Weakly nonlinear modeling of the early stages of bypass transition

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1. Motivation and objectives

Under ideal conditions, boundary layer transition occurs in a six-stage process described, for example, by Stuart (1965). The first stage is that of linear instability, and its onset is predicted accurately by two-dimensional normal mode solutions of the Orr-Sommerfeld equation. In the Blasius case, the critical Reynolds number based on displacement thickness is approximately $Re_{\delta^*} = 520$ and the exponentially growing modes observed at slightly larger values of Re_{δ^*} are known as Tollmien-Schlichting instabilities.

In many important engineering applications, however, transition to turbulence is known to occur at sub-critical Reynolds numbers, and in extreme cases the Tollmien-Schlichting stage may be entirely bypassed. Responsibility for this bypass phenomenon may be linked, for example, to surface roughness or structural vibrations, but in this report we shall be concerned with free-stream turbulence as the source. The reason is that the application of primary concern here is to turbomachinery, where a high level of free-stream turbulence is often produced by upstream stages.

We also wish to investigate the effect of streamwise pressure gradients on bypass transition. The adverse pressure gradient case is of most concern because transition on compressor blades, as well as flow over airplane wings, usually takes place in a region of adverse pressure gradient. However, transitional flows in the presence of a favorable pressure gradient are also of interest. This is particularly true in low-pressure turbines and occurs occasionally in compressors as well.

Experimental studies of the influence of free-stream turbulence on transition reveal the presence of longitudinal vortices with a quasi-periodic structure in the spanwise direction; these are sometimes termed Klebanoff modes. At some point, what may loosely be termed secondary instabilities are observed, and these cause a breakdown of the organized structures into turbulence. The secondary instabilities are usually attributed in the Blasius case to a distortion of the velocity profile such that it develops inflection points (see, *e.g.*, Wundrow & Goldstein (1998)).

Nonlinearity is an essential feature of bypass transition, and the process is clearly of such complexity that only a numerical simulation could describe all the stages. However, DNS simulations are time consuming, and there are many parameters in the problem that can be varied. Analytical methods are needed to suggest optimal initial conditions and also to provide insights that can be helpful in interpreting the results of both experiments and numerical simulations.

An idealization that has proved useful in numerical simulations of bypass transition is to consider an initial disturbance comprised of a pair of oblique modes inclined at equal and opposite angles to the primary flow direction. This has been done in studies of the Blasius boundary layer by Joslin, Streett & Chang (1993) and also by Berlin, Lundbladh & Henningson (1994). In both studies, Orr-Sommerfeld modes were used as initial conditions. However, they were superimposed in the latter paper in such a way that the vorticity component normal to the wall was zero and a smaller Reynolds number was used in the simulations. Experiments on "oblique transition" were reported recently by Elofsson (1998), and comparisons with the numerical simulations were encouraging.

The objective of the research reported here is to formulate nonlinear analyses to be employed in conjunction with numerical simulations of boundary layer transition influenced by free-stream turbulence. Following discussions with Professors Sanjiva Lele and Paul Durbin, it was decided to represent the perturbations at lowest order by modes belonging to the continuous spectrum of the Orr-Sommerfeld equation. Grosch & Salwen (1978) noted that a patch of vorticity in the free stream can be expanded in terms of these eigenfunctions. Their speed of propagation is close to the free-stream value, and their amplitude is largest around the edge of the boundary layer and very small within the boundary layer. These features are exhibited clearly by the computations of some spatially damped eigenfunctions for a Blasius boundary layer reported by Jacobs & Durbin (1998). The weakly nonlinear approach involves a perturbation about a superposition of modes belonging to the continuous spectrum of the Orr-Sommerfeld equation, so the following section consists of an outline and preliminary results for that problem.

2. Accomplishments

We wish to investigate the evolution of free-stream disturbances to boundary layers with velocity profiles belonging to the Falkner-Skan family of similarity solutions. These solutions are obtained for flows in which the free-stream velocity varies with distance along the surface according to

$$U_e^*(x^*) = U_0^*\, x^{\beta_H/(2-\beta_H)}, \tag{1}$$

where an asterisk denotes a dimensional variable. The Hartree form of the governing equation yielding the velocity profile is

$$f''' + ff'' + \beta_H(1 - f'^{\,2}) = 0\,, \tag{2}$$

where $f'(\eta) = u/U_e^*$ and η is a similarity variable.

In linear stability calculations, it is usual (but not universal) to use the boundary layer displacement thickness in non-dimensionalizing the Orr-Sommerfeld equation, which then takes the form

$$\left(\bar{u} - \frac{\omega}{\alpha}\right)(\phi'' - \alpha^2\phi) - \bar{u}''\phi = \frac{1}{i\alpha Re_{\delta^*}}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi)\,, \tag{3}$$

where the perturbation stream function is given by $\hat{\psi} = \phi(y)\exp\{i(\alpha x - \omega t)\}$ and, for spatially evolving waves, ω is real and α is complex. Some care is required in employing the velocity profiles obtained from similarity solutions in (3) because the derivatives in $\bar{u}''$ are with respect to y, which is non-dimensionalized with respect to δ^*, whereas η is the independent variable in (2). The required relationship is

$$\bar{u}''(y) = \Delta^2 f'''(\eta)\,, \text{ where } \quad \Delta \equiv \int_0^\infty (1 - f')\, d\eta. \tag{4}$$

Equation (4) is analogous to (12b) of Obremski, Morkovin & Landahl (1969) who have presented in some detail the linear stability characteristics of Falkner-Skan profiles for different values of β_H. Their non-dimensionalization utilizes δ, the boundary layer thickness, as the length scale in the Orr-Sommerfeld equation rather than δ^*. This accounts for the different factors in treating the $\bar{u}''$ term.

What differentiates the continuous spectrum from the normal mode solutions of (3) is the asymptotic form of the free-stream boundary conditions. Whereas the Tollmien-Schlichting modes decay exponentially as $y \to \infty$, those of the continuous spectrum are required only to be bounded. Outside the boundary layer, $\bar{u} = 1$; as a result, (3) has constant coefficients, and four linearly independent solutions are readily obtained. One grows exponentially with increasing y and must be rejected. The eigenfunction is then a linear combination of the remaining three, two of which are oscillatory while the third decays exponentially. In the spatial case, it can be shown that all modes are damped (*i.e.,* $\alpha_i > 0$), and the details for the boundary conditions can be found in §2.3 of Grosch & Salwen (1978).

A collocation method has been used to obtain solutions of (3) for values of β_H, the pressure gradient parameter, ranging from zero to $\beta_H = -0.1988$, which corresponds to separation. Only two cases are shown here because of the preliminary nature of our work. Specifically, the sensitivity of the eigenfunctions to pressure gradient was found to depend on which feature is plotted. Whereas previous articles have shown separately the real and imaginary parts of ϕ, in this study examining the variation of $|\phi|$ turned out to be more informative.

In Fig. 1, the modulus of the spatial eigenfunction for a Blasius boundary layer is shown. The magnitude is seen to be very small for the roughly 1/3 of the boundary layer adjacent to the wall. The "penetration depth" is, nonetheless, noticeably greater than in the cases illustrated in Figs. 3 and 4 of the article by Jacobs & Durbin (1998). Our result is consistent with their prediction, based on an analysis of the two-layer Tietjens model, that at lower Reynolds numbers the penetration depth will be greater.

The computations reported here were done at $Re_{\delta^*} = 500$ to agree with the experiments of Boiko *et al.* (1994), whereas those of Jacobs & Durbin were at a considerably higher Reynolds number. However, our frequency is smaller than theirs, so further comparisons would be desirable to confirm these trends.

As shown in Fig. 2, when there is a moderate adverse pressure gradient, $\beta_H = -0.12$, the penetration depth is slightly less than in the Blasius case. However, near the edge of the boundary layer the magnitude of the oscillations around the far field value of $|\phi|$ is seen to be much greater. This suggests that the details of transition

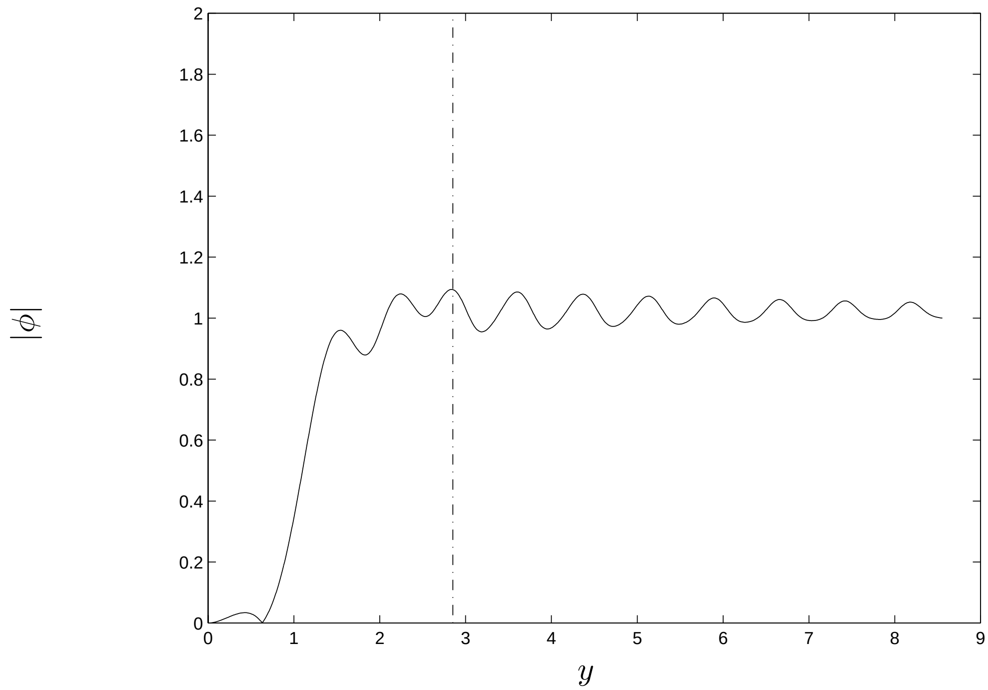

FIGURE 1. Modulus of the spatial eigenfunction for a Blasius boundary layer at $Re_{\delta^*} = 500$, $\omega = 0.18$, and $\alpha_r = 0.1799$.

induced by free-stream disturbances might be significantly different for boundary layers developing in an adverse pressure gradient.

3. Future plans

As discussed near the end of the introductory section, the longer term goal of this research is to formulate an amplitude expansion involving a perturbation about a linear state whose eigenfunction is derived from the continuous spectrum. The streamwise vortices and streaks observed in experiments can be modeled most simply by starting with a pair of oblique waves. A number of new features would be present in such a formulation, and certain mathematical difficulties must be addressed.

To simplify the discussion, let us consider first the simpler problem of formulating a weakly nonlinear analysis for a single plane wave. We expect the amplitude in the spatially evolving case to satisfy a Stuart-Landau equation having the form

$$\frac{dA}{dx} = -\alpha_i A + a_2 \, A \mid A \mid^2, \tag{5}$$

where a_2 is the Landau constant. When the basic disturbance is a Tollmien-Schlichting wave, the Landau constant is given by the ratio of two definite integrals. These integrals are obtained from imposing an orthogonality condition and the homogeneous boundary conditions. However, because the eigenfunctions of the continuous spectrum do not vanish as $y \to \infty$, the corresponding integrals do not exist, and an alternative solvability condition must be employed.

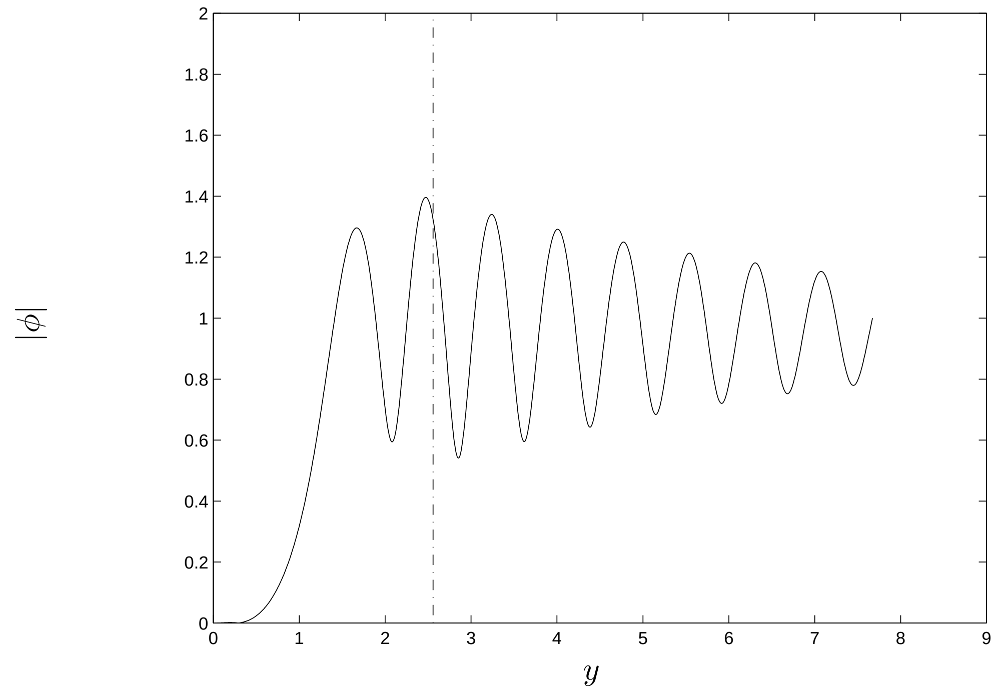

FIGURE 2. Spatial eigenfunction for an adverse pressure gradient boundary layer with $\beta_H = -0.12, Re_{\delta^*} = 500$, $\omega = 0.18$, and $\alpha_r = 0.1799$.

A second difficulty is that in the usual Stuart-Watson theory the perturbation is about a neutral mode. However, the modes of the continuous spectrum are weakly damped. The consequences of this may be minor, perhaps a slower rate of convergence of the amplitude expansion. A possible solution may be to include nonlinearity at the lowest order to obtain a neutral mode. In any case, the matter is one that must be considered.

Returning now to the pair of oblique modes as an initial perturbation, some aspects of the development can be anticipated from the paper by Benney (1961) which provides the most detailed description of the analysis leading to the Benney-Lin vortices. A plane wave in addition to a pair of oblique waves is considered in Benney (1961); however, setting the parameter $\mu = 0$ in §3 of his paper yields equations analogous to those anticipated in our analysis. Of particular significance is a sort of resonance that occurs between the waves and the mean flow at the first order beyond the linear problem. In the nearly-neutral case, this resonance was shown to produce a secondary flow whose u-component velocity grows like t^2 while the mean longitudinal vorticity has a growth proportional to t.

In the formulation under consideration here, the perturbation at lowest order is of the form

$$\begin{aligned}
u^{(1)} &= \{A(X)\,\hat{u}(y)\,e^{i(\alpha x-\omega t)} + c.c.\}\cos\beta z \\
v^{(1)} &= \{A(X)\,\hat{v}(y)\,e^{i(\alpha x-\omega t)} + c.c.\}\cos\beta z \\
w^{(1)} &= \{A(X)\,\hat{w}(y)\,e^{i(\alpha x-\omega t)} + c.c.\}\sin\beta z \\
p^{(1)} &= \{A(X)\,\hat{p}(y)\,e^{i(\alpha x-\omega t)} + c.c.\}\cos\beta z\,,
\end{aligned} \tag{6}$$

where α is now real and X is a slow variable in the streamwise direction. The quantity $\hat{v}(y)$ satisfies the Orr-Sommerfeld equation (3) with α^2 replaced everywhere by $\alpha^2 + \beta^2$. Once $\hat{v}$ has been determined, the continuity and vertical vorticity equations can be used to obtain the other velocity components, and the pressure perturbation is obtained from the $x-$component momentum equation.

It can be expected based on the considerations discussed above that at the next order a large mean flow response will occur. The amplitude equations will not arise until the following order in the amplitude expansion. One result that is hoped to be obtained after deriving and solving these equations is the obliqueness angle leading to the largest amplification rate. Because we are dealing with the sub-critical case here, an estimate of the amplitude of turbulence in the free stream required to promote instability will also come out of the analysis.

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