Large scale reaction front dynamics in anisotropic reaction-diffusion systems

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1. Motivation and objectives

The last decade has seen increasingly detailed development of the theory of traveling waves in reaction-diffusion systems (Debnatz, 1997). Due to its relative simplicity, special attention has been paid to the Fisher equation, or Kolmogorov-Petrovskii-Piskunov (KPP) equation, describing front propagation into an unstable state. The major feature of this process is that the dynamics of the reaction front are determined by the processes taking place at the leading edge of the front profile. However, in most cases the transport process is described by a diffusion approximation. As a result, the rate at which the wave propagates throughout the reaction-diffusion system can be overestimated. Physical reasoning supports this observation because the density field predicted by the diffusion approximation has higher tails than the density of the real transport process (Joseph & Preziosi, 1989). To deal with this problem several researchers have introduced the hyperbolic correction to the diffusion approximation, taking into account the finite speed of the transport process (Monin & Yaglom, 1987; Mendez & Camacho, 1997; Gallay & Raugel, 1998).

Recently we have presented a formulation of reaction front dynamics in terms of special relativity theory, where the diffusion wave speed plays the role of the speed of light (Fedotov, 1998, 1999). We have found that, in the long-time large-distance asymptotic limit, the Hamiltonian dynamical system associated with the reaction-diffusion system is similar to that of classical relativistic mechanics. This analogy is rooted in the fact that, in both cases, there is a finite propagation speed. In the solution of the generalized Fisher equation, ρ has the asymptotic form $\rho \sim \exp\left(G/\varepsilon\right)$, where ε is the small parameter describing the long-time large-distance limit and G obeys the relativistic Hamilton-Jacobi equation defining the Hamiltonian dynamical system associated with the reaction-diffusion system. One of the main advantages of this approach over conventional analyses of traveling wave solutions is that the front dynamics can be described by a first order partial differential equation rather than a second order equation (Evans & Souganidis, 1989).

It is quite natural to ask whether or not there exists a *general relativity analogy* and, if so, how both gravitational and electromagnetic fields associated with a reaction-diffusion system can be determined. It is the purpose of this paper to find the answer to this question and show that the function which determines the reaction front position can be derived from a variational principle of general relativity theory. The central result of this paper is that, for the anisotropic reaction-diffusion system, the reaction front position can be found exactly from the general relativity

Hamilton-Jacobi equation (Landau & Lifshitz, 1961). The diffusivity tensor determines the metric tensor of the 4-dimensional Riemannian space of general relativity, and the finite speed of diffusion waves can be regarded as the speed of light.

The theory presented in this paper is an approximate one analogous to the WKB approximation and the relation of geometric optics to electrodynamics. The mathematical basis of such theory is an asymptotic expansion in powers of a small parameter ε, which is the ratio of a characteristic width of traveling wave profile to the typical length scale of the problem. The basic equation governing the reaction front dynamic is for the lowest order terms. Due to this approximation, the theory cannot describe the phenomena involving interaction of reaction fronts, appearance of singularities, etc.

2. Reaction front propagation in the anisotropic reaction-diffusion system

Our analysis of reaction front propagation in anisotropic media begins with the transport equation for a scalar field $\rho(t, \mathbf{r})$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = U(\mathbf{r}) \rho f(\rho), \qquad \mathbf{r} = \left(x^1, x^2, x^3\right) \tag{1}$$

where $\mathbf{J}$ is the mass flux and the non-linear source term on the RHS of (1) is of Kolmogorov-Petrovskii-Piskunov-type (Freidlin, 1996), that is

$$\max_{\rho \in [0,1]} f(\rho) = f(0) = 1, \qquad f(1) = 0, \tag{2}$$

The reaction rate parameter $U(\mathbf{r})$ is assumed to be a function of the space coordinate $\mathbf{r}$. The reason for taking this dependence into account is that it might induce the phenomenon of wave front jump (Freidlin, 1996).

It is well known that in an anisotropic medium the direction of the mass flux $\mathbf{J}$ is, in general, not that of the gradient of the scalar field $\nabla \rho$. The component of the mass flux vector $\mathbf{J}(t, \mathbf{r})$ can be written as

$$J^i(t, \mathbf{r}) = \frac{1}{2} \sum_{k=1}^{3} D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k}, \tag{3}$$

where $D^{ik}(\mathbf{r})$ is the diffusivity tensor, which in our analysis may depend on the spatial coordinate $\mathbf{r}$. However, this classical approach to the transport process, in which one expresses an instantaneous dependence of flux on gradient, is not sufficiently accurate and may result in an overestimation of the speed of propagating fronts (Fedotov, 1999). In order to investigate the role of inertia effects in the anisotropic transport process described by Eq. (1), we introduce the relaxation time τ in a such way that the component of the flux $\mathbf{J}$ can be determined by the Cattaneo's law (Joseph & Preziosi, 1989)

$$J^i(t, \mathbf{r}) = \frac{1}{2\tau} \sum_{k=1}^{3} \int_0^t \exp\left(\frac{t-s}{\tau}\right) D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k}(s, \mathbf{r}) \, ds. \tag{4}$$

Here the matrix D^{ik} is assumed to be symmetric and positive definite. If inertia is neglected ($\tau = 0$), then we have Fick's law (3), and Eq. (1) together with (3) can be written in the form of the classical Fisher-KPP-equation.

Equations (1) and (4) can be rewritten as a single equation for ρ

$$\frac{\partial \rho}{\partial t} = \frac{1}{2\tau} \sum_{i,k=1}^{3} \int_0^t \exp\left(\frac{t-s}{\tau}\right) \frac{\partial}{\partial x^i} D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k}(s, \mathbf{r})\, ds + U(\mathbf{r})\, \rho f(\rho). \quad (5)$$

This equation incorporates the combined effects of anisotropic diffusion with finite velocity, exponential growth, and non-linear saturation. It should be noted that the initial flux $J^i(0, \mathbf{r})$ is assumed to be zero.

To analyze the reaction front dynamics corresponding to (5), the initial distribution for ρ has to be specified. It is well known that the propagation rate may vary from the minimum velocity value to infinity depending on the initial condition (Freidlin, 1996). Here we assume the initial distribution to be in the form of the indicator function χ_{S_0} of the set S_0

$$\rho(0, \mathbf{r}) = \chi_{S_0} = \begin{cases} 1, & \text{if } \mathbf{r} \in S_0, \\ 0, & \text{otherwise,} \end{cases} \quad . \quad (6)$$

This initial condition ensures that the reaction front propagates at the minimum velocity. To avoid unnecessary complications, S_0 is assumed to be the convex set. For example, the set S_0 can be a ball of a radius R/ε such that

$$\rho(0, \mathbf{r}) = \begin{cases} 1, & \text{if } ,\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 \leq \frac{R^2}{\varepsilon^2} \\ 0, & \text{otherwise,} \end{cases} \quad \varepsilon << 1. \quad (7)$$

One can see from (7) that the initial distribution involves a small parameter ε that plays a very important role in what follows.

It is well known that the hyperbolic scaling procedure $t \to t/\varepsilon$, $\mathbf{r} \to \mathbf{r}/\varepsilon$ yields the large-scale geometric front propagation for the Fisher-KPP equation (Freidlin, 1996). The behavior of the rescaled field $\rho^\varepsilon(t, \mathbf{r}) = \rho(t/\varepsilon, \mathbf{r}/\varepsilon)$ may be explained in terms of a simple geometric picture. Since the nonlinear function $\rho^\varepsilon f(\rho^\varepsilon)$ on the RHS of (5) is equal to zero only if $\rho^\varepsilon = 0$ and $\rho^\varepsilon = 1$, we may argue that in the limit $\varepsilon \to 0$ the solution ρ^ε converges to the indicator function of the set S_t (Evans & Souganidis, 1989)

$$\lim_{\varepsilon \to 0} \rho^\varepsilon(t, \mathbf{r}) = \chi_{S_t} = \begin{cases} 1, & \text{if } \mathbf{r} \in S_t, \\ 0, & \text{otherwise.} \end{cases}$$

The boundary of the set S_t can be regarded as a reaction front describing the interface dynamics between the stable ($\rho^\varepsilon = 1$) and unstable ($\rho^\varepsilon = 0$) phases.

After hyperbolic scaling $t \to t/\varepsilon$, $\mathbf{r} \to \mathbf{r}/\varepsilon$, Eq. (5) can be rewritten as (see Appendix 1)

$$\varepsilon\tau \frac{\partial^2 \rho^\varepsilon}{\partial t^2} + \left(1 - \tau U(\mathbf{r}) f(\rho^\varepsilon) - \tau U(\mathbf{r}) \rho^\varepsilon \frac{df(\rho^\varepsilon)}{d\rho^\varepsilon}\right) \frac{\partial \rho^\varepsilon}{\partial t} =$$

$$\frac{\varepsilon}{2} \sum_{i,k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\mathbf{r}) \frac{\partial \rho^\varepsilon}{\partial x^k} + \frac{U(\mathbf{r})}{\varepsilon} \rho^\varepsilon f(\rho^\varepsilon) \tag{8}$$

while the initial condition has the form

$$\rho^\varepsilon(0, \mathbf{r}) = \begin{cases} 1, & \text{if } \mathbf{r} \in S_0, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Now we turn to the problem of finding the front dynamics for (8) and (9) in the limit $\varepsilon \to 0$.

3. Geometric optic approximation

Here we present a heuristic derivation of the Hamilton-Jacobi equation describing reaction front dynamics. Let us write down $\rho^\varepsilon(t, \mathbf{r})$ in exponential form

$$\rho^\varepsilon(t, \mathbf{r}) = \exp\left(-\frac{G^\varepsilon(t, \mathbf{r})}{\varepsilon}\right), \qquad G^\varepsilon(t, \mathbf{r}) \geq 0, \tag{10}$$

where the non-negative function G^ε describing the logarithmic asymptotic of the concentration field plays a very important role. It follows from (10) that as long as the function $G(t, \mathbf{r}) = \lim_{\epsilon \to 0} G^\varepsilon(t, \mathbf{r})$ is positive ($G(t, \mathbf{r}) > 0$), the rescaled field $\rho^\varepsilon(t, \mathbf{r}) \to 0$ as $\varepsilon \to 0$. So the boundary of the set S_t (see (7)), described above as the reaction front position, is nothing else but the boundary of the set where $G(t, \mathbf{r}) > 0$. Therefore, we may argue that the reaction front position can be determined as

$$\delta S_t = \left\{ \mathbf{r} \in R^3 : G(t, \mathbf{r}) = 0 \right\}.$$

Now we are in a position to determine the function $G(t, \mathbf{r})$. First let us find an equation for $G^\varepsilon(t, \mathbf{r})$. Inserting (10) into (8), we find that $G^\varepsilon(t, \mathbf{r})$ satisfies the non-linear PDE

$$\tau \left(\frac{\partial G^\varepsilon}{\partial t}\right)^2 - (1 - \tau U(\mathbf{r}) f) \frac{\partial G^\varepsilon}{\partial t} - \frac{1}{2} \sum_{i,k=1}^{3} D^{ik}(\mathbf{r}) \frac{\partial G^\varepsilon}{\partial x^i} \frac{\partial G^\varepsilon}{\partial x^k} + U(\mathbf{r}) f\left(e^{-\frac{G^\varepsilon(t,\mathbf{r})}{\varepsilon}}\right) =$$

$$\varepsilon \left[\tau \frac{\partial^2 G^\varepsilon}{\partial t^2} - \frac{1}{2} \sum_{i,k=1}^{3} \left(D^{ik}(\mathbf{r}) \frac{\partial^2 G^\varepsilon}{\partial x^i \partial x^k} + \frac{\partial G^\varepsilon}{\partial x^k} \frac{\partial D^{ik}(\mathbf{r})}{\partial x^i}\right)\right] - U(\mathbf{r}) \tau \frac{\partial G^\varepsilon}{\partial t} \frac{df}{d\rho^\varepsilon} e^{-\frac{G^\varepsilon(t,\mathbf{r})}{\varepsilon}}. \tag{11}$$

Since

$$\lim_{\epsilon \to 0} f\left(e^{-\frac{G^\varepsilon(t,\mathbf{r})}{\varepsilon}}\right) = 1, \qquad \lim_{\epsilon \to 0} \exp\left(-\frac{G^\varepsilon}{\varepsilon}\right) = 0 \tag{12}$$

provided $G^\varepsilon(t, \mathbf{r}) > 0$ it follows from (11) that the limiting function

$$G(t, \mathbf{r}) = -\lim_{\epsilon \to 0} \varepsilon \ln \rho^\varepsilon(t, \mathbf{r}) \tag{13}$$

obeys the nonlinear PDE of the first order

$$\left(\frac{\partial G}{\partial t}\right)^2 - \left(\frac{1}{\tau} - U(\mathbf{r})\right)\frac{\partial G}{\partial t} - \frac{1}{2\tau}\sum_{i,k=1}^{3} D^{ik}(\mathbf{r})\frac{\partial G}{\partial x^i}\frac{\partial G}{\partial x^k} - \frac{U(\mathbf{r})}{\tau} = 0 \qquad (14)$$

provided

$$G(t, \mathbf{r}) > 0.$$

If $\tau = 0$, then

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sum_{i,k=1}^{3} D^{ik}(\mathbf{r})\frac{\partial G}{\partial x^i}\frac{\partial G}{\partial x^k} + U(\mathbf{r}) = 0. \qquad (15)$$

Freidlin (1996) was the first to show that in the long-time large-distance limit the traveling wave solution to the classical Fisher-KPP equation is equivalent to the solution of the Hamilton-Jacobi equation (15). The function $G(t, \mathbf{r})$ can be found from the variational problem

$$G(t, \mathbf{r}) = \min\left\{\int_0^t L\left(\mathbf{r}(s), \frac{d\mathbf{r}}{ds}(s)\right) ds : \qquad \mathbf{r}(0) \in \Omega_0, \qquad \mathbf{r}(t) = \mathbf{r}\right\},$$

where L is the Lagrangian function of classical mechanics

$$L = \frac{1}{2}\sum_{i,k=1}^{3} D_{ik}(\mathbf{r})\frac{dx^i}{ds}\frac{dx^k}{ds} - U(\mathbf{r}).$$

The reaction rate parameter $U(\mathbf{r})$ plays the role of potential energy, the matrix $D_{ik}(\mathbf{r}) = \left(D^{ik}(\mathbf{r})\right)^{-1}$ determines the positive definite quadratic form of the kinetic energy, and x^i may be regarded as the generalized coordinates. It is quite remarkable that the concepts of classical mechanics lead to a new formulation of reaction front dynamics for the reaction-diffusion system.

Our problem now is to find a solution to Eq. (14) that can be considered a generalization of (15) and possibly to find a new interpretation of the phenomenological parameters $U(\mathbf{r})$, $D^{ik}(\mathbf{r})$ and τ.

4. General relativity Hamilton-Jacobi equation

The interesting feature of Eq. (14) is that it can be rewritten in the form of the Hamilton-Jacobi equation for a relativistic charged particle $(e = 1)$ in the presence of both gravitational and electromagnetic fields

$$\sum_{\alpha,\beta=1}^{4} g^{\alpha\beta}\left(\frac{\partial G}{\partial z^\alpha} - \frac{1}{c}A_\alpha\right)\left(\frac{\partial G}{\partial z^\beta} - \frac{1}{c}A_\beta\right) + m^2 c^2 = 0, \qquad (16)$$

where the new four-dimensional radius vector z^α is determined as

$$z^0 = ct, \qquad z^i = x^i, \qquad i = 1, 2, 3. \tag{17}$$

Here we have introduced the 4-potential of the electromagnetic field A_α such that the space components of A_α ($\alpha = 1, 2, 3$) forming the vector potential of the field are zero

$$A_1 = A_2 = A_3 = 0 \tag{18}$$

and the time component A_0 forming the scalar potential is

$$A_0 = -\varphi, \qquad \varphi\left(\mathbf{r}\right) = \frac{1}{2}\left(U\left(\mathbf{r}\right) - \frac{1}{\tau}\right). \tag{19}$$

The contravariant metric tensor $g^{\alpha\beta}$ has the following form

$$g^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & d^{11} & d^{12} & d^{13} \\ 0 & d^{21} & d^{22} & d^{23} \\ 0 & d^{31} & d^{32} & d^{33} \end{pmatrix}, \tag{20}$$

where the contravariant tensor d^{ik} is determined as follows

$$d^{ik}\left(\mathbf{r}\right) = \frac{D^{ik}\left(\mathbf{r}\right)}{\max_{i,k,\mathbf{r}} D^{ik}\left(\mathbf{r}\right)} \tag{21}$$

The mass $m\left(\mathbf{r}\right)$ and "speed of light" c are

$$m\left(\mathbf{r}\right) = \frac{1}{2c^2}\left(U\left(\mathbf{r}\right) + \frac{1}{\tau}\right), \qquad c^2 = \frac{\max_{i,k,\mathbf{r}} D^{ik}\left(\mathbf{r}\right)}{2\tau}. \tag{22}$$

The fact that Eq. (14), which governs the dynamics of the reaction front, can be rewritten as the general relativity Hamilton-Jacobi equation is of basic importance for us. It allows us to write down the solution of (14) as in Landau & Lifshitz textbook (1961)

$$G = \min\left\{-mc\int ds + \frac{1}{c}\int A_i dz^i\right\} \tag{23}$$

where ds is the line element of the 4-dimensional Riemannian space of general relativity

$$-\left(ds\right)^2 = \sum_{\alpha,\beta=1}^{4} g_{\alpha\beta} dz^\alpha dz^\beta \tag{24}$$

and $g_{\alpha\beta}$ is the covariant metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & d_{11} & d_{12} & d_{13} \\ 0 & d_{21} & d_{22} & d_{23} \\ 0 & d_{31} & d_{32} & d_{33} \end{pmatrix} \tag{25}$$

Here the covariant tensor d_{ik} is

$$d_{ik}d^{kj} = \delta_i^j \tag{26}$$

where δ_k^j is the familiar Kronecker delta.

The components d_{ik} can be found by using standard procedure; that is, if we set $d = | d_{ik} |$ and let C_{ik} be the cofactor of d_{ik} in d, then $d_{ik} = C_{ik}/g$.

The explicit solution (23) can be rewritten in terms of the Lagrangian function L

$$G(t, \mathbf{r}) = \min \left\{ \int_0^t L\left(\mathbf{r}(s), \frac{d\mathbf{r}}{ds}(s)\right) ds : \quad \mathbf{r}(0) \in \Omega_0, \quad \mathbf{r}(t) = \mathbf{r} \right\}, \tag{27}$$

where

$$L = -mc^2 \sqrt{1 - \frac{1}{c^2} \sum_{i,k=1}^3 d_{ik}(\mathbf{r}) \frac{dx^i}{ds}\frac{dx^i}{ds}} - \varphi. \tag{28}$$

In terms of the phenomenological parameters $U(\mathbf{r})$, $D^{ik}(\mathbf{r})$ and τ, the Lagrangian function L takes the form

$$L = -\frac{1}{2}\left(U(\mathbf{r}) + \frac{1}{\tau}\right)\sqrt{1 - \frac{2\tau}{\max_{i,k,\mathbf{r}} D^{ik}(\mathbf{r})} \sum_{i,k=1}^3 d_{ik}(\mathbf{r}) \frac{dx^i}{ds}\frac{dx^i}{ds}} - \frac{1}{2}\left(U(\mathbf{r}) - \frac{1}{\tau}\right). \tag{29}$$

Thus expression (27) provides an explicit solution to the reaction position problem for the generalized Fisher-KPP equation (5) with the initial condition (6). The exact formula for reaction front position and its propagation rate can be obtained when the reaction rate parameter U and the diffusion tensor D^{ik} are constant.

5. Explicit formula for reaction front position

Let us denote by $H(\mathbf{r}, \mathbf{p})$ the Hamiltonian function associated with the variational problem (27). By using the Legendre transformation

$$H(\mathbf{r}, \mathbf{p}) = \max_{\mathbf{k}} (\mathbf{p} \cdot \mathbf{k} - L(\mathbf{r}, \mathbf{k})) \tag{30}$$

we can find

$$H(\mathbf{r}, \mathbf{p}) = \sqrt{m^2(\mathbf{r})c^4 + c^2 \sum_{i,k=1}^3 d^{ik}(\mathbf{r}) p_i p_k} + \varphi(\mathbf{r}). \tag{31}$$

The optimal trajectories giving the minimum to the functional (27) satisfy the Hamilton equations

$$\frac{dx^i}{ds} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{ds} = -\frac{\partial H}{\partial x^i}. \tag{32}$$

When the parameters $m^2(\mathbf{r}), d^{ik}(\mathbf{r})$ and $\varphi(\mathbf{r})$ are independent from the space coordinate $\mathbf{r}$, we have

$$\frac{dx^i}{ds} = const, \qquad p_i = const. \tag{33}$$

It follows from this that the optimal trajectories are straight lines. Taking into account the boundary conditions in (27), we can find

$$x^i_{opt}(s) = \frac{x^i - y^i}{t}s + y^i, \qquad \mathbf{y} \in \delta S_0. \tag{34}$$

When these trajectories are substituted into (27), we obtain

$$G(t, \mathbf{r}) = \min_{\mathbf{y} \in S_0} \left\{ -mc^2 t \sqrt{1 - \frac{1}{c^2 t^2} \sum_{i,k=1}^{3} d_{ik}(x_i - y_i)(x_k - y_k)} - \varphi t \right\}. \tag{35}$$

Let us denote by $l^2_{\min}(\mathbf{r}, S_0)$ the following expression

$$l^2_{\min}(\mathbf{r}, S_0) = \min_{\mathbf{y} \in S_0} \sum_{i,k=1}^{3} d_{ik}(x_i - y_i)(x_k - y_k). \tag{36}$$

Then

$$G(t, \mathbf{r}) = -mc^2 t \sqrt{1 - \left(\frac{l_{\min}(\mathbf{r}, S_0)}{ct}\right)^2} - \varphi t. \tag{37}$$

It is clear from (37) that the theory is valid as long as

$$1 - \left(\frac{l_{\min}(\mathbf{r}, S_0)}{ct}\right)^2 \geq 0.$$

This condition has a very simple physical interpretation: relativity theory forbids the particle from propagating at a speed which exceeds the velocity of light c. As a result, an inequality $l_{\min}(\mathbf{r}, S_0) \leq ct$ must hold. It follows from (26) that the reaction front position $\delta S_t = \{\mathbf{r} \in R^3 : G(t, \mathbf{r}) = 0\}$ at time t can be represented as

$$\delta S_t = \{\mathbf{r} \in R^3 : l_{\min}(\mathbf{r}, S_0) = ut\}, \tag{38}$$

where

$$u = c\sqrt{1 - \left(\frac{1 - \tau U}{1 + \tau U}\right)^2} = \frac{\sqrt{2 \max_{i,k} D^{ik} U}}{1 + \tau U}, \qquad \tau U \leq 1. \tag{39}$$

We can also give the asymptotic behavior of scalar field $\rho^\varepsilon(t, \mathbf{r})$ in terms of $l_{\min}(\mathbf{r}, S_0)$

$$\lim_{\varepsilon \to 0} \rho^\varepsilon(t, \mathbf{r}) = \begin{cases} 1, & \text{if } l_{\min}(\mathbf{r}, S_0) > ut \\ 0, & \text{otherwise}, \end{cases} \tag{40}$$

6. Summary and future work

We have extended the classical treatment of the Fisher-KPP equation due to Freidlin (1996) to include the phenomenon of diffusion with a finite velocity. The main result is that in the long-time large-distance asymptotic limit the Hamiltonian dynamical system associated with the anisotropic reaction diffusion system has a structure identical to that of general relativity theory. We have shown that the function determining the position of reaction front and its speed is nothing else but the action functional for a particle in both gravitational and electromagnetic fields. The metric tensor of the 4-dimensional Riemannian space of general relativity has been determined through the diffusivity tensor, while the speed of light corresponds to the finite speed of diffusion waves. The mass of the relativistic particle and scalar potential have been found to be functions of reaction rate coefficient and relaxation time. For the constant values of reaction rate function and diffusivity tensor, the analogy with the general relativity theory has allowed us to find the explicit formula for the reaction front position and its speed.

An important application of the result of this paper may be the propagation of a reaction front in a turbulent combustion flow (Bray, 1990). It is well known that the macroscale equations for turbulent heat/mass transport involve effective anisotropic transport processes with a finite velocity (Monin & Yaglom, 1987). It is also of great interest to analyze the reaction front dynamics in a slowly varying medium when the phenomenon of the reaction front jump might happen.

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Appendix 1

The system (1),(4) can be rewritten as follows

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial J^i}{\partial x^i} = U(\mathbf{r}) \rho f(\rho), \qquad \mathbf{r} \in R^3, \tag{A1}$$

$$\frac{\partial J^i}{\partial t} = -\frac{J^i}{\tau} - \frac{1}{\tau} \sum_{k=1}^{3} D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k}. \tag{A2}$$

It is easy to see that an expression for J^i in (4) is just a solution of the differential equation (A2) under the initial condition $J^i(0, \mathbf{r}) = 0$.

By differentiating the first equation with respect to time t and the second one with respect to the space coordinate x^i we obtain

$$\frac{\partial^2 \rho}{\partial t^2} + \sum_{i=1}^{3} \frac{\partial^2 J^i}{\partial t \partial x^i} = U(\mathbf{r}) \left(f(\rho) + \rho \frac{df(\rho)}{d\rho} \right) \frac{\partial \rho}{\partial t}, \qquad \mathbf{r} \in R^3, \tag{A3}$$

$$\frac{\partial^2 J^i}{\partial x^i \partial t} = -\frac{1}{\tau} \frac{\partial J^i}{\partial x^i} - \frac{1}{\tau} \sum_{k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k}. \tag{A4}$$

By using (A1) the last equation can be rewritten as

$$\sum_{i=1}^{3} \frac{\partial^2 J^i}{\partial x^i \partial t} = -\frac{1}{\tau} \left(-\frac{\partial \rho}{\partial t} + U(\mathbf{x}) \rho f(\rho) \right) - \frac{1}{\tau} \sum_{i,k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k}. \tag{A5}$$

Substitution of the expression for $\sum_{i=1}^{3} \frac{\partial^2 J^i}{\partial x^i \partial t}$ given by (A5) into (A3) and multiplication by τ give

$$\tau \frac{\partial^2 \rho}{\partial t^2} + \left(1 - \tau U(\mathbf{r}) f(\rho) - \tau U(\mathbf{r}) \rho \frac{df(\rho)}{d\rho} \right) \frac{\partial \rho}{\partial t} =$$

$$\sum_{i,k=1}^{3} \frac{\partial}{\partial x^i} D^{ik}(\mathbf{r}) \frac{\partial \rho}{\partial x^k} + U(\mathbf{r}) \rho f(\rho)$$

After hyperbolic scaling $t \to t/\varepsilon$, $\mathbf{r} \to \mathbf{r}/\varepsilon$, this equation can be rewritten as (8).