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Critical states in 2D disk-shaped type-II superconductors in periodic external magnetic field

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Following the procedure of Mikheenko and Kuzovlev, we present analytical solutions of field and current patterns in thin film disk-shaped type-II superconductors in perpendicular time-varying periodic external magnetic fields. We also calculate the magnetic moment and effective susceptibility. The analysis is carried out within the framework of the critical state model assuming a constant critical current. Our results are compared to that of Mikheenko and Kuzovlev; and we discuss the discrepancies.

1. Introduction

There has been a lot of interest in the critical state model [1,2] in disk-shaped type-II superconductors with external magnetic fields applied perpendicular to the disk [3–8]. Frankel [3] and Däumling and Larbalestier [4] employed numerical methods to obtain field and current patterns in disk-shaped superconductor films assuming constant critical currents and also Anderson–Kim [9] type critical currents. Däumling and Larbalestier [4] found that in superconductor films, the characteristic screening field is $\sim j_c d$, where j_c is the critical current density, and d is the thickness of the film; thus an external field is considered a weak field if it is not much larger than the characteristic field $j_c d$. However, for superconductor films, even weak magnetic fields can penetrate into the film if it is applied perpendicular to the film plane, because of large demagnetization effects. Device application has been an area where weak magnetic fields are often considered [10–12]; constant critical currents are usually assumed in applying the critical state model. Assuming constant critical currents, Mikheenko and Kuzovlev [8] recently have found analytical solutions for current and field patterns in 2D superconducting disks in external fields in the framework of the critical state model. Considering a disk-shaped superconducting sample in a time-varying periodic external field, Mikheenko and Kuzovlev construct the current configuration in 2D disk-shaped superconductors by analogy to that of a long cylinder [8]. They believe that, in a periodic external field, two flux penetration regions can form with critical currents flowing in opposite direction, i.e. $+I_c$ in the region $b<r<R$, and $-I_c$ in the region $a<r<b$, where R is the radius of the sample, where $I_c=j_c d$.

In this paper, we present solutions to the field and current patterns in thin film 2D disk-shaped superconductors in a periodic external field; we also give analytical solutions to the magnetic moment and the effective magnetic susceptibility. We show that the analogy of current patterns between a thin film disk and a long cylindrical superconductor is *not* valid. We also compare our results to that of Mikheenko and Kuzovlev, and discuss the discrepancies.

We follow the procedures by Mikheenko and Kuzovlev [8], so we can compare our results with theirs.

2. Theory

We consider a disk-shaped superconductor film with external magnetic field H applied along the z-direction, which is perpendicular to the film plane, after the sample is zero-field cooled (ZFC).

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We assume that the film thickness d is approximately less than 2λ, where λ is the London penetration depth; the factor of 2 is included since shielding currents flow on both sides of the film. Circulating currents in the film plane are treated as having uniform density in the thickness direction of the superconductor. We also assume that the external magnetic field H is weak enough so that the critical current in the film is independent of the local density of trapped vortices, or

$$j_c \times d = \text{constant}\,, \tag{1}$$

where j_c is the critical current density, and d is the thickness of the film. However, the actual situation may be more complicated. Under the above assumptions, an elegant analytical solution can be found for circulating currents $I(r)$ and the magnetic moment M in 2D disk-shaped superconductors.

Shielding currents only flow in the circumferential direction because of the symmetry of the sample. The magnetic moment M therefore is

$$M = \frac{1}{2c}\int_0^R \int_0^{2\pi} rI(r) r\,\mathrm{d}r\,\mathrm{d}\phi\,. \tag{2}$$

The radial direction field $H_r(r, z)$ and the z direction field $H_z(r, z)$ are [13], respectively,

$$H_r(r, z) = \frac{2}{c}\int_0^R G_r(r, \rho, z) I(\rho)\,\mathrm{d}\rho\,, \tag{3}$$

and

$$H_z(r, z) = \frac{2}{c}\int_0^R G_z(r, \rho, z) I(\rho)\,\mathrm{d}\rho\,, \tag{4}$$

where the functions G_r and G_z are

$$G_r(r, \rho, z) = \frac{1}{\sqrt{(\rho+r)^2+z^2}}\left[-K(k) + \frac{\rho^2+r^2+z^2}{(\rho-r)^2+z^2}E(k)\right],$$

and

$$G_z(r, \rho, z) = \frac{1}{\sqrt{(\rho+r)^2+z^2}}\left[K(k) + \frac{\rho^2-r^2-z^2}{(\rho-r)^2+z^2}E(k)\right],$$

where $k^2 = 4\rho r/[(\rho+r)^2+z^2]$, and K and E are complete elliptic integrals of the first and the second kind.

2.1. The Meissner state

In this state, there is no flux penetration of any kind, or

$$H_z(r, z=0) = 0, \qquad r<R\,.$$

A thin film disk can be treated as an extremely flat ellipsoid whose z direction axis shrinks to zero. Following the calculations presented by Landau [13], one obtains,

$$H_r(r, z\to 0) = -\frac{2}{\pi} H_0 \operatorname{sign}(z) \frac{r}{\sqrt{R^2-r^2}}\,. \tag{5}$$

The surface current $I(r)$ is

$$\frac{4\pi}{c} I(r) = H_r(r, z=0^+) - H_r(r, z=0^-)\,,$$

or

$$I(r) = -\frac{c}{4\pi} H_0 F\left(\frac{r}{R}\right), \tag{6}$$

where the function F is defined as [8]

$$F(x) = \begin{cases} \dfrac{4}{\pi}\dfrac{x}{\sqrt{1-x^2}}, & \text{if } x<1\,, \\ 0, & \text{if } x>1\,. \end{cases}$$

The M can now be calculated,

$$M(H_0) = -\frac{2}{3\pi} H_0 R^3\,;$$

a well-known result [13]. The magnetic field in the film plane generated by the shielding current (6) is

$$H_z(r, z=0) = H_0 Q(r/R)\,,$$

where the function Q is defined as [8]

$$Q(x) = \begin{cases} -1, & \text{if } x<1, \\ \dfrac{2}{\pi}\left(-\arcsin\dfrac{1}{x} + \dfrac{1}{\sqrt{x^2-1}}\right), & \text{if } x>1\,. \end{cases}$$

2.2. The intermediate state

First, let us consider a case where the external field is increased monotonically from zero to the maximum field H_0 after the sample becomes superconducting. For the sake of simplicity, only field and current patterns of the maximum field H_0 are considered. We assume that magnetic vortices penetrate into the sample symmetrically to a radius of $a(H_0)$ in the external field. Thus the critical current $j_c d$ flows in the ring $a<r<R$, while the magnetic field remains free of trapped vortices in the inner ring $r<a(H_0)$.

Mikheenko and Kuzovlev [8] suggested that the shielding current $I(r)$ flowing in the sample can be viewed as a linear combination of currents of eq. (6),

$$I(r) = -\frac{c}{4\pi} H_0 \int_0^R F\left(\frac{r}{l}\right) W(l, H_0)\,\mathrm{d}l\,, \tag{7}$$

where $W(l, H_0)$ is the weight function. Substituting eq. (7) into eq. (4), we have,

$$\int_0^R W(l, H_0)\,\mathrm{d}l = 1\,. \tag{8}$$

Equation (7) can also be understood [8] in this way: the current configuration $(c/4\pi)F(r/l)$ with $a<l<R$, generates a constant uniform unit magnetic field inside the ring $r<l$; the total field generated in the ring $r<a$ by the circulating currents is thus $-H_0 \int_a^R W(l, H_0)\,\mathrm{d}l$, which cancels the applied magnetic field H_0 in the inner ring $r<a$, or $-H_0 \int_a^R W(l, H_0)\,\mathrm{d}l + H_0 = 0$. The total magnetic field generated by shielding currents (7) and the external field in the film plane is

$$H_z(r, z=0) = H_0 \int_a^r [Q(r/l)+1]\, W(l, H_0)\,\mathrm{d}l\,, \quad a<r<R\,. \tag{9}$$

The weight function W can be obtained from the fact that a constant critical current flows in the ring $a<r<R$, that is

$$\int_r^R F(r/l)\, W(l, H_0)\,\mathrm{d}l = \frac{4\pi}{cH_0} j_c d(|H_z(r)|)\,, \qquad a<r<R\,. \tag{10}$$

The solution to the above equation has been found by Mikheenko and Kuzovlev:

$$W(r, H_0) = \frac{2\pi j_c d}{cH_0} \frac{1}{r\sqrt{1-(r/R)^2}}\,. \tag{11}$$

Equation (8) can be solved for $a(H_0)$, yielding

$$a = R/\cosh\left(\frac{H_0}{H_c}\right), \tag{12}$$

where $H_c = (2\pi/c) j_c d$ is the characteristic critical field. The magnetic moment M of the 2D disk-shaped superconductor film is

$$M = -\frac{2}{3\pi} H_c R^3 \frac{1}{2}\left[\arccos\frac{a}{R} + \frac{a}{R}\sqrt{1-(a/R)^2}\right] = -\frac{2}{3\pi} H_0 R^3 S\left(\frac{H_0}{H_c}\right), \tag{13}$$

and the shielding current is

$$I(r) = -I_c \frac{2}{\pi} \arctan\left[\frac{r}{R}\left(\frac{R^2-a^2}{a^2-r^2}\right)^{1/2}\right] \theta(a-r) - I_c \theta(r-a)\,, \tag{14}$$

where θ is the step function; the function S is defined to be same as in reference [8]:

$$S(x) = \frac{1}{2x}\left[\arccos\frac{1}{\cosh(x)} + \frac{\sinh|x|}{\cosh^2(x)}\right].$$

We now consider the case when the external field is monotonically decreased to a field H ($|H| \leqslant H_0$), after having been increased monotonically from zero to the maximum field H_0. Analytical solutions to current and field patterns can be found for this case. If we notice that

$$H = H_0 + H^*, \qquad H^* = H - H_0\,,$$

the problem can be treated as a superposition of the following two case.

(1) In the ZFC sample, the external field is increased monotonically from zero to H_0.
The sample has a critical current of I_c.

(2) In the ZFC sample, the external field is decreased monotonically from zero to $H - H_0$. However, the critical current flowing in the sample is $2I_c$.

The magnetic moment and shielding currents of the second case can be easily found by substituting $-I_c$ with $2I_c$ (or $-H_c$ with $2H_c$), and H_0 with H^* (or $H-H_0$) into solutions for the first case:

$$M=-\frac{2}{3\pi}H^*R^3S\left(\frac{H^*}{-2H_c}\right)$$
$$=-\frac{2}{3\pi}(H-H_0)R^3S\left(\frac{H-H_0}{-2H_c}\right), \quad (15)$$

$$I(r)=2I_c\frac{2}{\pi}\arctan\left[\frac{r}{R}\left(\frac{R^2-b^2}{b^2-r^2}\right)^{1/2}\right]\theta(b-r)$$
$$+2I_c\theta(r-b), \quad (16)$$

and

$$b=R/\cosh\left(\frac{H^*}{-2H_c}\right)=R/\cosh\left(\frac{H-H_0}{-2H_c}\right), \quad (17)$$

where b is the radius to which the vortices have penetrated into the disk sample in the second case. The critical current $2I_c$ in the second case can be understood this way: a shielding current $-I_c$ is induced in the ring $a<r<R$ in the first case, and a shielding current $+2I_c$ is induced in the ring $b<r<R$ in the second case; the net shielding current of the sample is $+I_c$ in the ring $b<r<R$ after summing the currents from both cases, satisfying the critical state model and the critical current criterion $|I(r)|\leqslant I_c$.

Superposing solutions of the magnetic moment and current patterns for case 1 and case 2 give the exact analytical solutions of the magnetic moment and the shielding current,

$$M=-\frac{2}{3\pi}H_0R^3S\left(\frac{H_0}{H_c}\right)$$
$$-\frac{2}{3\pi}(H-H_0)R^3S\left(\frac{H-H_0}{-2H_c}\right), \quad (18)$$

$$I(r)=-I_c\frac{2}{\pi}\arctan\left[\frac{r}{R}\left(\frac{R^2-a^2}{a^2-r^2}\right)^{1/2}\right]\theta(a-r)$$
$$-I_c\theta(r-a)$$
$$+2I_c\frac{2}{\pi}\arctan\left[\frac{r}{R}\left(\frac{R^2-b^2}{b^2-r^2}\right)^{1/2}\right]\theta(b-r)$$
$$+2I_c\theta(r-b). \quad (19)$$

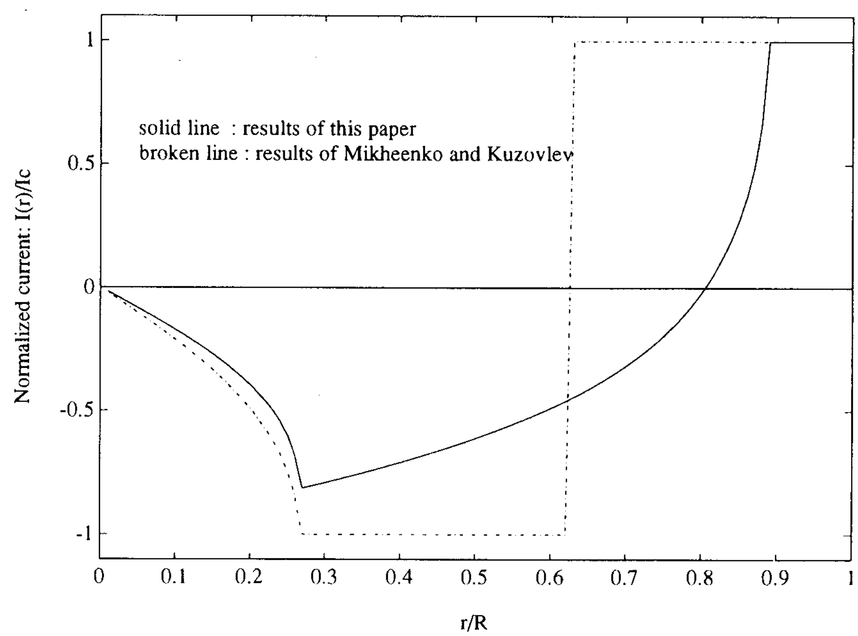

Fig. 1. Normalized current patterns in a 2D disk-shaped superconductor calculated using eq. (14) for $H_0=2H_c$, and $H=H_c$. Corresponding field patterns are shown in fig. 2. Mikheenko and Kuzovlev's results are also shown for comparison.

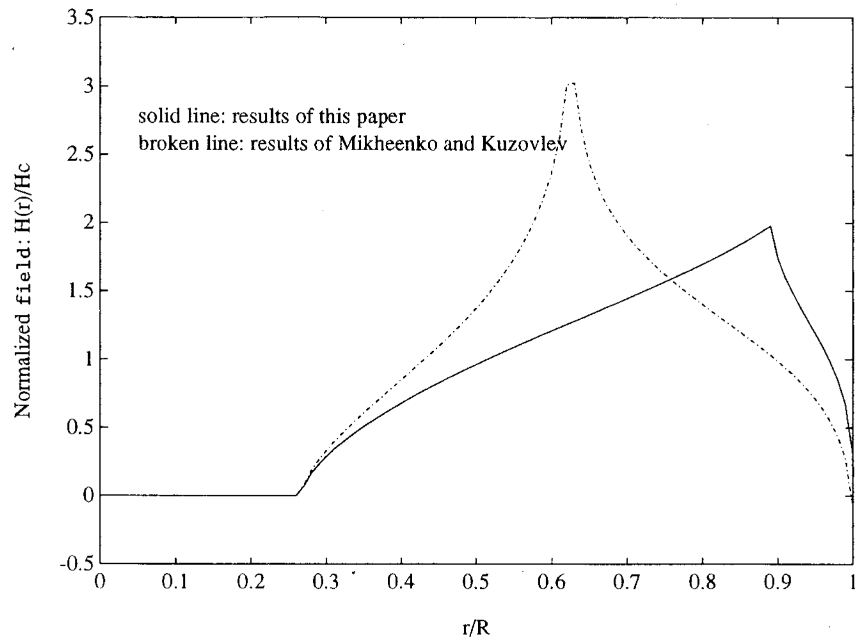

Fig. 2. Normalized field patterns in a 2D disk-shaped superconductor calculated using eqs. (9) and (14) for $H_0=2H_c$, and $H=H_c$. Corresponding current patterns are shown in fig. 1. Mikheenko and Kuzovlev's results are also shown for comparison.

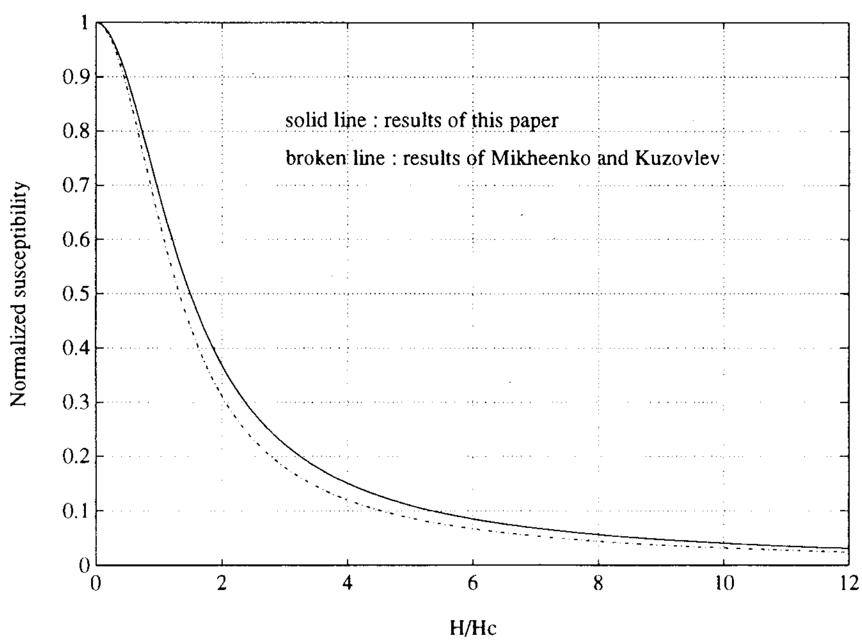

Fig. 3. Normalized effective magnetic susceptibility $\chi_{\text{eff}}(H_0/H_c)/\chi_{\text{eff}}(0)$, calculated using eq. (21) for a 2D superconducting disk sample. Mikheenko and Kuzovlev's results are also shown for comparison.

Also, following the definition of shielding currents in eq. (7), we obtain the weight function,

$$W(r,H_0)=\frac{2\pi I_c}{cH_0}\left(\frac{1}{r\sqrt{1-(r/R)^2}}-\frac{2\theta(r-b)}{r\sqrt{1-(r/R)^2}}\right). \tag{20}$$

The z direction magnetic field in the film plane of the sample can be calculated using eqs. (4) and (20). As the external field is further reduced to the minimum field $-H_0$, the final state of the superconductor is the same as if the external field had been decreased *monotonically* to $-H_0$ from zero directly for the ZFC sample, which is expected.

If the external field is increased after the minimum field $-H_0$ is reached, current and field patterns can be calculated in a similar way as presented above. We now have $H=-H_0+H^*$, where $H^*=H+H_0$. The magnetic moment, shielding currents, and the weight function can be obtained by substituting I_c with $-I_c$, and H_0 with $-H_0$ (or simply using $H^*=H+H_0$) into the corresponding relations for the external field. Once the external field H_0 is reached, if the field is reduced we recover the solutions of eqs. (18) and (19). We have now complete analytical solutions to the field and current patterns for 2D disk-shaped superconductors in a periodic external field. When using a time-varying periodic field in experiments, one usually measured the effective magnetic susceptibility $\chi_{\rm eff}$, which is averaged over a period [8]. In this case, we find

$$\chi_{\rm eff}=\frac{4}{\pi}\chi_0\int_0^\pi \cos\Psi\sin^2\frac{\Psi}{2}\,S\!\left(\frac{H_0}{H_c}\sin^2\frac{\Psi}{2}\right)\mathrm{d}\Psi, \tag{21}$$

where $\Psi=\omega t$ and $H=H_0\cos(\omega t)$. Equation (21) shows that $\chi_{\rm eff}$ decreases as $\sim 1/H_0$ at large H_0, which is different from the $\sim 1/H_0^{3/2}$ dependence as concluded to by Mikheenko and Kuzovlev.

3. Discussion

A superconducting disk is an extreme case of a cylinder whose length is much smaller than that of its radius; therefore, the end effects are no longer negligible, but in fact constitute the dominant factor in this problem. Since circulating currents only flow in

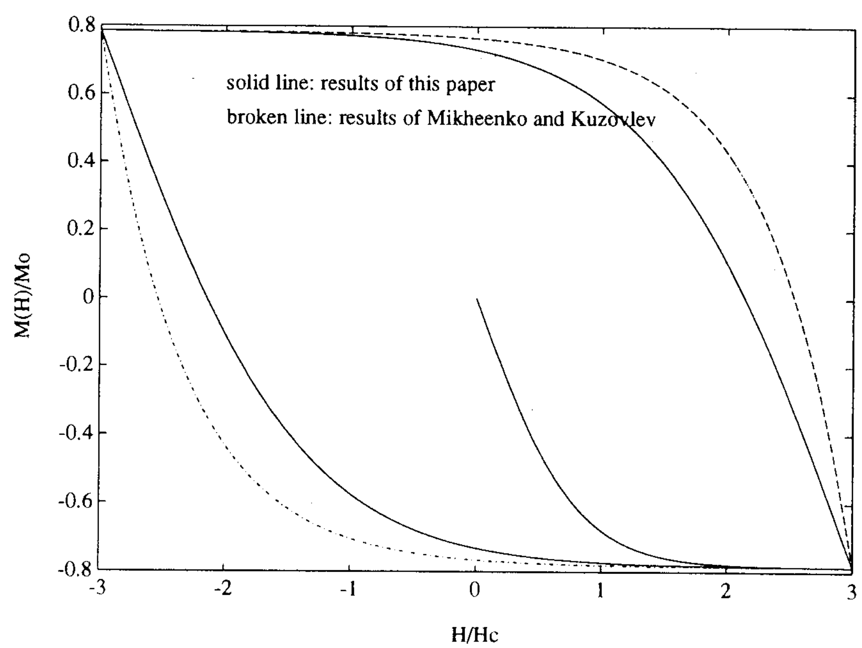

Fig. 4. Magnetic moment $M(H)$ calculated using eq. (18) for a 2D superconducting disk-shaped superconductor in a time-varying periodic weak external magnetic field; external fields are applied after the sample becomes superconducting. Here $M_0=(2/3\pi)H_cR^3$.

circumferential direction, we have

$$j_\phi = \frac{c}{4\pi}(\partial H_r/\partial z - \partial H_z/\partial r) .$$

$|\partial H_r/\partial z|$ is much larger than $|\partial H_z/\partial r|$ for weak external fields except at the center of the disk; the circulating current mainly comes from the term $\partial H_r/\partial z$, rather than from $\partial H_z/\partial r$, in contrast to the case of a long cylindrical sample. In 2D disk-shaped superconductors, as the external field is decreased from the maximum field H_0, the current flowing in the outer region $b<r<R$ is the critical current $+I_c$; but the current $I(r)$ flowing in the inner region $a<r<b$ is neither $+I_c$ nor $-I_c$; it is a function of location and must satisfy the critical current constraint $|I(r)|<I_c$. Considerable shielding currents also flow in the vortex-free region $r<a$, which is very different from the case of a long cylindrical sample.

Figure 1 shows an example of current patterns for $H_0=2H_c$ and $H=H_c$ calculated using eq. (20); fig. 2 shows the z direction field patterns in the film plane for the same condition calculated using eqs. (9) and (20). Current and field patterns calculated using the weight function of Mikheenko and Kuzovlev are also shown for comparison in figs. 1 and 2; their results show an abrupt jump in the current which is unphysical in a 2D disk-shaped superconductor. Figure 3 illustrates our theoretical result of the effective susceptibility given by eq. (21) and that of Mikheenko and Kuzovlev; our results are generally larger than theirs. Figure 4 shows the magnetic hysteresis loop of a 2D disk-shaped superconductor in a time-varying periodic external field for $H_0=3H_c$ using eq. (18); our hysteresis loop is thinner than that of Mikheenko and Kuzovlev; the abrupt jump in their shielding currents is the reason for the fattening of their magnetic hysteresis loop, because the abrupt jump in shielding currents artificially enhances the shielding capability of the thin film 2D superconductor disk.

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