

Thermal convection in a twisted magnetic field

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The onset of convection in a layer of an electrically conducting fluid heated from below is considered in the case when the layer is permeated by horizontal magnetic field of strength B_0 the orientation of which varies sinusoidally with height. An approximate critical value of the Rayleigh number for the onset of convection is derived. The width δ of the convection rolls, $\delta \sim \sqrt{(h_t/B_0)\sqrt{\rho\mu\lambda\nu}}$ is found to be independent of the height of the layer once the right hand side in this relationship is sufficiently small. Applications to the penumbral filaments of sunspots are briefly discussed.

1. Introduction

Convection driven by thermal buoyancy in the presence of an imposed magnetic field has been studied theoretically and experimentally for several decades. For an early review we refer to the book of Chandrasekhar (1961). Usually the case of an imposed homogeneous vertical magnetic field is treated which exerts a strongly inhibiting influence on convection. A most famous example are sunspots which appear dark because the heat transport by convection in the solar atmosphere is almost completely suppressed at the spot by the emerging radial magnetic field.

Homogeneous horizontal magnetic fields exert a far lesser influence on convection. Two-dimensional convection rolls aligned with the magnetic field do not feel any effect at all and the critical value of the Rayleigh number R for the onset of convection is the same as in the non-magnetic case. Since two-dimensional convection rolls aligned with the magnetic field are subject to three-dimensional instabilities, however, the effect of the horizontal magnetic field is felt by the three-dimensional forms of convection realized at higher values of R (Busse & Clever 1989).

Another situation in which a horizontal magnetic field affects convection in an electrically conducting fluid is that of a twisted magnetic field which changes its direction as function of the vertical coordinate. Such a field is accompanied by a current density. In the following we shall consider the case when the electric current is directed parallel to the magnetic field such that a static “force-free” configuration exists. The onset of convection in such a configuration is the topic of this paper.

It has long been known that magnetic fields of sunspots exhibit a torsion which is equivalent to a twist in the local approximation. In the penumbra of large sunspots where the magnetic field becomes nearly horizontal convection appears to assume the form of thin roll like structures called filaments. The small wavelength of these rolls is usually attributed to the influence of the vertical component of the magnetic field. But the twist of the magnetic field may exert an even stronger effect. This possibility will be explored in the following sections in terms of a simple model which offers the advantage that the boundary conditions in the vertical direction do not seem to be significant.

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2. Mathematical Formulation of the Problem

We consider a layer of height d of an electrically conducting fluid heated from below and cooled from above. The fluid is permeated by a horizontally homogeneous magnetic field of the form

$$\vec{B}_0 = B_0(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \quad (2.1)$$

where \vec{i} and \vec{j} denote the unit vectors in the horizontal x - and y -directions of a cartesian system of coordinates. The direction of the z -coordinate is opposite to gravity. Since the field (2.1) is force-free, i.e. $(\nabla \times \vec{B}_0) \times \vec{B}_0 = 0$, a motionless static solution of the problem exists.

Using d as length scale, d^2/ν as time scale where ν denotes the kinematic viscosity of the fluid, and $(T_2 - T_1)P/R$ as scale of the deviation Θ of the temperature from its static distribution we obtain the equations of motion for the dimensionless velocity vector \vec{u} and the heat equation for Θ in the following form

$$(\partial_t + \vec{u} \cdot \nabla)\vec{u} = -\nabla\pi + \vec{k}\Theta + \nabla^2\vec{u} + \vec{B} \cdot \nabla\vec{B} \quad , \quad (2.2)$$

$$0 = \nabla \cdot \vec{u} \quad , \quad (2.3)$$

$$P(\partial_t + \vec{u} \cdot \nabla)\Theta = R\vec{k} \cdot \vec{u} + \nabla^2\Theta \quad (2.4)$$

where \vec{k} is the unit vector in the z -direction and the Prandtl number and the Rayleigh number are defined by

$$P = \frac{\nu}{\kappa}, \quad R = \frac{\alpha(T_2 - T_1)gd^3}{\nu\kappa} \quad (2.5)$$

Here κ and α denote the thermal diffusivity and the coefficient of thermal expansion of the fluid and g is the acceleration of gravity. T_1 and T_2 are the temperatures at the lower and upper boundaries which are positioned at $z = \pm 0.5$. In order to treat the problem in its simplest physically realistic form the Boussinesq approximation has been assumed in which the density ρ is regarded as a constant except in connection with the gravity term where its dependence on the temperature has been taken into account. All terms which can be written as gradients in equation (2.2) have been combined into $\nabla\pi$.

The general representation for solenoidal vector fields in terms of poloidal and toroidal components can be used to write the dimensionless magnetic field in the form

$$\vec{B} = \frac{B_0 d}{\nu\sqrt{\rho\mu}}(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z + \frac{\nu}{\lambda}(\nabla \times (\nabla h \times \vec{k}) + \nabla g \times \vec{k})) \quad . \quad (2.6)$$

where λ is the magnetic diffusivity which is defined as the inverse of the product of the electrical conductivity σ and the magnetic permeability μ . By taking the vertical components of the equation of induction,

$$\frac{\lambda}{\nu}\nabla^2\vec{B} = \partial_t\vec{B} - \nabla \times (\vec{u} \times \vec{B}) \quad (2.7)$$

and of its curl we obtain the equations for h and g

$$\nabla^2\Delta_2 h = (\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla u_z \quad (2.8)$$

$$\nabla^2\Delta_2 g = (\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla(\partial_y u_x - \partial_x u_y - \gamma u_z) \quad (2.9)$$

where all terms multiplied by ν/λ have been neglected since we shall consider only the limit $\nu \ll \lambda$ which is appropriate for liquid metals, but also for solar plasmas. Δ_2 denotes

the horizontal Laplacian, $\Delta_2 = \partial_{xx}^2 + \partial_{yy}^2$. The term $\vec{B} \cdot \nabla \vec{B}$ in equation ((2.2) can now be evaluated,

$$\vec{B} \cdot \nabla \vec{B} = Q[(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla(\nabla \times (\nabla h \times \vec{k}) + \nabla g \times \vec{k}) + \Delta_2 h \gamma(\vec{i} \sin \gamma z + \vec{j} \cos \gamma z)] . \quad (2.10)$$

where Q is the Chandrasekhar number,

$$Q = \frac{B_0^2 d^2}{\nu \rho \mu \lambda} \quad (2.11)$$

In the following we shall restrict the analysis to the linear problem of the onset of steady convection in which case the left hand sides of equations (2.2)-(2.4) can be neglected. Oscillatory onset of convection is possible in the presence of a magnetic field, but not in the limit $\nu \ll \lambda$ when the time derivative in equation (2.7) can be dropped.

By taking the z -component of the double curl of equation (2.2), i. e. by operating with $\vec{k} \cdot \nabla \times (\nabla \times \dots)$ onto it, we find

$$\nabla^4 u_z + \Delta_2 \Theta - Q[(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla(\nabla^2 \Delta_2 h + \gamma^2 \Delta_2 h)] = 0 . \quad (2.12)$$

With the help of equations (2.4) and (2.8) Θ and h can be eliminated from this equation,

$$\begin{aligned} & \nabla^6 u_z - R \Delta_2 u_z - Q[(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla(\nabla^2 - \gamma^2)u_z - \\ & \quad 2\gamma^2[(\vec{i} \sin \gamma z + \vec{j} \cos \gamma z) \cdot \nabla]^2 u_z + \\ & \quad 4\gamma(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla(\vec{i} \sin \gamma z + \vec{j} \cos \gamma z) \cdot \nabla \partial_z u_z] \\ & = -Q[\gamma^4(\vec{i} \cos \gamma z - \vec{j} \sin \gamma z) \cdot \nabla \Delta_2 h + 2\gamma^3(\vec{i} \sin \gamma z + \vec{j} \cos \gamma z) \cdot \nabla \Delta_2 \partial_z h] . \end{aligned} \quad (2.13)$$

On the right hand side terms involving h are still left. We shall neglect these terms by making the assumption that the parameter γ is sufficiently small such that terms multiplied by γ^n with $n \geq 3$ are negligible in comparison with those with a lower power of n .

Equation (2.13) admits solutions of the form

$$u_z = f(z) \exp\{ia(\vec{i} \sin \chi - \vec{j} \cos \chi) \cdot \vec{r}\} \quad (2.14)$$

where χ denotes the angle by which the axis of the convection roll described by (2.14) is turned away from the positive x -axis towards the negative y -axis. The Rayleigh number R for onset of convection will be minimized when convection sets in at a height z_0 such that the angle χ satisfies $\chi = \gamma z_0$. In this case the dominant term multiplied by Q in equation (2.13) vanishes for $z = z_0$. It does not vanish for $z \neq z_0$ and it is appropriate to use a Taylor expansion, $\cos \gamma z = \cos \gamma z_0 - (z - z_0)\gamma \sin \gamma z_0 + \dots$, and likewise for $\sin \gamma z$. Without losing generality we may assume $\chi = z_0 = 0$ in which case the ordinary differential equation for $f(z)$,

$$\left[\left(\frac{d^2}{dz^2} - a^2 \right)^3 + Ra^2 + Q\gamma^2 a^2 \left(z^2 \left(\frac{d^2}{dz^2} - a^2 \right) + 2 + 4z \frac{d}{dz} \right) \right] f(z) = 0 \quad (2.15)$$

is obtained where terms up to the order γ^2 have been taken into account. In the next section an approximate solution of this equation is derived.

3. Derivation of an Approximate Solution

For the solution of equation (2.15) the ansatz

$$f(z) = \sum_{n=0}^{\infty} z^{2n} A_n \exp\{-c^2 z^2\} \quad (3.1)$$

will be made. It will be anticipated that the parameter c can be chosen sufficiently large such that the boundary conditions at $z = \pm 0.5$ will not affect the solution. Since equation (2.15) is linear homogeneous $A_0 = 1$ can be assumed. The solution procedure can be understood most readily when the crudest approximation, $A_n = 0$ for $n \geq 1$, is inspected. Equation (2.15) yields in this case

$$-120\xi^3 - 36\xi^2 - 6\xi - 1 + (R + 2Q\gamma^2)/a^4 + a^2 z^2 \{720\xi^4 + 144\xi^3 + 12\xi^2 - (1 + 10\xi)\gamma^2 Q/a^2\} + o(z^4) = 0 \quad (3.2)$$

where the definition $\xi = c^2/a^2$ has been used and where terms of the order z^4 have not been denoted explicitly since we shall neglect them in first approximation. The wavy bracket yields an equation for the determination of c^2 as a function of $\gamma^2 Q$, while the z -independent terms determine R in dependence on a and $\gamma^2 Q$. The onset of convection will occur when R reaches a minimum as a function of a^2 . This minimum is determined by the relationship

$$60\xi^3 - 3\xi - 1 = 0 \quad (3.3)$$

which yields the unique positive root $\xi_c = 0.31961$. The wavy bracket of equation (3.2) then yields

$$c^2 = \sqrt{\gamma^2 Q}/5.600 \quad (3.4)$$

The corresponding critical value R_c of the Rayleigh number is given by $R_c = 71.560c^4$. This relationship shows that the onset of convection does not depend on the external length scale d . For a fixed value of the temperature gradient, $(T_2 - T_1)/d$, a new Raleigh number R^* can be defined with the natural length scale c^{-1} of convection, $R^* = R/c^4$. Its critical value R_c^* is thus given by

$$R_c^* = R_c/c^4 = 31.356R_c/\gamma^2 Q = 71.560 \quad (3.5)$$

corresponding to the critical wavenumber $a_c^* = a_c/c = \xi_c^{-1/2} = 1.768$. The result (3.5) also demonstrates that the assumptions of small γ and large c can be readily satisfied when a sufficiently large value of the Chandrasekhar number is used.

In order to obtain a more accurate solution of equation (2.15) higher order terms in the representation (3.1) must be taken into account. Some preliminary results are shown in table 1.

Table 1						
Truncation	ξ_c	R_c^*	α_c^*	$A_1/\sqrt{\gamma^2 Q}$	$R_c/\gamma^2 Q$	$c^2/\sqrt{\gamma^2 Q}$
$n \geq 1$	0.31961	71.560	1.768	0	2.282	0.4226
$n \geq 2$	0.320	95.68	1.769	-0.5596	1.5617	0.3575

4. Conclusion

Convection rolls in the presence of a twisted magnetic field in sunspots may serve as a model for penumbral filaments. The theory does not strongly depend on the property that the magnetic is horizontal. An extension of the analysis to the case of a moderately inclined magnetic field should yield only minor changes in the results. Since torsions of sunspot fields are known to vary it is tempting to predict a dependence of the wavelength of filaments on the strength of the magnetic field and on its twist. Using the relationship $a_c^* = \xi^{-1/2}$ we obtain for the dimensional width δ (=half wavelength) of filaments

$$\delta = (h_t \pi \sqrt{\rho \mu \lambda \nu} / B_0)^{1/2} 0.632 \quad (4.1)$$

where h_t denotes the height in meters over which the orientation of the magnetic field changes by 180 deg. It will be of interest to look for correlations between the width of filaments and the torsion of the magnetic field in sunspots.

REFERENCES

- BUSSE, F.H. & CLEVER, R.M. 1989 Travelling Wave Convection in the Presence of Horizontal Magnetic Field. *Phys. Rev. A* **40**, 1954-1961.
- CHANDRASEKHAR, S. 1961 Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford.