. \tag{10c}$$

The projection step (10) is not solved in this work by building a Neumann problem for the pressure since that would require the imposition of a non-physical boundary condition for this quantity. The alternative is to consider (10) as a Div-Grad problem (see *e.g.* Quartapelle 1993, Azaiez 1994) that amounts to first discretizing in space Eqs. (10), then decoupling the velocity and pressure by performing block gaussian elimination on the discretized system in order to build an equation for the pressure. In this way, no pressure boundary conditions are needed. This way of solving the projection step has been considered in finite-difference approximations (Fortin *et al.* 1971, Kim & Moin 1985), finite-element methods (Donea *et al.* 1982, Gresho & Chan 1990), and spectral methods (Botella, 1997). However, since the Div-Grad problem is a saddle-point problem, the discrete approximations of the velocity and the pressure have to be compatible of avoiding pressure oscillations.

3.2 Spatial discretization using a B-spline collocation method

3.2.1 Equal-order discretization

The possibility of pressure checkerboarding in a spline-collocation method is addressed by considering the following one-dimensional version of the Div-Grad problem:

$$\sigma u(x) + \frac{d}{dx}p(x) = f(x) \quad \text{in } \Lambda =]0,1[, \tag{11a}$$

$$\frac{d}{dx}u(x) = g(x) \quad \text{in } \Lambda =]0,1[, \tag{11b}$$

$$u(0) = 0, \quad u(1) = 0, \tag{11c}$$

where σ is a constant and the functions f and g are given force terms. The velocity and pressure are approximated with the same smoothest splines basis characterized by $(k, m = k-2, \boldsymbol{\xi})$

$$u(x) = \sum_{j=1}^{N} u_j B_j(x), \quad p(x) = \sum_{j=1}^{N} p_j B_j(x), \tag{12}$$

where, for brevity, the superscript k is dropped. The collocation points $\{x_i,\, i = 1, \dots, N\}$ are the maximum of the B-splines, and the equal-order spline-collocation discretization of (11) is

$$\sigma \sum_{j=2}^{N-1} u_j B_j(x_i) + \sum_{j=1}^{N} p_j B_j'(x_i) = f(x_i), \quad i = 2, \dots, N-1, \tag{13a}$$

$$\sum_{j=2}^{N-1} u_j B_j'(x_i) = g(x_i), \quad i = 1, \dots, N, \tag{13b}$$

with the homogeneous boundary conditions (11c) giving the coefficients $u_1 = u_N = 0$. Note that Eqs. (13) amounts to discretizing (11a)-(11b) at the inner collocation points $\{x_i,\, i = 2, \dots, N-1\}$ and adding the boundary condition $\nabla.\mathbf{v}_{|\partial\Omega} = 0$. This boundary condition can also be found in the influence matrix method of Kleiser & Schumann (1980) for equal-order Chebychev discretization.

The system (13) takes the following sparse matrix form

$$\sigma \mathcal{M}_e U + \widetilde{\mathcal{D}}_e P = F, \tag{14a}$$

$$\mathcal{D}_e U = G, \tag{14b}$$

where U and P are vectors representing the unknown coefficients of the velocity and the pressure respectively and $\mathcal{M}_e$ is the (non-diagonal) mass matrix; note that in a collocation method the first derivative matrix of the pressure $\widetilde{\mathcal{D}}_e$ is not the transpose of the divergence matrix $\mathcal{D}_e$. By performing block-Gaussian eliminations, the velocity can be decoupled from the pressure as

$$\sigma \mathcal{M}_e U = F - \widetilde{\mathcal{D}}_e P, \tag{15a}$$

$$\frac{1}{\sigma} \mathcal{A}_e P = \frac{1}{\sigma} \mathcal{D}_e \mathcal{M}_e^{-1} F - G, \tag{15b}$$

where $\mathcal{A}_e = \mathcal{D}_e \mathcal{M}_e^{-1} \widetilde{\mathcal{D}}_e$ is the $N \times N$ matrix called the Uzawa operator of the Div-Grad problem. It is worthwhile to note that, since the inverse of the mass matrix is dense, the pressure operator $\mathcal{A}_e$ is a dense matrix. This block factorization is not recommended for solving practical problems, but is used for the purpose of evaluating the collocation discretization. As it will be shown later, this discretization yields one spurious pressure mode.

3.2.2 The "staggered splines" discretization

Efficient techniques for preventing spurious oscillations of the pressure are now well established for the classical numerical methods. Among others, we cite the use of a staggered finite-difference grid (Kim & Moin, 1985), of pressure elements of one degree less than the velocity in the finite-element method (*e.g.* Gresho & Sani, 1998), and the $P_N \times P_{N-2}$ discretization in spectral methods (*e.g.* Botella, 1997). To our knowledge, this issue has not been previously addressed for B-spline discretizations.

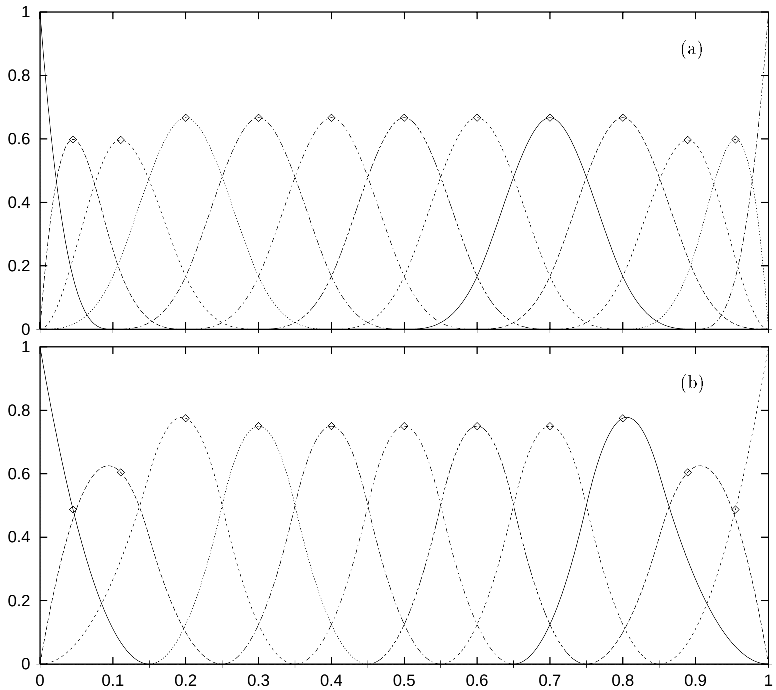

FIGURE 3. Staggered B-spline bases for (a) the velocity and (b) the pressure on evenly distributed breakpoints (+), with $k = 4$ and $l = 10$. The collocation points ($\diamond$) are the maximum of the velocity B-splines.

In the following, we describe a constructive method for building a basis for the pressure that prevents checkerboarding for a smoothest spline collocation method. This compatible basis is constructed from the velocity basis, which is given by the smoothest spline basis $(k, m = k-2, \boldsymbol{\xi})$. The pressure basis $(k^p, m^p, \boldsymbol{\xi}^p)$ is constructed by requiring that p be represented by a smoothest spline of order one less than for the velocity, *i.e.*:

$$k^p = k - 1, \quad m^p = m - 1. \tag{16}$$

As for the equal-order discretization, the collocation points are the maximum of the N velocity B-splines where, by using (5),

$$N = l + k - 1. \tag{17}$$

The collocation discretization is defined by requiring that the collocation points at the boundary $x_1 = 0$ and $x_N = 1$ be used for the boundary conditions and the $N-2$ inner collocation points be used for discretizing the equations; as a result, in order to close the system, the number N^p of pressure B-splines must be equal to $N-2$. The number of pressure breakpoints l^p is then, by using (17),

$$l^p = l - 1.$$

Now, the pressure basis is completely characterized by defining $\boldsymbol{\xi}^p$ as

	size	dim(ker($\mathcal{A}_i$))	$-\lambda_{\min}$	$-\lambda_{\max}$
Equal-order discretization, $i = e$	N	2	9.8	$18\,N^2$
Staggered discretization, $i = s$	N-2	1	9.8	$1.5\,N^2$

Table 2. Properties of the operator $\mathcal{A}_i$, $i = e$ or s.

N	26	56	106	166
Equal-order discretization	$3.7 \cdot 10^{-3}$	$6.7 \cdot 10^{-6}$	$1.0 \cdot 10^{-7}$	$6.2 \cdot 10^{-9}$
Staggered discretization	$7.6 \cdot 10^{-3}$	$6.6 \cdot 10^{-6}$	$7.4 \cdot 10^{-8}$	$4.3 \cdot 10^{-9}$

Table 3. Error on the velocity given by the two formulations with splines of order 6.

$$\xi_j^p = (\xi_j + \xi_{j+1})/2, \quad j = 1, \ldots, l\,, \tag{18}$$

which amounts to staggering the pressure breakpoints with respect to those of the velocity basis. The compatible B-spline bases are plotted in Fig. 3 for $k = 4$ and an evenly spaced distribution of breakpoints $\boldsymbol{\xi}$. In this case, it can be observed that the inner collocation points are also the maximum of the pressure basis functions except for the first and last $k-1$ B-splines.

The staggered collocation approximation of (11) yields, in a fashion similar to the previous case, the $(N-2) \times (N-2)$ pressure operator $\mathcal{A}_s = \mathcal{D}_s \mathcal{M}_s^{-1} \widetilde{\mathcal{D}}_s$, where the subscript s refers to operators obtained from the staggered bases.

3.2.3 Discussion

The spectral properties of $\mathcal{A}_i$, $i = e$ or s yielded by the equal-order and the staggered discretization respectively has been investigated by numerically calculating its eigenvalues. As a matter of fact, the dimension of the kernel of $\mathcal{A}_i$ gives the number of spurious pressure modes. Furthermore, the determination of $\lambda_{\min}$ and $\lambda_{\max}$, the eigenvalue of $\mathcal{A}_i$ with minimal and maximal modulus respectively, gives an estimation of the spectral condition number $\chi(\mathcal{A}_i) = \lambda_{\max}/\lambda_{\min}$ that is relevant for the iterative inversion of this operator. The main properties of the operator $\mathcal{A}_i$ for an even distribution of breakpoints and $k = 6$ is given in Table 2.

The operator $\mathcal{A}_e$ proves to possess two zero eigenvalues, one corresponding to the constant pressure mode, the other testifying of the presence of a spurious pressure mode in the equal-order discretization. The extension of this method in two dimensions would give three spurious modes, as is observed in the equal-order Chebyshev discretization of the Div-Grad problem (Azaiez *et al.*, 1994). On the other hand, the dimension of the kernel of $\mathcal{A}_s$ is equal to one, showing that no spurious modes occur in the staggered discretization. Note also that for both discretizations the other eigenvalues are complex with negative real part, and the condition number $\chi(\mathcal{A}_i)$ is proportional to N^2. It is expected that, for any given distribution of breakpoints $\boldsymbol{\xi}$, the staggered discretization is free of spurious modes, and this has been checked for the following Chebyshev distribution of breakpoints,

$$\xi_j = \left(\cos(\pi(l+1-j)/l) + 1\right)/2, \quad j = 1, \ldots, l+1. \tag{19}$$

B-spline order	u	p	p'
k even	k	$k-2$	$k-2$
k odd	$k-1$	$k-2$	$k-2$

Table 4. Order of accuracy of the staggered collocation discretization with B-splines of order k.

The Div-Grad problem (11) with $\sigma = 10$ has been solved with both collocation methods for the solution

$$u = x(1-x)\cos(33x), \quad p = \cos(14x), \tag{20}$$

with velocity B-splines of order $k = 6$, on evenly spaced breakpoints. The error on the velocity with respect to N is given in Table 3, showing that both discretizations give a similar sixth order accuracy. Data on the pressure approximation given by the equal-order collocation method are unavailable since this quantity is plagued by the spurious mode. The accuracy of the staggered approximation has been evaluated with numerical tests on the solution (20), and the order of accuracy observed numerically is summarized in Table 4.

3.3 Two-dimensional numerical results

3.3.1 Iterative solution of the Div-Grad problem

We consider here solutions to the following Div-Grad problem:

$$\sigma \mathbf{v} + \nabla p = \mathbf{f}, \tag{21a}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{21b}$$

$$\mathbf{v} \cdot \mathbf{n}_{|\partial\Omega} = \mathbf{0}, \tag{21c}$$

in the domain $\Omega =]0,1[^2$, with the staggered splines collocation method. The two-dimensional discretization is a straightforward extension of the method described in the previous section: the velocity and pressure spline approximations are expressed in tensor product B-splines bases,

$$\mathbf{v} = \sum_{i,j=1}^{N} \mathbf{v}_{i,j}\, B_i(x) B_j(y), \quad p = \sum_{i,j=1}^{N-2} p_{i,j}\, \widetilde{B}_i(x) \widetilde{B}_j(y),$$

where $\{B_i(x),\, i = 1, \ldots, N\}$ and $\{\widetilde{B}_i(x),\, i = 1, \ldots, N-2\}$ are the one-dimensional compatible B-spline bases of order k introduced in the previous section. Equations (21) are discretized on the collocation grid $\{(x_i, y_j) = \max_{i,j} B_i(x) B_j(y);\, i,j = 1, \ldots, N\}$, leading to the discrete system

$$\sigma \mathcal{M} U + \widetilde{\mathcal{D}} P = F, \tag{22a}$$

$$\mathcal{D} U = 0. \tag{22b}$$

The pressure equation associated with this system is

$$\frac{1}{\sigma}\mathcal{A}P = \frac{1}{\sigma}\mathcal{H}, \quad \text{with}\, \mathcal{A} = \mathcal{D}\mathcal{M}^{-1}\widetilde{\mathcal{D}} \text{ and } \mathcal{H} = \mathcal{D}\mathcal{M}^{-1}F. \tag{23}$$

As in the one-dimensional case, the operator $\mathcal{A}$ is dense. However this system can be solved by using an Uzawa-type iterative algorithm (Cahouet & Chabard, 1988, Quarteroni & Valli 1994), which involves only the solution of sparse elliptic problems. A prototype Uzawa algorithm is given by the following iteration step, where P^m is given and m refers to the number of the iterations:

$$\text{Solve} \quad \sigma\mathcal{M}U^m = F - \widetilde{\mathcal{D}}P^m, \tag{24a}$$

$$\text{Compute the residual} \quad R^m = \frac{1}{\sigma}(\mathcal{H} - \mathcal{A}P^m) = \mathcal{D}U^m, \tag{24b}$$

$$\text{Solve} \quad \mathcal{C}(P^{m+1} - P^m) = -\rho R^m, \tag{24c}$$

where $\mathcal{C}$ is a preconditioner of $\mathcal{A}$ and ρ an acceleration parameter. This algorithm allows us to solve (22) without explicitly building the dense matrix $\mathcal{A}$; the convergence criterion $|R^m| < \epsilon$ permits us to control the degree to which the velocity is divergence free. If $\mathcal{C} = \mathcal{I}$ and the value of ρ is computed dynamically with the steepest descent formula, the above algorithm reduces to the gradient method applied to (23). As a matter of fact, this basic Uzawa algorithm can be accelerated by any iterative method, see *e.g.* Cahouet & Chabard (1988) for a conjugate gradient version.

Straightforward applications of the Uzawa algorithm for solving (22) have proven to give poor results: the spline-collocation matrices are not symmetric so that iterative methods such as conjugate gradient failed to converge. Our attention has been drawn towards Krylov subspace methods (Saad 1996, Barrett 1996 and references therein) such as CGS and Bi-CGSTAB. These methods are, however, known to converge poorly or to be unstable when the system is not well conditioned. As a matter of fact, a preconditioner of the system is $\mathcal{C} = \mathcal{A}_L$ where

$$\mathcal{A}_L = \mathcal{D}\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}. \tag{25}$$

In this equation, $\mathcal{M}_L^{-1}$ refers to the inverse of the diagonal matrix obtained by summing the row of the mass matrix and putting the result on the diagonal; such a matrix is called the lumped mass matrix in the finite-element method (Gresho & Sani, 1998). The operator $\mathcal{A}_L$ is sparse, with a bandwidth $\simeq 2k^2$. The condition number $\chi(\mathcal{C}^{-1}\mathcal{A})$ of the unpreconditioned ($\mathcal{C} = \mathcal{I}$) and preconditioned ($\mathcal{C} = \mathcal{A}_L$) Div-Grad system (22) has been evaluated numerically for both the case of an uniform grid and the stretched grid given by the Chebyshev distribution (19). The relevant results are reported in Table 5. The preconditioned system proves to have a condition number independent of the number of splines N; however, its value increases with the order k of the B-splines.

Grid	Uniform		Chebyshev	
Precond.	$\mathcal{C}=\mathcal{I}$	$\mathcal{C}=\mathcal{A}_L$	$\mathcal{C}=\mathcal{I}$	$\mathcal{C}=\mathcal{A}_L$
$k=4$	$0.21/h^2$	9.4	$3\cdot 10^{-3}/h^4$	9.0
$k=6$	$0.14/h^2$	66	$3\cdot 10^{-3}/h^4$	57

Table 5. Condition number $\chi(\mathcal{C}^{-1}\mathcal{A})$ of the Div-Grad problem, where $h=1/N-2$ refers to the inverse of the number of unknowns in each direction.

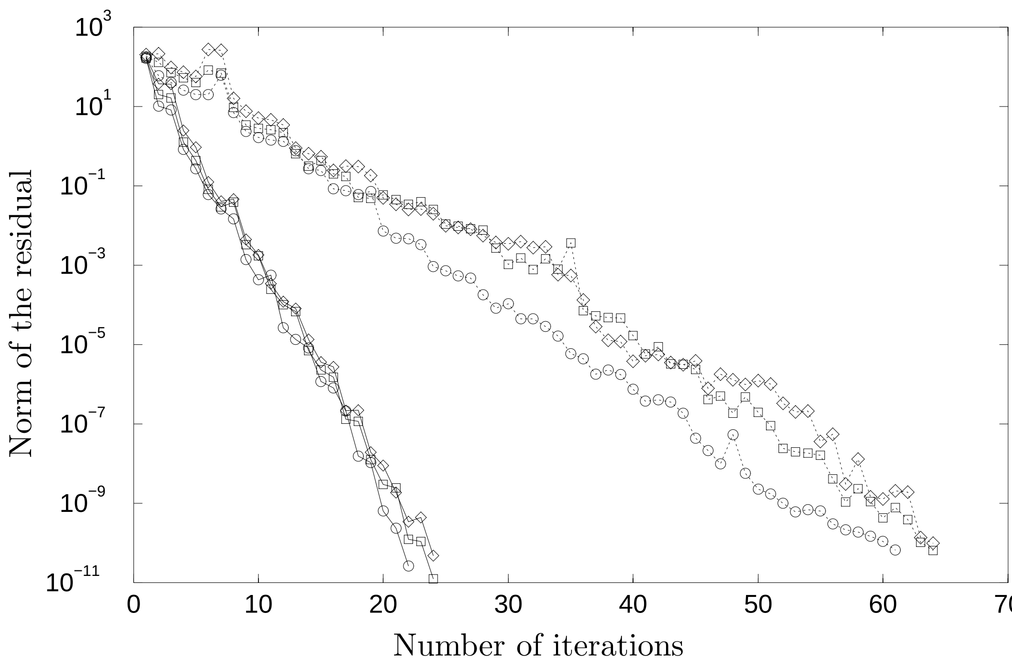

FIGURE 4. Preconditioned Uzawa Bi-CGSTAB algorithm: Norm of the residual vs. the number of iterations for various discretizations defined by $N=l+k-1$ with $l=25$ (○), $l=50$ (□), $l=80$ (◇) for splines of order $k=4$ (solid line) and $k=6$ (dotted line).

An Uzawa algorithm accelerated by Bi-CGSTAB iterations has been developed (Botella, 1999). A single step of this algorithm requires in particular the inversion of two mass matrix problems for each component of the velocity and two inversions of the preconditioner. An example of application of this algorithm with $\mathcal{C}=\mathcal{A}_L$ is given by the solution of problem (22) with $\sigma=1$, for the solution

$$\mathbf{v}=\mathbf{rot}\sin 4\pi x\sin 4\pi y,\quad p=\cos 4\pi x\cos 4\pi y,$$

on a regular grid. The convergence history of the residual for splines of order 4 and 6, obtained with the start vector $P^0=0$, is reported in Fig. 4. This figure shows in particular that the preconditioned algorithm converges in a number of steps independent of N, although this number is quite high for $k=6$. If no preconditioner is used, it has been observed that such an iterative method, or an equivalent one accelerated by CGS, does not converge for splines of order 6 when $N\geq 15$. At last, we mention that it has been verified that an $O(N^{-6})$ accuracy on the velocity and an $O(N^{-4})$ error on the pressure is obtained, in accordance with the one-dimensional results shown in Table 4.

3.3.2 Navier-Stokes results

In this part, we describe ongoing work on approximating the Navier-Stokes equations (8), and more precisely on "cheap" ways to solve these equations, by which we mean that would not require performing Uzawa iterations for solving the projection step. For the sake of the clarity of the discussion, the staggered-splines collocation discretization of scheme (9)-(10) is written in the simplified form of the following first order scheme, where the nonlinear terms are omitted:

$$\mathcal{M}\frac{(\widetilde{U}-U^n)}{\Delta t} - \mathcal{K}\widetilde{U} + \widetilde{\mathcal{D}}P^n = F^{n+1}, \tag{26}$$

where $\mathcal{K}$ is the viscous diffusion matrix, and

$$\mathcal{M}\frac{(U^{n+1}-\widetilde{U})}{\Delta t} + \widetilde{\mathcal{D}}(P^{n+1}-P^n) = 0, \tag{27a}$$

$$DU^{n+1} = 0. \tag{27b}$$

We mention, however, that the numerical results discussed in this section have been obtained with the second order scheme where the nonlinear terms are discretized in the convective form. These equations are to be solved with the staggered splines discretization of order $k = 4$ or 6. Since Eq. (26) is a second-order problem for $\widetilde{U}$, the expected spatial order of accuracy of the above scheme is $k-2$ for both velocity and pressure. The solution of Eqs. (27) by the Uzawa algorithm requires solving linear systems involving the preconditioner $\mathcal{A}_L$ more than 20 times to get a "decent" solution. Since the sparsity and the conditioning of $\mathcal{A}_L$ is roughly equivalent to the pressure operator of finite-difference approximations, this way of solving the Navier-Stokes equations, with the consistent mass matrix $\mathcal{M}$ in Eq. (27a), is very costly and cannot be in any way competitive with finite-difference methods.

The consistent mass matrix has been a relevant issue in the finite-element method since its first application to CFD (Gresho & Sani, 1998). An ad-hoc approximation that is commonly used is to "lump" the mass matrix (*i.e.* $\mathcal{M}$ is replaced by $\mathcal{M}_L$ in Eqs. (26)-(27)), which causes severe loss of accuracy for time-dependent problems. So far, the most satisfying alternative to the CM scheme (26)-(27) is the "projection 2" scheme of Gresho & Chan (1990) that uses a semi-consistent mass matrix approximation, *i.e.* that consists in lumping the mass matrix in front of the pressure gradient only. This scheme that introduces a modified form of the pressure gradient in the first step as

$$\mathcal{M}\frac{\widetilde{U}-U^n}{\Delta t} - \mathcal{K}\widetilde{U} + \mathcal{M}\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}P^n = F^{n+1}, \tag{28a}$$

$$\mathcal{M}_L\frac{U^{n+1}-\widetilde{U}}{\Delta t} + \widetilde{\mathcal{D}}(P^{n+1}-P^n) = 0, \tag{28b}$$

$$DU^{n+1} = 0, \tag{28c}$$

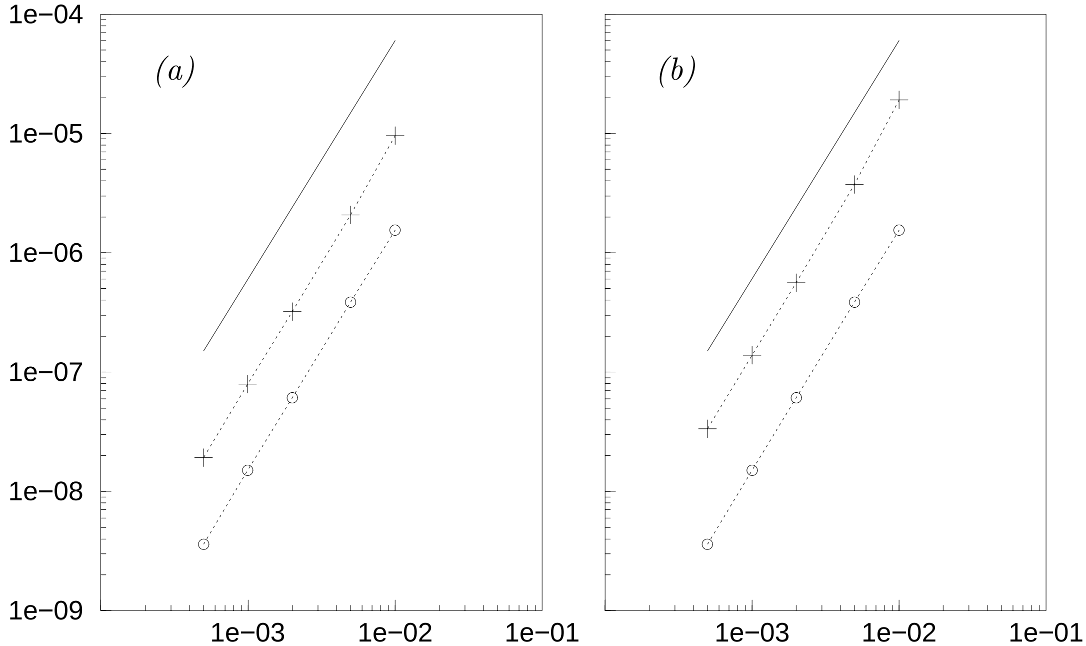

FIGURE 5. Error on the splines coefficients of u (○) and p (+) vs. Δt for the decaying vortices flow computed with *(a)* the CM scheme and *(b)* the SCM scheme; ——— : reference line of slope 2. The spatial resolution is defined by $k = 4$ and $l = 25$, the reference solution being the solution computed with $\Delta t = 10^{-4}$.

is referred to in the following as SCM. Equations (28b)-(28c) yield the pressure equation

$$\mathcal{A}_L(P^{n+1} - P^n) = D\widetilde{U}/\Delta t, \tag{29}$$

which amounts to inverting the operator $\mathcal{A}_L$ defined by (25) only once for every time-cycle. This would yield a cost-effective method when compared to the CM scheme where $\mathcal{A}_L$ is used as a preconditioner of the projection step and is thus inverted at each Uzawa iteration.

The time accuracy of the CM and SCM schemes has been evaluated on the decaying vortices flow discussed by Le & Moin (1991). Fig. 5 compares the time-accuracy of these schemes at $t = 1.5$. Both methods give an identical second-order accuracy for the velocity. The pressure is also second-order accurate, although the error is higher for the SCM scheme. The properties of the SCM scheme have then been investigated on the regularized driven cavity flow (Peyret & Taylor, 1983), with the spatial resolution defined by $k = 6$ and $N = 65$ on the Chebyshev distribution of breakpoints given by (19), and the time step $\Delta t = 5 \times 10^{-3}$. The steady-state solution at $Re = 10,000$ is displayed in Fig. 6. Starting from this solution, the unsteady flow at $Re = 12,000$ is computed on nearly half a million time-steps. Figure 7 displays the time-evolution of the kinetic energy, showing that the periodic solution is reached at $t \simeq 1500$, with a period $T = 3.095 \pm \Delta t$. This value of T compares well with the period $T = 3.085 \pm 5 \times 10^{-3}$ computed by Shen (1991) with a Chebyshev-Tau scheme with a $N = 65$ truncation. This result shows that

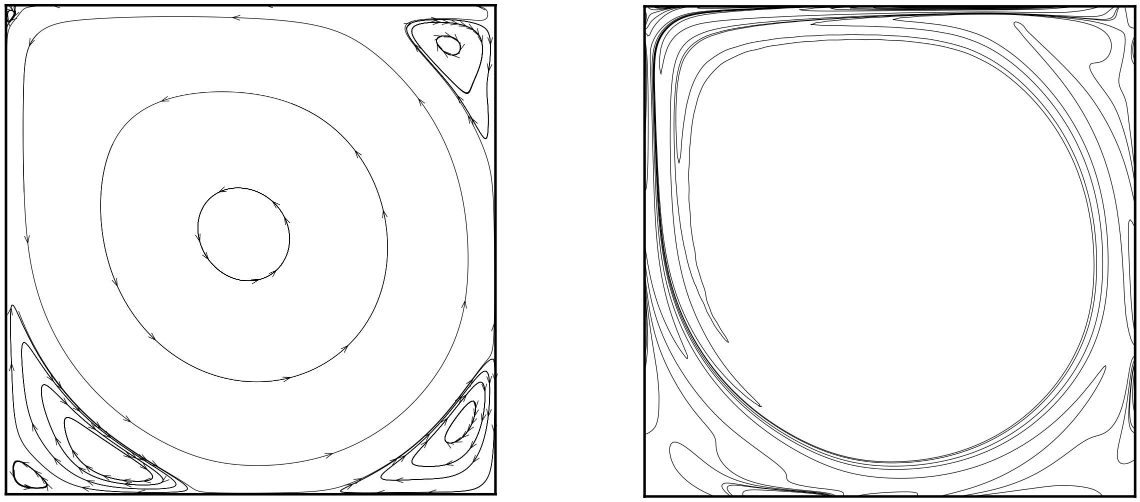

FIGURE 6. Streamlines (left) and vorticity lines (right) for the steady-state driven cavity flow at $Re = 10,000$.

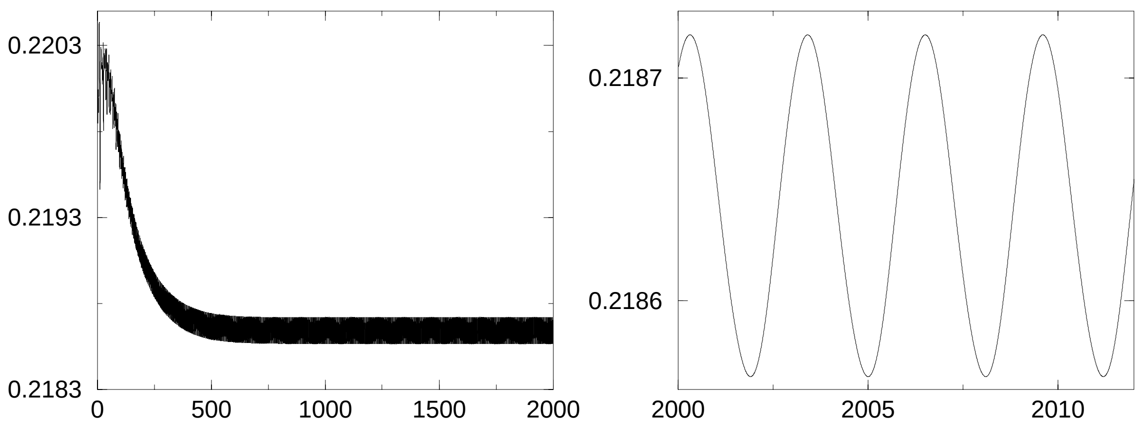

FIGURE 7. Time-evolution of the kinetic energy for the driven cavity flow at $Re = 12,000$.

the staggered-spline method with the SCM scheme reproduces spectral results with the same coarse resolution and that it is able to conserve kinetic energy on a long time integration. Figure 8(a) compares the spatial accuracy of the CM and SCM schemes with splines of order 4 on the driven cavity flow at $Re = 100$, the reference solution being obtained from a fully converged Chebyshev solution (Botella, 1997). As expected, the CM solution gives a second-order accuracy on both the velocity and the pressure. The SCM scheme yields an identical accuracy on the velocity, but one order of accuracy is nevertheless dropped on the approximation of the pressure. This loss of accuracy is certainly caused by the lumped mass matrix approximation used in this scheme.

The SCM scheme has demonstrated satisfying stability properties and, for low order splines ($k = 4$), good spatial accuracy. However, as observed on the decaying vortices flow computed with $k = 6$ (Fig. 8(b)), the high order accuracy that would

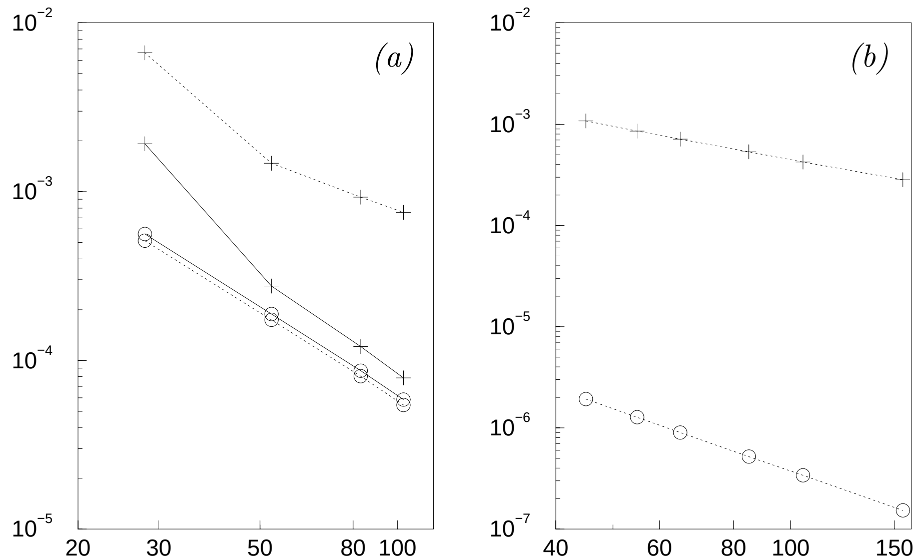

FIGURE 8. Spatial accuracy of u ($\circ$) and p (+) vs. the number of splines N: (a) Driven cavity flow at $Re = 100$, CM (solid line) and SCM (dashed line) schemes with $k = 4$; (b) Decaying vortices flow at $t = 0.45\pi$ for SCM with $k = 6$.

have been expected is destroyed by the lumped approximation, yielding only second-order accuracy on the velocity, and first order on the pressure.

This loss of accuracy can be explained when investigating the truncation error of the SCM scheme (28)

$$\mathcal{M}\frac{U^{n+1}-U^n}{\Delta t} - \mathcal{K}U^{n+1} + \widetilde{\mathcal{D}}P^{n+1} - \Delta t\mathcal{K}\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}(P^{n+1}-P^n) + \mathcal{E}_L = F^{n+1}, \quad (30)$$

where in addition to the standard $O(\Delta t^2)$ splitting error, the "lumping" error

$$\mathcal{E}_L = (\mathcal{M}-\mathcal{M}_L)\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}P^{n+1} = O(h), \quad (31)$$

that appears is a spatial error.

Possible ways to improve the SCM scheme are discussed in Dukowicz & Dvinsky (1991) and Jacobs (1994); a tempting alternative, proposed in the latter reference, would be to simply replace the term $\mathcal{M}\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}P^n$ in Eq. (28a) by $\widetilde{\mathcal{D}}P^n$. The lumping error of the resulting scheme, designated in the following as SCM$'$, reduces to

$$\mathcal{E}_L = (\mathcal{M}-\mathcal{M}_L)\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}(P^{n+1}-P^n) = O(h\Delta t), \quad (32)$$

so that this error is now part of the temporal error of the scheme. Computations of the decaying vortices flow discussed above with SCM$'$ give accurate results for values of Δt as low as 5×10^{-3} but are found unstable when Δt is further decreased. A similar behavior has been found for the computation of unsteady Stokes flows, so that the instability is not related to nonlinear effects. It has been observed in Gresho & Chan (1990) that the discrete projector associated to Eq. (28b)-(28c) does annihilate the modified pressure gradient $\mathcal{M}\mathcal{M}_L^{-1}\widetilde{\mathcal{D}}P^n$ of the SCM scheme. On the other hand, it can be shown analytically that the "correct" pressure gradient of the SCM′ scheme is only annihilated up to the splitting error when $h \ll \Delta t$, explaining the instability when Δt is decreased. Note also that this condition is opposite to the standard CFL condition, so that this scheme is of no practical interest. Thus far, attempts to improve the stability of the SCM′ scheme has proven unsuccessful. From both the analytical point of view and in numerical tests, "moving" the $O(h)$ lumped error into the temporal error results in unstable schemes. Let us mention that in this category belongs the three-step scheme proposed, but never implemented, by Dukowicz & Dvinsky (1991). Through a change of dependent variables, it can be shown that this scheme is equivalent to one of the unstable schemes mentioned above.

4. Conclusions and open questions

In this report, a B-spline collocation discretization for solving the Navier-Stokes equations has been introduced and evaluated. Although no mathematical justifications are presented here, the staggered collocation method has proven to be an efficient method for preventing spurious pressure oscillations, and yields compatible bases of arbitrary order k. In addition, it is valuable to mention that the staggered bases are also suitable for a Galerkin discretization.

The staggered discretization uses B-splines bases of maximum continuity, yielding high resolving power schemes. One of the issues raised in the first part of the report is that numerical methods should not only be evaluated on their asymptotic order of accuracy, but also on their resolving power. The C^1 Gauss-collocation method displays an optimal order of accuracy, but the low continuity of the basis induces a poor resolving power. As a result, its accuracy on the solution of differential problems is inferior to the one of the smoothest collocation method even if the latter is formally of lower order.

The low computational efficiency of the unsteady computations is related to the presence of a non-diagonal mass matrix. This problem represents a critical issue for finite-element type methods and is not related to the collocation approach chosen in this study. The lumping of the mass matrix is a widely used ad-hoc technique allowing finite-element transient calculations to be cost-effective. However, when calculating an implicit variable like the pressure in incompressible flows, this technique results in a loss of accuracy for high-order discretizations such as B-spline methods, as proved by the computations using the lumped "projection 2" scheme. To our knowledge, there does not exist yet a totally satisfying lumped fractional step scheme that would yield both a high-order and cost-effective finite-element type discretization for transient solutions of the Navier-Stokes equations.

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