# Ensemble-averaged LES of a time-evolving plane wake

By D. Carati[1] and M. M. Rogers

The ensemble-averaged dynamic procedure (EADP) introduced during the 1996 CTR Summer Program is tested on a time-evolving plane wake, an inhomogeneous flow that is statistically non-stationary. Convergence of the results with respect to the LES ensemble size is investigated, and it is found that an ensemble of as few as 16 realizations yields accurate converged results. New modeling concepts are tested in which quantities that explicitly require the knowledge of several realizations of the same flow are included.

## 1. Introduction

The idea of using a set of LES's for developing new concepts in subgrid-scale modeling was introduced during the 1996 CTR Summer Program (Carati, Wray & Cabot 1996). This method consists of generating several statistically equivalent LES's simultaneously, evaluating the subgrid-scale model constant by using information derived from the set of resolved velocity fields. Each of these fields evolves according to

$$\partial_t \overline{u}_i^r + \partial_j \overline{u}_j^r \overline{u}_i^r = -\partial_i \overline{p}^r + \nu_0 \nabla^2 \overline{u}_i^r - \partial_j \tau_{ij}^r \qquad r = 1, \ldots, R\,, \tag{1.1}$$

where $r$ is an index corresponding to the realization being considered and $R$ is the total number of realizations. Utilizing these $R$ realizations, an ensemble-averaged dynamic procedure (EADP) can be developed as an alternative to the volume-averaged (or plane-averaged) dynamic procedure. There are several advantages of the EADP. Foremost of these is that the method does not rely on any homogeneous flow directions for the computation of model terms. Hence, there is no theoretical limitation preventing its use in a fully inhomogeneous and non-stationary flow. Also, the EADP is well suited for parallel computing since the $R$ simulations only interact through the computation of the subgrid-scale model (see Fig. 1). The other terms in the equation can then be computed independently from the other realizations. It should also be noted that for statistically stationary flows the EADP is not more expensive than traditional LES because the ensemble greatly reduces the averaging time period required for converged statistics. In fact, if the different realizations are really independent, the CPU time required for collecting the statistics could even be reduced by using an ensemble of LES's. Finally, the EADP is theoretically

1 Université Libre de Bruxelles, (Brussels, Belgium).

appealing because it could provide useful information on the statistics of resolved velocity fields that might be used for building a bridge between LES and RANS.

Prior to this work, the EADP had only been tested in detail on homogeneous turbulence (both forced and decaying) with the Smagorinsky (1963) model. The feasibility of the method has been demonstrated, but its robustness for more complex flows and its potential advantages were not explored thoroughly. For homogeneous turbulence it was shown that good results were obtained with only $R = 16$ realizations. It is, of course, crucial to show that the number of realizations required for implementing this method does not increase dramatically for flows of greater complexity. Also, the knowledge of several realizations could be used not only to compute the Smagorinsky constant through the EADP, but also to explore new subgrid-scale models. The motivations for this work are thus to check the robustness of the method in an inhomogeneous flow and to demonstrate that new ensemble-based modeling concepts can be proposed and tested easily with the EADP.

The flow considered here is a time-evolving plane wake for which data from both direct numerical simulations (Moser & Rogers 1994, Moser, Rogers & Ewing 1997) and large-eddy simulations (Ghosal & Rogers 1997) are available. This flow is both statistically non-stationary and inhomogeneous and should thus be a more demanding test of the EADP than the homogeneous flows studied previously.

## 2. Subgrid-scale modeling

Subgrid-scale modeling for an ensemble of LES's is not more complicated than that for a single LES. As usual in LES, the model terms are assumed to depend both on instantaneous local quantities (such as the resolved strain-rate tensor) and universal parameters. In the context of an ensemble of LES's, it is natural to suppose that the instantaneous and local dependence of the model will also be realization dependent, while the universal parameter should be independent of the realization. Thus the model for the subgrid-scale tensor in each realization, $\tau_{ij}^r = \overline{u_i^r u_j^r} - \overline{u}_i^r \, \overline{u}_j^r$, can be expressed as

$$\tau_{ij}^r - \frac{1}{3}\tau_{kk}^r \delta_{ij} \approx C \mathcal{T}_{ij}^r[\overline{u}_l^r; \overline{G}] \,, \tag{2.1}$$

where $\mathcal{T}_{ij}^r$ is a tensorial functional of both the resolved field corresponding to the same realization index $r$ and the LES filter $\overline{G}$. In contrast, the parameter $C$ should be the same for all the realizations in the ensemble. The EADP prediction (Carati, Wray & Cabot 1996) $C_d$ for the parameter $C$ is given by

$$C_d = \frac{\sum_r L_{ij}^r \mathcal{M}_{ij}^r}{\sum_r \mathcal{M}_{ij}^r \mathcal{M}_{ij}^r} \,, \tag{2.2}$$

where $\mathcal{M}_{ij}^r = \widehat{\mathcal{T}_{ij}^r[\overline{u}_l^r; \overline{G}]} - \mathcal{T}_{ij}^r[\widehat{\overline{u}}_l^r; \widehat{\overline{G}}]$ and $L_{ij}^r = \widehat{\overline{u}_i^r \overline{u}_j^r} - \widehat{\overline{u}}_i^r \widehat{\overline{u}}_j^r$. Here, $\widehat{\overline{G}}$ is the filter obtained by successively applying the LES filter and a test filter $\widehat{G}$. In the present study, we have investigated three different models, all based on the eddy-viscosity concept. The first one is the classical Smagorinsky model.

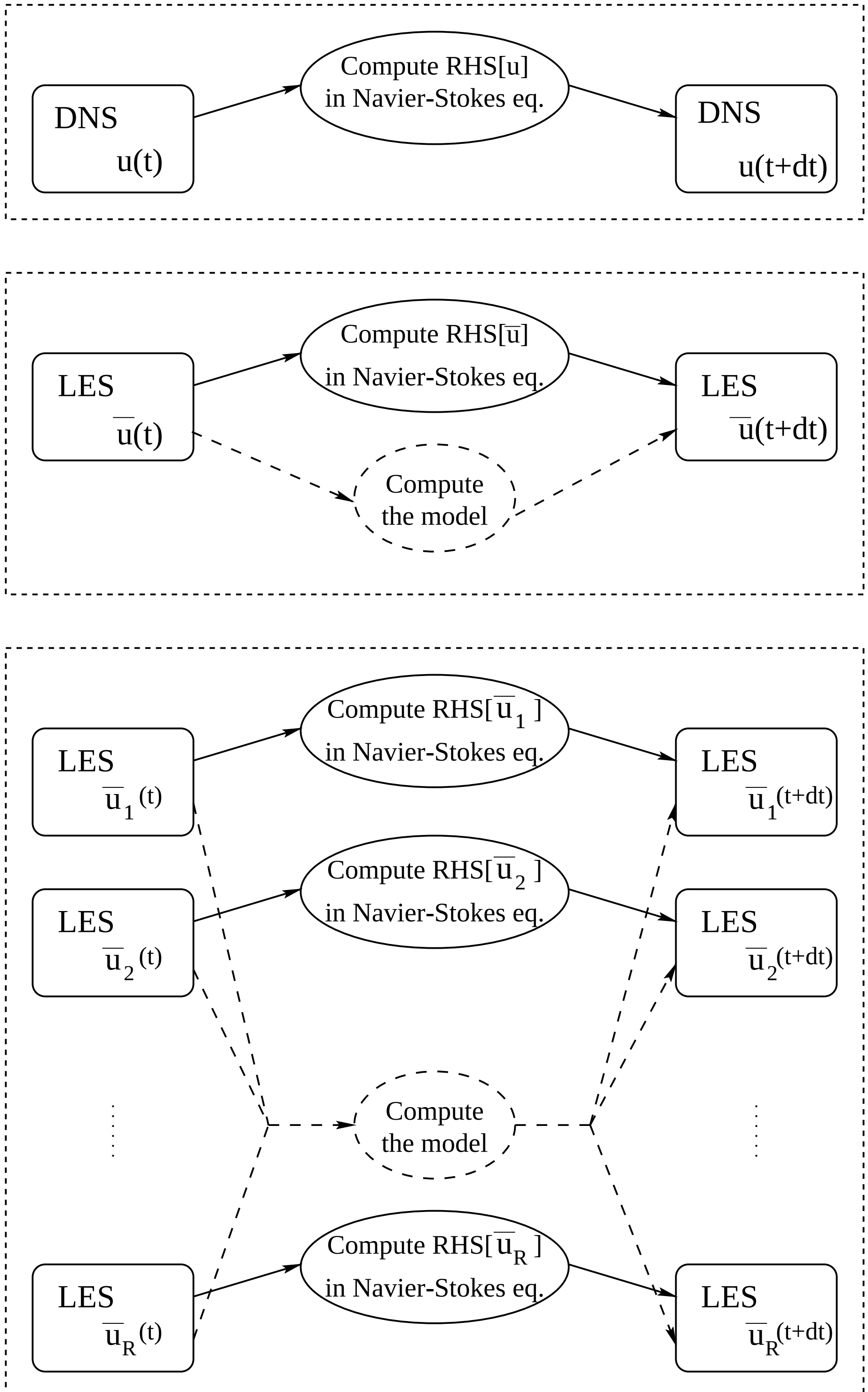


FIGURE 1. When a set of LES's are generated simultaneously, the different LES's are advanced in time through two types of terms. The first type, like in DNS (top) and traditional LES (middle), is given by the right-hand-side of the Navier-Stokes equation. For these terms no information is needed from the other fields. Information from the other fields is only required for the subgrid-scale model terms (bottom).

$$\textbf{Smagorinsky model:} \quad \mathcal{T}^r_{ij}[\overline{u}^r_l;\overline{G}] = -2\overline{\Delta}^2 \left(2\overline{S}^r_{kl}\overline{S}^r_{kl}\right)^{1/2} \overline{S}^r_{ij} \tag{2.3}$$

In the Smagorinsky model (2.3), the inertial range scaling for the eddy-viscosity $\nu_t \sim \overline{\Delta}^{4/3}\overline{\epsilon}^{1/3}$ has been expressed in terms of the resolved strain-rate tensor by using the approximation for the dissipation rate $\overline{\epsilon} \sim \nu_t \overline{S}^r_{kl}\overline{S}^r_{kl}$. This approximation is required in traditional LES because a separate equation for the dissipation rate is not usually computed. However, in LES based on the dynamic procedure, the product of $C$ and $\overline{\epsilon}^{1/3}$ can be predicted through the expression (2.2). This has motivated the use of models directly based on the inertial range scaling such as

$$\textbf{Model A:} \quad \mathcal{T}^r_{ij}[\overline{u}^r_l;\overline{G}] = -2\overline{\Delta}^{4/3} \overline{S}^r_{ij} \; . \tag{2.4}$$

The model parameter predicted by the dynamic procedure with model A (2.4) is not dimensionless, but this does not cause any difficulties. Finally, we have considered a third model for which the tensorial functional $\mathcal{T}^r_{ij}[\overline{u}^r_l;\overline{G}]$ not only depends on the particular realization $\overline{u}^r_l$, but also on the ensemble-averaged velocity field.

$$\textbf{Model B:} \quad \mathcal{T}^r_{ij}[\overline{u}^r_l;\overline{G}] = -2\overline{\Delta}^{4/3} \left(\overline{S}^r_{ij} - \langle \overline{S}_{ij}\rangle\right) , \tag{2.5}$$

where the brackets indicate ensemble-averaging over all realizations. The advantage of this last model is that it can represent the effects of backscatter in some realizations while maintaining an overall average dissipative effect (provided that the parameter $C$ is positive). In each realization the subgrid-scale dissipation is proportional to $\overline{S}^r_{kl}(\overline{S}^r_{kl} - \langle \overline{S}_{kl}\rangle)$, which can be either positive or negative. However, the mean of this quantity is $(\overline{S}^r_{kl} - \langle \overline{S}_{kl}\rangle)^2$, which is always positive.

Of course, the sign of $C$ will also determine the sign of the subgrid-scale dissipation since a negative $C$ corresponds to a negative eddy-viscosity. In order to avoid numerical instabilities, $C$ must then be set equal to a minimal positive value (clipping procedure, see Ghosal *et al.*, 1995) at points where the total viscosity (eddy plus molecular) is negative. For the Smagorinsky model, the stability condition

$$C\overline{\Delta}^2 \left(2\overline{S}^r_{kl}\overline{S}^r_{kl}\right)^{1/2} + \nu_0 > 0 \tag{2.6}$$

depends on the realization. This is an undesirable property since $C$ is supposed to be a universal flow characteristic for all members of the ensemble. An alternative formulation in which $C$ is indeed the same for all realizations results from the following stability condition

$$C\overline{\Delta}^2 \max_r \left\{ \left(2\overline{S}^r_{kl}\overline{S}^r_{kl}\right)^{1/2} \right\} + \nu_0 > 0 \; . \tag{2.7}$$

In the limit of an infinite number of realizations, the maximum of the resolved strain-rate tensor amplitude would be almost unbounded. Hence, for the Smagorinsky model, it is reasonable to simply impose $C > 0$. For model A, however, the situation is different. The stability condition is the same in each realization

$$C\overline{\Delta}^{4/3} + \nu_0 > 0 \; , \tag{2.8}$$

resulting in the model parameter $C$ being given by $C = \max\{C_d, -\nu_0 \overline{\Delta}^{-4/3}\}$. For simplicity, the same condition has been used for model B.

## 3. Application of the EADP to a time-evolving plane wake

In a previous study, the EADP has been successfully implemented for homogeneous turbulence in both forced (stationary) and decaying (non-stationary) situations. Here, we propose to investigate a flow with the added complexity of an inhomogeneous direction. The pseudospectral direct numerical simulation of the plane wake considered here has been described in detail by Moser & Rogers (1994) and Moser, Rogers & Ewing (1997). The spatial dependence of the independent variables is represented in the periodic streamwise and spanwise directions by Fourier basis functions and the cross-stream dependence is represented by a class of Jacobi polynomials on a mapped infinite domain. Up to $512 \times 195 \times 128$ modes are required to accurately resolve the turbulence. The Reynolds number based on the integrated mass flux deficit,

$$\mu = -\int_{-\infty}^{+\infty} (U(y) - U_\infty)\, dy, \tag{3.1}$$

is $R_e = \mu/\nu = 2000$. In a time-evolving plane wake, the integrated mass flux deficit is constant.

LES's of the same flow using the dynamic procedure and a filtered DNS field as an initial condition have been reported by Ghosal & Rogers (1997). The simulations were pseudospectral like the DNS, but the spatial dependence of the vorticity in the inhomogeneous cross-stream direction is represented in terms of Fourier modes on a finite domain. The appropriate non-periodic velocity field is then calculated using the method of Corral & Jimenez (1995). The number of modes used in the LES's was $64 \times 48 \times 16$ and the same number of modes and same numerical method have been adopted for the LES's examined here. Thus each LES mode represents up to 260 DNS modes.

### *3.1 The initial conditions*

In order to justify ensemble-averaging, the $R$ velocity fields should be *statistically equivalent* and *statistically independent*. Carati *et al.* (1996) have proposed that acceptable initial conditions for LES $v(\mathbf{x}, 0) = v_0(\mathbf{x}; w_l)$ should be generated using random numbers $w_l$ and should satisfy some constraints: $P_s[v_0] = p_s, \quad s = 1, \ldots S$. For example, the constraints could be obtained by matching the mean velocity profile, the energy spectrum, etc. Proposed definitions for "statistically equivalent" and "statistically independent" were also given. Two LES's are statistically equivalent if the domain of the flow and the boundary conditions are exactly the same and if the initial conditions satisfy the same set of constraints. Two LES's are statistically independent if the initial conditions are generated with uncorrelated random numbers $w_l$. For the time-evolving plane wake, a large number of quantities are measured, and any number of them might be considered as constraints that need to be maintained by all realizations (e.g. profiles of mean velocity, turbulent kinetic energy, enstrophy, etc.). The question then becomes: is it possible to create $R$ independent

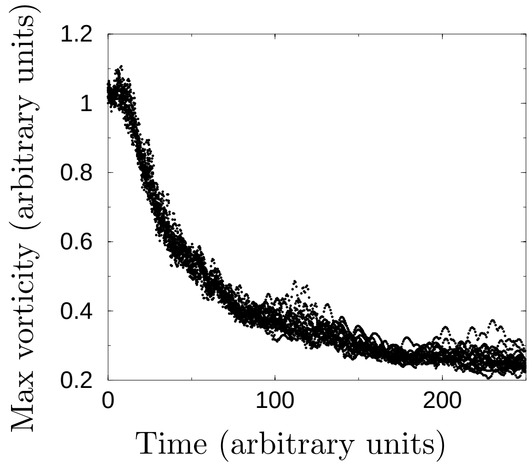


FIGURE 2. Maximum grid-point value of the $x$-component of the resolved vorticity, ${\omega_x}^r_{\max}$, as a function of time for each of the 16 realizations.

initial LES fields that maintain all the relevant quantities in the plane wake from one single filtered DNS field? Since the observed quantities of interest are obtained through $(x-z)$-plane averaging, they are invariant under the transformation

$$\overline{u}_i(x,y,z,t_0) \longrightarrow \overline{u}_i(x+\delta_x, y, z+\delta_z, t_0) . \tag{3.2}$$

Thus by using $R$ values of $(\delta^r_x, \delta^r_z)$, $R$ different (but statistically identical) initial velocity fields can be produced from the filtered DNS field by shifting in the $(x-z)$ plane. These initial fields clearly satisfy the requirement that the LES realizations be statistically equivalent because the initial values of any plane-averaged quantity are identical. However, this procedure does not produce statistically independent initial conditions, even with random choices for $(\delta^r_x, \delta^r_z)$, because the two fields are identical and simply shifted in space. Without the subgrid model terms, this correlation would maintain itself in time. However, the model terms will have the desirable effect of de-correlating the different members of the ensemble. This results because the universal model terms act at the same $(x-z)$ location in all the realizations, not at the same relative position in the shifted flows. An example of this de-correlation is given in Fig. 2, where the maximum grid-point value of the streamwise vorticity component ${\omega_x}^r_{\max}$ for each of the 16 realizations is plotted as a function of time (model A has been used to generate this plot). Because this maximum is computed on a grid that has been shifted by a random (nonintegral multiple of the grid-spacing) shift, ${\omega_x}^r_{\max}$ is not the same for all the realizations, even at $t=0$. The fairly rapid spreading of the values associated with the different realizations suggests that the different LES fields de-correlate fairly quickly.

If a very large number of realizations is used, the coefficient $C$ obtained through the EADP is independent of $x$ and $z$. It reduces to the value of $C$ obtained by the plane-averaged dynamic procedure. In this limit, the various realizations will not diverge and the EADP, with the peculiar construction of the initial conditions presented above, will degenerate into a collection of LES based on the plane-averaged dynamic procedure. A more sophisticated procedure for building the initial conditions would then be needed if a large number of realizations were required for

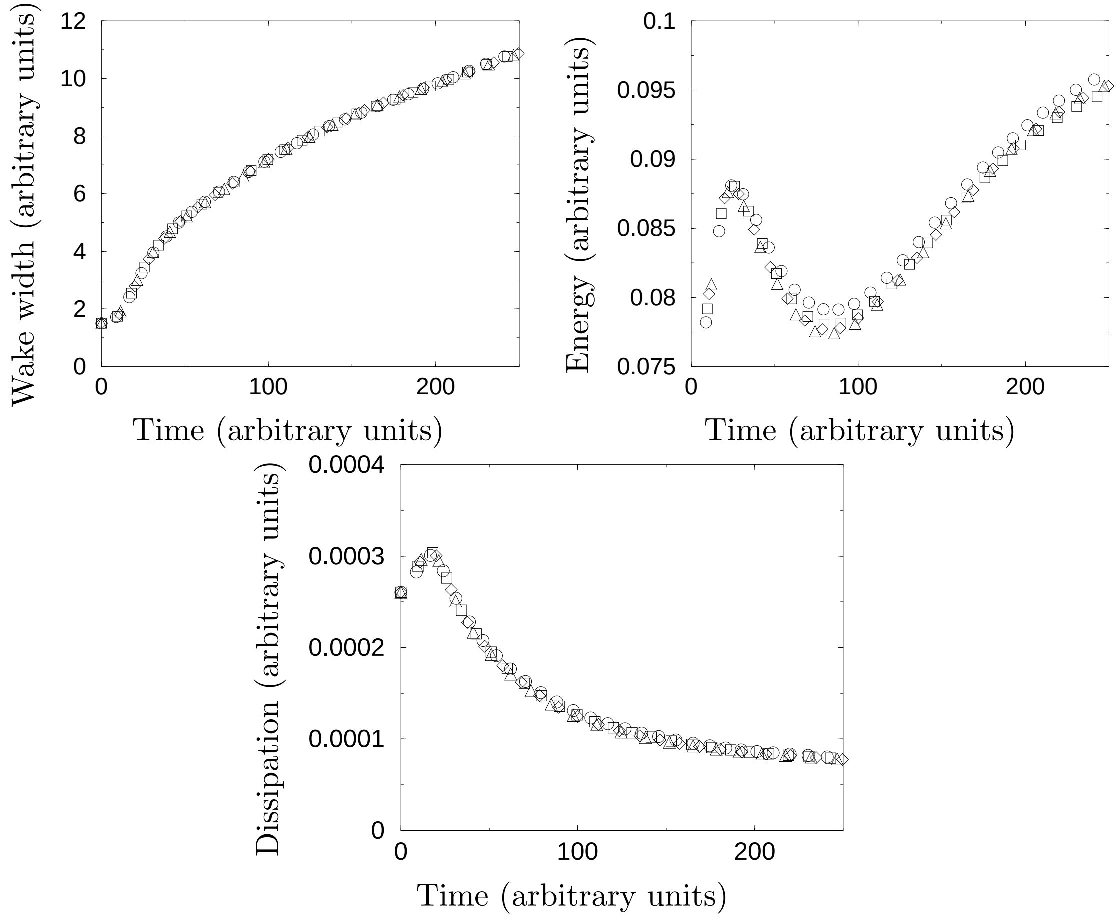


FIGURE 3. Convergence of the ensemble-averaged evolutions of the wake width (top-left), resolved turbulent kinetic energy density integrated in $y$ (top-right) and resolved turbulent kinetic energy dissipation integrated in $y$ (bottom). Various ensemble sizes are compared: $R = 4$, ○ ; $R = 8$ □ ; $R = 16$, ◇ ; and $R = 32$, △ .

statistical convergence. However, it will be seen that $R = 16$ realizations are adequate for satisfactorily converged statistics, and this issue is irrelevant in the present study.

### *3.2 Tests of convergence*

In order to test the convergence of the EADP results for increasing values of $R$, two types of tests were performed. First, the ensemble-averaged values of several relevant quantities in the time-evolving wake flow have been compared for various ensemble sizes. In particular, the results for *i*) the wake width, *ii*) the turbulent kinetic energy density integrated in $y$, and *iii*) the turbulent kinetic energy dissipation integrated in $y$ are compared for $R = 4$, 8, 16, and 32. As can be seen from Fig. 3, the values obtained with 16 and 32 realizations are almost indistinguishable for all three quantities.

Second, the influence of the ensemble size on the computed eddy-viscosity has been examined. The profile of the mean eddy-viscosity and the fraction of grid points for which the eddy-viscosity has been clipped according to the criterion (2.8)

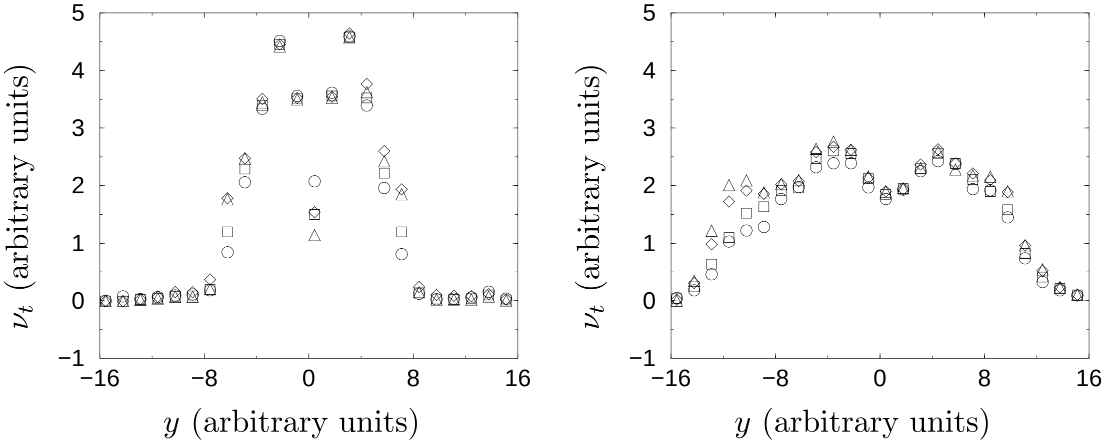


FIGURE 4. Convergence of the eddy-viscosity profile for $t = 0$ (left) and $t = 250$ (right). Various ensemble sizes are compared: $R = 4$, ○ ; $R = 8$, □ ; $R = 16$, ◇ ; and $R = 32$, △ . The eddy-viscosity is normalized by the molecular viscosity.

are compared for the same values of $R$ in Figs. 4 and 5. As seen in Fig. 4, the eddy-viscosity profile depends only weakly on the number of realizations for values of $R$ between 4 and 32, and the profiles are nearly identical for $R = 16$ and $R = 32$. As expected, the fraction of grid points requiring clipping of the model coefficient $C$ rapidly decreases with $R$ (Fig. 5). The total fraction of clipped points integrated in $y$ is less than 1% for $R = 16$ during the entire simulation. This, combined with the very small change in most of the ensemble-averaged quantities as $R$ is increased from 16 to 32, supports the adoption of $R = 16$ as a reasonable ensemble size for both model testing and production LES. Because this value of $R$ is the same as that required for the simulation of homogeneous turbulence, it seems reasonable to hope that $R = 16$ provides an adequate ensemble size for the EADP in even more complicated geometries.

The comparison between various ensemble sizes is presented here only for model A (2.4). However, the same conclusions concerning the convergence of the results and the appropriate value of $R$ are obtained when either the Smagorinsky model or model B (2.5) is used as well.

### *3.3 Comparison of models*

As mentioned in the introduction, an important motivation for developing the EADP is the possibility of investigating new concepts in subgrid-scale modeling. Here, the filtered DNS of Moser, Rogers, & Ewing (1997) is compared with the LES predictions of Ghosal & Rogers (1997) and the predictions of the models presented in Section 2. We have also added the results of a LES without a subgrid-scale model. In all cases, and in agreement with the conclusion of the preceding section, the simulations for the EADP have been performed with $R = 16$.

The first important conclusion is that the plane-averaged and ensemble-averaged dynamic procedures lead to indistinguishable results when they are applied with the same model. For instance, in the LES of Ghosal & Rogers (1997), the plane-averaged dynamic procedure has been implemented with the standard Smagorinsky

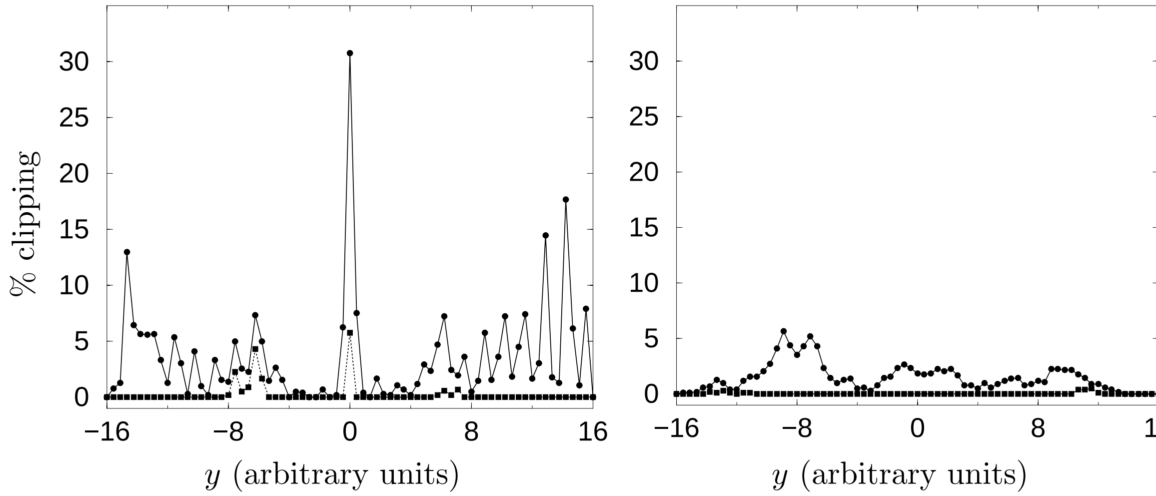


FIGURE 5. Profile of the fraction of grid points requiring clipping of the coefficient $C$ at $t = 0$ (left) and $t = 250$ (right). Two ensemble sizes are compared: $R = 4$, ● ; and $R = 32$, ■ .

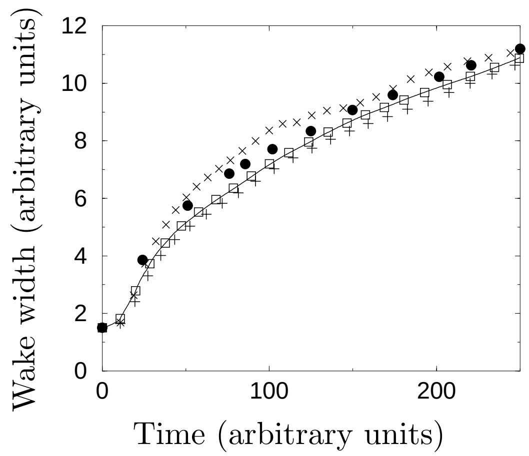


FIGURE 6. The wake width evolution obtained from the filtered DNS, ●; the Smagorinsky model, —; Model A, □ ; Model B, +; and no model, ×.

model. Their results are identical to those obtained when the Smagorinsky model is used with the EADP. In the following comparison, the Smagorinsky case will refer to both the EADP and the plane-averaged LES of Ghosal & Rogers.

The evolution of the wake width is illustrated in Fig. 6. This quantity is dominated by large-scale flow features and consequently is not strongly affected by the models. Actually, the prediction of the LES without a subgrid-scale model (an under-resolved DNS) provides a reasonable approximation to the value obtained by filtering the DNS data.

The turbulent kinetic energy density integrated in $y$ is more difficult to predict using LES. As can be seen in Fig. 7, not using a subgrid-scale model results in poor prediction of resolved energy density. Model A leads to almost the same result as the Smagorinsky model. This is a general feature of the dynamic procedure

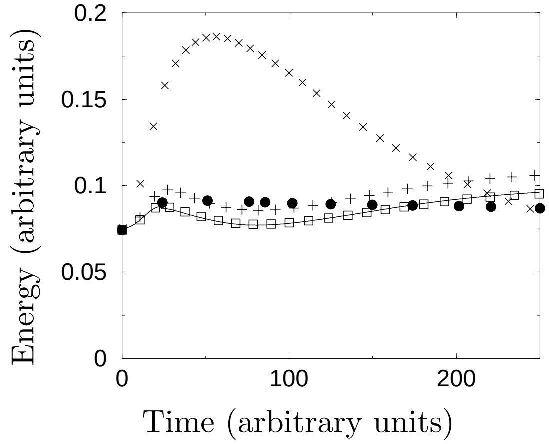


FIGURE 7. The evolution of the resolved turbulent kinetic energy density integrated in $y$ obtained from the filtered DNS •; the Smagorinsky model, —; Model A, □ ; Model B, +; and no model, ×.

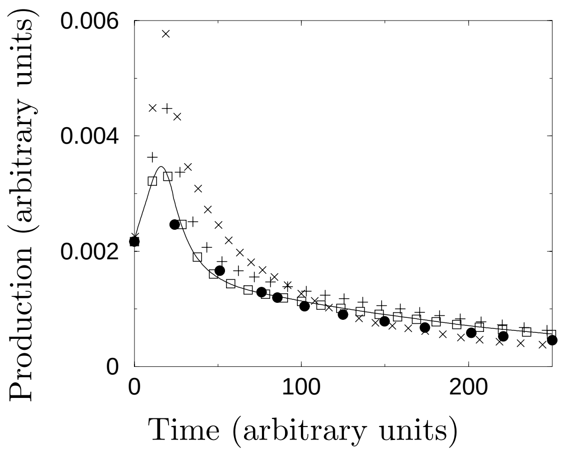


FIGURE 8. The evolution of the resolved turbulent kinetic energy production integrated in $y$ obtained from the filtered DNS •; the Smagorinsky model, —; Model A, □ ; Model B, +; and no model, ×.

that has been noted previously (Wong & Lilly 1994; Carati, Jansen & Lund 1995). However, within the dynamic procedure approach the model A is computationally much cheaper to implement than the Smagorinsky model, and this motivates the use of the scaling (2.4) for the eddy-viscosity instead of (2.3). The model B, from which the ensemble-averaged resolved strain-rate has been removed, leads to results that better fit the DNS data in the early stages of the simulation. At later times, however, this model is further from the filtered DNS values than model A and the Smagorinsky model. In general the predictions of all three models seem comparable.

The evolution of the turbulent kinetic energy production integrated in $y$ is presented in Fig. 8. The no-model LES prediction for the resolved energy production is much too high in the early stage and too low at later times. Again, model A leads to almost the same result as the Smagorinsky model. Model B systematically over-predicts the energy production. However, it would be rather speculative to draw any definitive conclusion regarding which model (A or B) is better from the results presented here.

## 4. Conclusions

The motivations for the present study were *i*) the determination of the required number of realizations for ensemble-averaged determination of subgrid LES terms and *ii*) the demonstration of new modeling concepts than can be implemented within the framework of the EADP.

Simulations with $R = 16$ lead to results that do not differ significantly from those obtained with $R = 32$. Hence, $R = 16$ is a reasonable choice for the ensemble size. This is the same value recommended by Carati *et al.* (1996) for homogeneous turbulence, suggesting that this might be an adequate ensemble size for converged results even in more complex flows. This is, of course, a major encouragement for further developing the EADP methodology.

All three subgrid-scale models employed with the EADP procedure resulted in comparable predictions of various filtered DNS statistics, whereas not using any model provided inadequate estimates of quantities other than the mean velocity profile. Predictions using the Smagorinsky model and the EADP procedure are identical to those made by conventional plane-averaged evaluation of the Smagorinsky model terms. Model A leads to results that are very similar to those predicted by the Smagorinsky model. Hence the present study suggests that, in the context of the dynamic procedure, the Smagorinsky model should be abandoned in favor of model A, which is computationally much cheaper. We have also introduced a new model (model B), which explicitly requires ensemble-averaged statistics to predict the subgrid terms (although the same model could be implemented without an ensemble if the flow has a homogeneous direction for averaging). Having an ensemble of LES's opens up many new possibilities for subgrid-scale modeling. The subgrid-scale tensor has traditionally been modeled in terms of the resolved strain-rate tensor $\overline{S}^r_{ij}$. With an ensemble of LES realizations, it is possible to build up new models based on quantities that explicitly require an averaging procedure such as the ensemble-averaged resolved strain-rate tensor $\langle\overline{S}^r_{ij}\rangle$ (as in model B). Another tensor that could be an interesting ingredient in subgrid-scale modeling is the second-order velocity correlation $\langle(\overline{u}^r_i - \langle\overline{u}^r_i\rangle)(\overline{u}^r_j - \langle\overline{u}^r_j\rangle)\rangle$.

Considering the rapid development of parallel computers, the use of an ensemble of statistically equivalent and independent LES's can be regarded as a very promising technique. This technique can be implemented with fairly small ensemble sizes. Original modeling concepts that cannot be implemented in fully inhomogeneous flows by conventional LES techniques are possible within the framework of the EADP and warrant further examination.

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