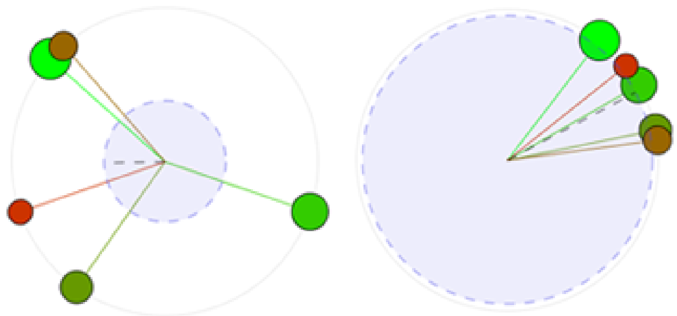
The Kuramoto Model: From Asynchrony to Synchrony



Phases of coupled oscillators with weak (left) and strong (right) coupling. Color and ball-size indicate the oscillators' different intrinsic frequences; dashed circle and marker indicate the order parameter's magnitude and phase (i.e., vector strength) [Kuramoto84,Wordsworth??].

Features

— **Sinusoidal phase-coupling (instead of pulse-coupling)**
— **Accounts for heterogeneity (unlike rate models)**

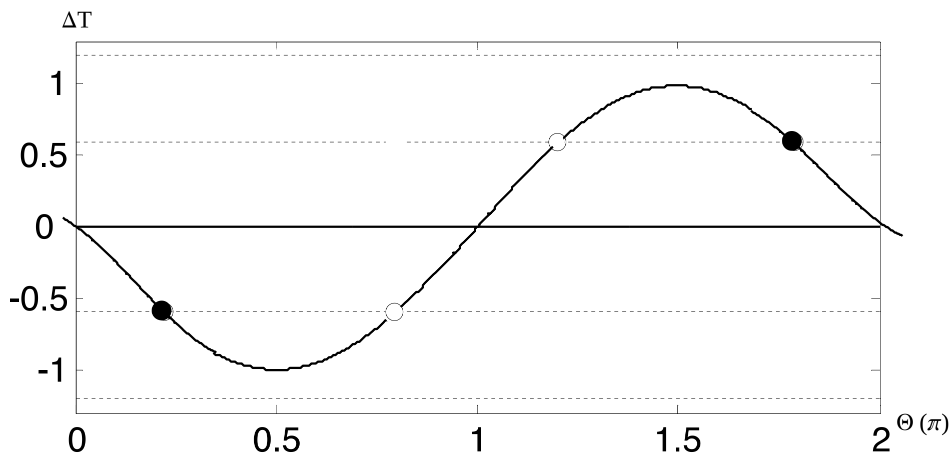
Assumptions

— **Coupling is global (allows mean-field approach)**
— **Coupling is weak (doesn't change oscillation's amplitude)**

Results

— **Synchrony emerges for $K > K_c$ (critical coupling strength)**
— **Like a phase transition from liquid to solid**

Kuramoto's model (1984)



The Kuramoto model's sinusoidal phase-coupling corresponds to a PRC that is a flipped sinusoid. To obtain the Kuramoto model's coupling strength, K, we must multiply the PRC's maximum advance/delay, ΔT_{max}, by the network's total spike rate.

Instead of pulse-coupling, this model uses phase-coupling:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin[\theta_j-\theta_i], \qquad i = 1...N$$

Instead of vector strength, an order parameter is defined:

$$r e^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

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Mean Field Description

Consider multiplying the order parameter by $e^{-i\theta_i}$:

$$r\, e^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}, \qquad i = 1 \ldots N$$

whose imaginary part equals:

$$r \sin(\psi-\theta_i) = \frac{1}{N}\sum_{j=1}^{N}\sin[\theta_j-\theta_i], \qquad i = 1 \ldots N$$

substitute into the phase equation:

$$\dot{\theta}_i = \omega_i + K\, r\, \sin[\psi-\theta_i], \qquad i = 1 \ldots N$$

This result shows that the order parameter's magnitude (r) and phase (ψ) summarizes the effect all the oscillators have on any particular oscillator.

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Steady-state analysis

If the oscillator's synchronize, their phases will change at the same rate, and so will the order parameter's phase (ψ). Thus we will have:

$$\psi[t] = \Omega\, t + \psi[0]$$

where Ω is the locked frequency. It is convenient to redefine the i^{th} oscillator's phase θ_i in this rotating reference frame.

That is:

$$\theta_i = \phi_i - \psi[t] = \phi_i - \Omega\, t - \psi[0]$$
$$\Longleftrightarrow \phi_i = \theta_i + \Omega\, t + \psi[0]$$

where ϕ_i is it's absolute phase. Thus, the original equation

$$\dot{\phi}_i = \omega_i + K\, r\, \mathrm{Sin}[\psi[t] - \phi_i]$$

is transformed into

$$\dot{\theta}_i = \omega_i - \Omega - K\, r\, \mathrm{Sin}[\theta_i]$$

Therefore, in steady-state (i.e., $\dot{\theta} = 0$), we have

$$\omega_i = \Omega + K\, r\, \mathrm{Sin}[\theta]$$

Hence, only oscillators whose natural frequencies fall in the range

$$\Omega - K\, r < \omega_{locked} < \Omega + K\, r$$

will lock, with their phases distributed in the range

$$-\pi / 2 < \theta_{locked} < \pi / 2$$

Relating coupling strength to frequency-density

To obtain the critical coupling strength, K_c, it is convenient to treat drop ω_i's subscript and treat it as a continuous variable instead, which is true if N is very large (continuum limit). In that case, the order parameter is given by

$$r = \frac{1}{N} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\theta}\, N\, g[\theta]\, d\theta$$

where is $N g(\theta)\, d\theta$ is the number of oscillators with phases between θ and $\theta + d\theta$. This number is given by $N g(\omega)\, d\omega = N g(\omega)\, (d\omega / d\theta)\, d\theta$, where $g(\omega)$ is the probability density function of the oscillators' natural frequencies. Thus, we have

$$r = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\theta}\, g[\omega[\theta]]\, \frac{d\omega}{d\theta}\, d\theta$$

We obtain $d\omega/d\theta$ from the steady-state solution: As the $\sin(\theta)$ term is odd, it drops out, leaving

$$\omega = \Omega + K\, r\, \mathrm{Sin}[\theta] \Rightarrow \frac{d\omega}{d\theta} = K\, r\, \mathrm{Cos}[\theta]$$

Substituting into the previous equation and replacing $e^{i\theta}$ with $\cos(\theta) + i \sin(\theta)$ yields

$$r = K\, r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g[\omega[\theta]]\, (\mathrm{Cos}[\theta] + i\, \mathrm{Sin}[\theta])\, \mathrm{Cos}[\theta]\, d\theta$$

If $g(\omega)$ is symmetric around Ω, the $\sin(\theta)$ term drops out, and dividing by r leaves us with

$$1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g[\omega[\theta]]\, \mathrm{Cos}^2[\theta]\, d\theta$$

$$\Longleftrightarrow K = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g[\omega[\theta]]\, \mathrm{Cos}^2[\theta]\, d\theta\right)^{-1}$$

This is interesting: r dropped out and we are left with an inverse relationship between the coupling strength and the frequency density Ñ if the density is low, the coupling strength must be high.

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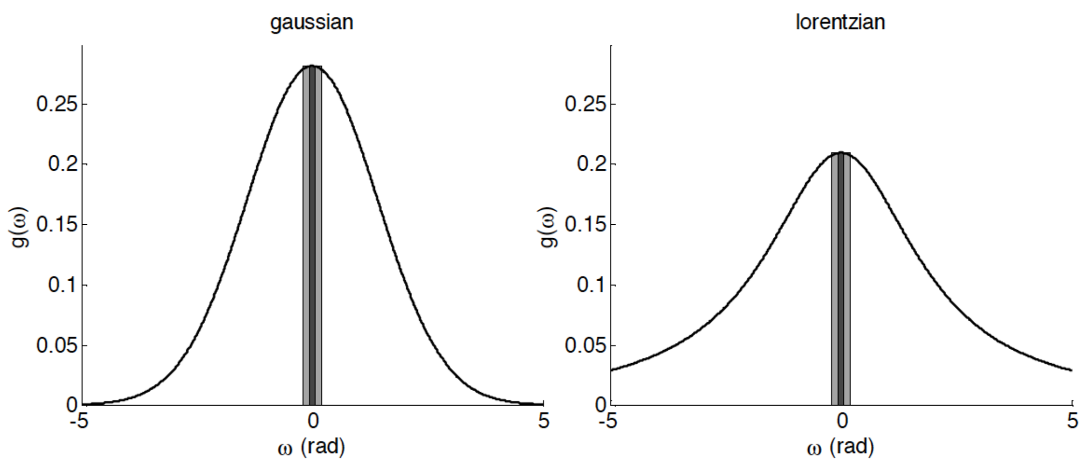
Critical coupling (K_c)



The oscillators whose frequencies are space most closely Ñ where the probability density peaks Ñ are the first ones to synchronize.

To find the critical coupling, K_C, we need to pick the most favorable conditions for synchrony. This corresponds to $g(\omega)$'s peak, where the oscillators are spaced most closely in frequency. These oscillators will be the first to synchronize, locking at the frequency Ω where $g(\omega)$ peaks. As they span a small range, we can set $g(\omega) = g(\Omega)$, and obtain

$$1 = K_C\, g[\Omega] \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{Cos}^2[\theta]\, d\theta \Rightarrow K_C = \frac{2}{\pi\, g[\Omega]}$$

We can obtain an approximate result for higher coupling strengths by expanding $g(\omega)$ to second-order. This yields a result that describes r's initial growth:

$$g\,(K\, r\, \sin(\theta)) \approx g\,(\Omega) + \frac{1}{2}\, \ddot{g}\,(0)\,(K\, r\, \sin(\theta))^2$$

This yields the following result for r's initial growth:

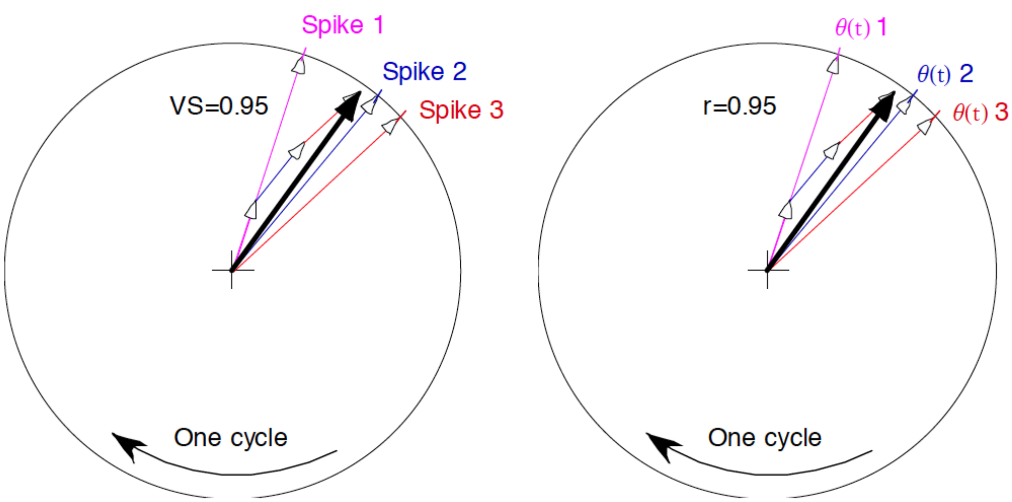
$$r = \sqrt{\frac{16}{\pi K_C{}^3 \ddot{g}[\Omega]}\left(1 - \frac{K_C}{K}\right)}$$

For the Lorentzian distribution, $g(\omega) = \frac{\gamma}{\pi(\gamma^2+\omega^2)}$, $\ddot{g}(\Omega) = \frac{2}{\pi\gamma^3}$ for which the solution is:

$$r = \sqrt{\left(1 - \frac{K}{K_C}\right)}$$

In fact, this solution holds for the whole distribution.

Vector Strength (VS) versus Order Parameter (r)



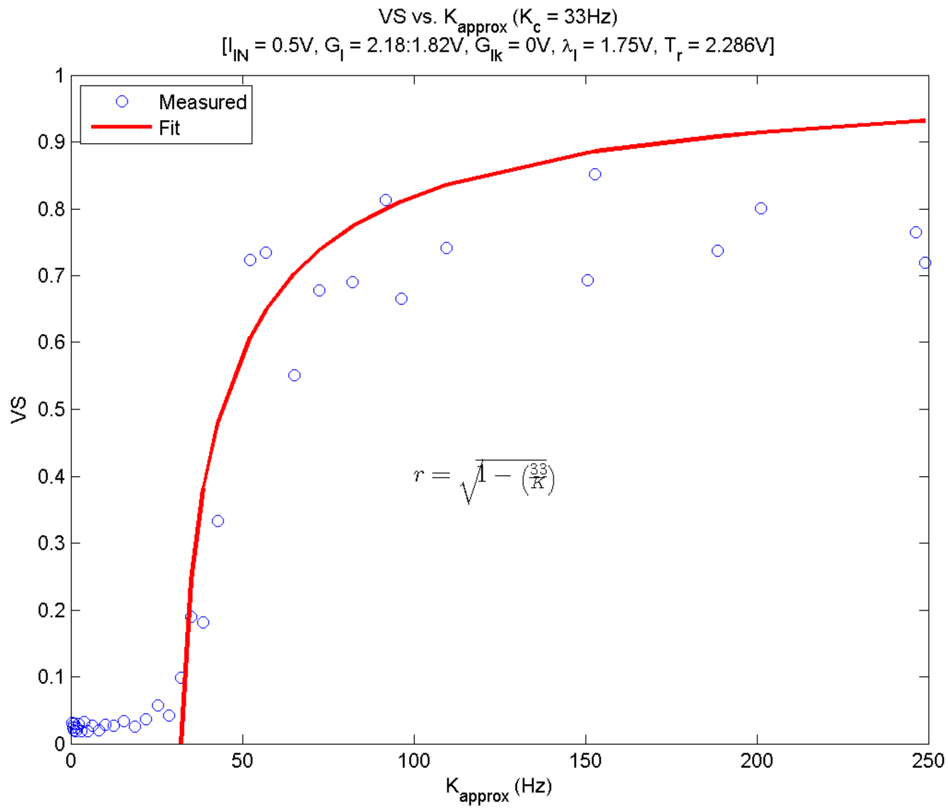
In many cases, VS and r are similar.

Both quantify synchrony in terms of vector sums of phase differences.

Vector strength sums spike phases across time, over at least on period.

The order parameter sums neuron phases at a point in time.

Does Kuramoto Apply?



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Does Kuramoto Apply? Yes, but ...