Problems: Tue 6/27

1. Sketch the graph of $f(x) = 1 + 2\cos x$.

2. Sketch the graph of $g(x) = \dfrac{3x+1}{x}$.

Hint: $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$.

3. Sketch the graph of the following functions.

(a) $F(x) = |\sin x|$.

(b) $G(x) = \sin(|x|)$.

4. Find the domain of $h(x) = \dfrac{\tan x}{2^x \log_3(x)}$.

Even & Odd Functions

Def: Let $f(x)$ be a function.

We say f is *even* if: $f(-x) = f(x)$. (Symmetry in y-axis)

We say f is *odd* if: $f(-x) = -f(x)$. (Symmetry in the origin)

5. Determine whether the following polynomials are even, odd, or neither.

(a) $p(x) = x^5 + 2x^3 + 7x$

(b) $q(x) = x^4 - x$

(c) $r(x) = x^6 - 3x^2 + 1$

Do you see a pattern? How can you quickly tell whether a polynomial is even, odd, or neither?

6. Are there any functions that are *both* even and odd? If so, which ones?

7. If f and g are even functions, is $f + g$ also even?

Problems: Discontinuities: Thu 6/29

1A. Sketch the graph of $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$

1B. Sketch the graph of $g(x) = \dfrac{x^2}{x}$.

2A. Sketch the graph of $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$

2B. Sketch the graph of $F(x) = \dfrac{|x|}{x}$.

3A. Sketch the graph of $H(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x < 0 \\ \sin(x) & \text{if } x \geq 0. \end{cases}$

3B. Sketch the graph of $K(x) = \sin\left(\dfrac{1}{x}\right)$.

Problems: Continuity

4. (a) Show that $f(x) = 2^x\left(x^3 - 5\right)$ is continuous on $(-\infty, \infty)$.

(b) Show that $g(x) = \dfrac{e^x}{\sin x}$ is continuous at $x = \dfrac{\pi}{2}$.

(c) Show that $h(x) = \cos(\ln x)$ is continuous on $(0, \infty)$.

5. Prove that the equation $2x + e^x = 3$ has a solution in the interval $(0, 1)$.

Problems: Mon 7/3

1. Let $f(x) = \dfrac{x-2}{x^2-2x}$.

(a) Sketch the graph of $f(x)$.

(b) Find $f(2)$, if it exists.

(c) Find $\lim_{x\to 2} f(x)$, if it exists.

(d) Is $f(x)$ continuous at $x = 2$? Give complete justification.

(e) Is $f(x)$ continuous at $x = 0$? Give complete justification.

(f) Is $f(x)$ continuous at $x = 3$? Give complete justification.

Problems: Wed 7/5

1. Evaluate $\lim_{x \to 2\pi} \frac{x^3}{\cos x}$

2. Evaluate $\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$

3. Evaluate $\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t}$

4. Show that $f(x) = \begin{cases} x^2 + 2 & \text{if } x > 0 \\ 2 - x & \text{if } x \leq 0 \end{cases}$ is continuous at $x = 0$.

5. Evaluate $\lim_{x \to 0} \frac{|x|}{x}$

6. Evaluate $\lim_{x \to 0} x^8 \arctan(x)$.

7. Evaluate $\lim_{x \to 0} x^8 \arctan\left(\frac{1}{x}\right)$.

Problems: Thu 7/6

1. Show that $f(x) = \dfrac{e^x}{\sin x}$ is continuous at $x = \dfrac{\pi}{2}$.

2. Show that $g(x) = 2^x\left(x^3 - 5\right)$ is continuous on $(-\infty, \infty)$.

3. Show that $F(x) = \begin{cases} \dfrac{\sin(\pi x)}{2 - x} & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ \ln(x^2) & \text{if } x > 1 \end{cases}$ is continuous on $(-\infty, \infty)$.

4. Prove that the equation $2x + 3^x = 4$ has a solution in the interval $(0, 1)$.

Problems: Vertical Asymptotes: Mon 7/10

1. Evaluate $\lim_{x\to 1-} \frac{x-2}{(x-1)^2}$ and $\lim_{x\to 1+} \frac{x-2}{(x-1)^2}$.

2. Evaluate $\lim_{x\to 3+} \ln(x^2-9)$.

3. Evaluate $\lim_{x\to 2\pi-} x\csc x$ and $\lim_{x\to 2\pi+} x\csc x$.

4. Find all vertical asymptotes of $h(x) = \frac{x^3-x}{x^2-6x+5}$.

Problems: Horizontal Asymptotes

5. Evaluate $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$

6. Evaluate $\lim_{x\to-\infty} \frac{x+2}{\sqrt{9x^2+1}}$

7. Evaluate $\lim_{x\to\infty} \frac{\sin^2 x}{x^3}$

Problems: Tue 7/11

1. Evaluate $\lim_{x\to\frac{\pi}{2}-} e^{\tan x}$ and $\lim_{x\to\frac{\pi}{2}+} e^{\tan x}$.

2. Find all vertical and horizontal asymptotes of $h(x) = e^{\frac{3}{x-2}}$.

3. Evaluate $\lim_{x\to\infty} \left(\sqrt{x^2+2x} - x\right)$.

4. Evaluate $\lim_{x\to 0} x^4 e^{|\cos(1/x)|}$

5. Let $f(x) = \begin{cases} x\sin(1/x) & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$

Is $f(x)$ continuous or discontinuous at $x = 0$? Fully justify your answer.

6. Evaluate $\lim_{x\to\infty} [\ln(2+\sin x) - \ln(x)]$.

Problems: Wed 7/12

Problem 1: Find the equation of the tangent line to $y = x^2$ at $x = 3$.

Solution: Let $f(x) = x^2$. The point is $(3, f(3)) = (\quad)$. By the Point-Slope Formula, the tangent line is:

The slope of the tangent line at $x = 3$ is:

$$m = f'(3) =$$

Problem 2: Find the equation of the tangent line to $y = \sqrt{x}$ at $x = 4$.

Solution: Let $f(x) = \sqrt{x}$. The point is $(4, f(4)) = (\quad)$. By the Point-Slope Formula, the tangent line is:

The slope of the tangent line at $x = 4$ is:

$$m = f'(4) =$$

Problem 3: Prove that if $f(x)$ and $g(x)$ have derivatives, then

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

Solution: Use the definition of derivative:

$$(f+g)'(x)=\lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}$$

$$=$$

$$=f'(x)+g'(x).$$

□

Problems: Mon 7/17

In problems #1 to #5, find the derivative of the given function.

1. $f(x) = x\cos(x) + \dfrac{3}{x^2}$.

2. $g(x) = x^2 e^x \sec(x)$.

3. $h(x) = 4\cos(x)\tan(x)$.

4E. $q(x) = 3x^3\left(\dfrac{1}{x^3} - x^5\right)\tan(x)$.

5. $r(x) = \dfrac{\cot x}{x^3+1} + \dfrac{5}{\sqrt[3]{x}}$

6E. Let $p(x) = x^{\pi}e^x + \dfrac{\sqrt[2]{x}}{\sqrt[4]{x}}$.
Find the equation of the tangent line to $y = p(x)$ at $x = 1$.

7. Suppose $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$.
Find $f'(0)$.

Problems: Tue 7/18

In all problems, find the derivative of the given function.

1. $F(x) = (4x - x^2)^{100}$

2. $f(x) = \sqrt[3]{1 + \tan x}$

3. $f(x) = \dfrac{1}{(x^2+1)^4}$

4. $f(x) = \sqrt{\dfrac{x}{x^2+4}}$

5. $f(x) = \sin(\cos(\tan x))$

6E. $r(x) = \sec(2x)\ln(\sin^2 x)$

7E. $g(x) = \log_3(\sec(10^{\pi x}))$

8E. $f(x) = \cos^{2017}(x \arctan x + 4\pi)$

Problems: Wed 7/19

Set A: Review

0. (HW #2) Find the constant k for which the function

$$f(x) = \begin{cases} 0.5x & \text{if } 0 \le x \le 1 \\ \sin(kx) & \text{if } 1 < x \le 5 \end{cases}$$

is continuous on the interval $[0, 5]$.

8E. Let $f(x) = \cos^{2017}(x \arctan x + 4\pi)$. Find $f'(x)$.

Set B: New Problems

1. Let $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 0$ and $f'(1) = 1$.
Find $h'(1)$.

2. Suppose $f(x)$ is one-to-one and $f(4) = 5$ and $f'(4) = 3$.
Find $(f^{-1})'(5)$.

3. (a) Find the 50th derivative of $g(x) = 2^x$
(b) Find the 99th derivative of $h(x) = \sin x$.
(c) Find a formula for the nth derivative of $f(x) = x^{-1}$

4. Find the values of x at which the curve $y = x^4 + 4x^3 - 8x^2 - 48x + 1$ has a horizontal tangent line.

Challenge. Suppose that $f(x)$ is a differentiable function satisfying three properties:
(a) $f(a + b) = f(a)f(b)$ for all real numbers a, b.
(b) $f(0) = 1$
(c) $f'(0) = 17$.
Find the function $\dfrac{d}{dx} \ln(f(x))$.

Problems: Thu 7/20

1. Let $F(x) = \begin{cases} 2x^2 & \text{if } x < 0 \\ 3x & \text{if } x \geq 0. \end{cases}$

(a) Show that $F(x)$ is continuous on $(-\infty, \infty)$.
(b) Show that $F(x)$ is **not** differentiable at $x = 0$.

2. Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that $f(x)$ is **not** differentiable at $x = 0$.

3. Let $g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(a) Show that $g(x)$ is differentiable for $x \neq 0$, and find $g'(x)$ for $x \neq 0$.
(b) Is $g(x)$ differentiable at $x = 0$? If so, find $g'(0)$.
(c) Is $g(x)$ continuous at $x = 0$?

For Fun

4. (a) Sketch the graph of a function which is continuous everywhere, but not differentiable at exactly two points.

(b) Sketch the graph of a function which is continuous everywhere, but not differentiable at infinitely many points.

4. (c) Are there functions which are continuous everywhere, but differentiable at **no** point?

If you say Yes: Can you describe such a function?

If you say No: Give a reason why. (i.e.: You must argue: If a function is continuous everywhere, then it must be differentiable somewhere.)

Challenge. Suppose that $|f(x)| \leq x^2$ for all real numbers x.

Show that $f(x)$ is differentiable at $x = 0$, and find $f'(0)$.

Problems: Mon 7/24

1. Sketch the solution sets of the following equations.
(a) $xy - x^2 = 0$
(b) $(y - x)^2 = 1$
(c) $\sin(x^2 + y^2) = 0$

2. In each of the following, regard y as an implicit function of x. Find dy/dx.
(a) $4\cos(x)\sin(y) = 1$
(b) $y = \ln(x^2 + y^2)$
(c) $e^y = x - y$

3. Suppose $x^4 + y^4 = 16$, and regard y as an implicit function of x. Find the second derivative y''.

4. Consider the astroid $x^{2/3} + y^{2/3} = 1$. Draw the segment of the tangent line to the astroid at the point (a, b) cut off by the x- and y-axes.
(a) Find the endpoints of this line segment.
(b) Show that this line segment has length 1, no matter what point (a, b) on the astroid is chosen.

Problems: Tue 7/25

Set A

4. Consider the astroid $x^{2/3} + y^{2/3} = 1$. Draw the segment of the tangent line to the astroid at the point (a, b) cut off by the x- and y-axes.

(a) Find the endpoints of this line segment.

(b) Show that this line segment has length 1, no matter what point (a, b) on the astroid is chosen.

Set B

In each of the following problems, find y'.

1. $y = (\cos x)^x$

2. $y = (\tan x)^{1/x}$

3. $y = (2x + 1)^5 (x^4 - 3)^6$

4. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

Problems: Wed 7/26

1. (a) Approximate $\sqrt{17}$ by using linear approximation.
(b) Is your approximation in (a) larger or smaller than the actual value?

2. Let $f(x) = e^{3x}$.

(a) Find the linear approximation (linearization) of $f(x)$ at $x = 0$.
Conclude that for $x \approx 0$, we have $e^{3x} \approx 1 + 3x$.

(b) Find the quadratic approximation of $f(x)$ at $x = 0$.
Conclude that for $x \approx 0$, we have $e^{3x} \approx 1 + 3x + \frac{9}{2}x^2$.

3. Consider the equation $\cos(x) + 10x = 2$.
(a) Show that this equation has a solution $x \in (0, \frac{\pi}{2})$.
(b) Approximate this solution by using linear approximation at $x = 0$.

Problems: Thu 7/27

Find the intervals on which the given function is increasing/decreasing. Identify any turning points.

1. $f(x) = 2x^3 + 3x^2 - 36x.$

2. $g(x) = \dfrac{x^2}{x^2 + 3}$

3. $h(x) = x^2 \ln(x)$

4. $p(x) = \ln(x^4 + 27)$

Problems: Mon 7/31

In each problem:

(a) Find the intervals on which the given function is increasing/decreasing. Identify any turning points.

(b) Find all the local maxima and local minima.

1. $f(x) = x^5 + x^4 - 3x^3 + 7$

2. $g(x) = \sqrt{x}\, e^{-x}$

3. $h(x) = \sqrt[3]{x}\,(x + \sqrt[3]{x})$

4. $p(x) = \sqrt{x^2 + 1} - x$

Problems: Tue 8/1

In each problem, sketch the graph of the function. Follow these steps:

(a) Find the intervals of increase and decrease. Identify any turning points.
(b) Identify local maxima and minima.
(c) Find the intervals where the function is concave up/down. Identify any inflection points.
(d) Find the vertical and horizontal asymptotes, if any.
(e) Sketch the function.

1. $f(x) = x^4 + 8x^3 + 200$

2. $g(x) = 3x^{2/3} - x$

3. $h(x) = x + \cos x$

4. $p(x) = e^{\arctan(x)}$

Problems: Wed 8/2

In each problem: Find the absolute maximum and absolute minimum values of the function on the given set.

1. $f(x) = (x^2 - 1)^3$ on $[-2, 3]$.

2. $g(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$.

3. $h(x) = x\sqrt{4 - x^2}$ on its domain.

Problems: Thu 8/3

1. A (right circular) cylinder is inscribed in a sphere of radius 3. Find the largest possible volume of such a cylinder.

2. A cylindrical can without a top is made to contain 64 cm^3 of liquid. Find the radius that will minimize the cost of the metal to make the can.

3. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

4. Find the area of the largest rectangle that can be inscribed in the ellipse $4x^2 + y^2 = 4$.

Problems: Mon 8/7

Set A

1. $\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)}$

2. $\lim_{x\to \pi} \frac{\sin x}{\cos x - 1}$

3. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

4. (a) $\lim_{x\to \infty} \frac{x}{\arctan(x)}$

(b) $\lim_{x\to 0} \frac{x}{\arctan(x)}$

Set B

1. $\lim_{x\to 0^+} x \ln x$

2. (a) $\lim_{x\to \infty} x^2 e^x$

(b) $\lim_{x\to -\infty} x^2 e^x$

3. $\lim_{x\to \frac{\pi}{2}-} (\sec x - \tan x)$

Problems: Tue 8/8

1. $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

2. $\lim_{x \to 0} x^{\tan x}$

3. Let $f(x) = \dfrac{\sin x}{x}$

(a) Find the vertical asymptotes of f, if there are any.
(b) Find the horizontal asymptotes of f, if there are any.

4. Let $g(x) = \sqrt{x} \ln x$.

(a) Find the vertical asymptotes of g, if there are any.
(b) Find the horizontal asymptotes of g, if there are any.
(c) Sketch the graph of g. Label any turning points and inflection points.

Problems: Wed 8/9

Steps:

- Draw a picture
- Label the variables
- Identify: "Given" info and "Wanted" info
- Use a geometric formula

1. A snowball melts so that its surface area decreases at a rate of 1 cm^2/min. Find the rate at which the radius decreases when the radius is 10 cm.

2. A trough is 10 feet long and its ends have the shape of isosceles triangles that are 3 feet across at the top and have a height of 1 foot.

If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 *inches* deep?

3. A triangle is changing in size. Its height is increasing at a rate of 1 cm/min, while its area is increasing at a rate of 2 cm^2/min.

At what rate is the base of the triangle changing when the height is 10 cm and the area is 100 cm^2?

4. The top of a ladder slides down a vertical wall at a rate of 0.15 m/sec. At the moment when the bottom of the ladder is 3 meters from the wall, it slides away from the wall at a rate of 0.2 m/sec.

How long is the ladder?

5. Water is leaking out of an inverted conical tank at a rate of 1000 cm^3/min. At the same time, water is being pumped into the tank at a constant rate. The tank has height 6 meters, and the radius at the top is 2 meters.

If the water level is rising at a rate of 20 cm/min when the height of the water is 2 meters, find the rate at which water is being pumped into the tank.

Geometric Formulas:

- Surface Area of Sphere: $A = 4\pi r^2$
- Area of Triangle: $A = \frac{1}{2}bh$
- Volume of Cone: $V = \frac{1}{3}\pi r^2 h$

Problems: Thu 8/10

Steps:

- Draw a picture
- Label the variables
- Identify: "Given" info and "Wanted" info
- Use geometric information (e.g.: similar triangles)

1. A spotlight on the ground shines on a wall 12 meters away. A man 2 meters tall walks from the spotlight toward the building at a speed of 1.5 m/sec.

How fast is the length of his shadow on the building decreasing when he is 4 meters from the building?

2. A kite 100 feet above the ground moves horizontally at a speed of 8 ft/sec. At what rate is the angle between the string and the horizontal decreasing when 200 feet of string has been let out?

3. Water is leaking out of an inverted conical tank at a rate of 1000 cm^3/min. At the same time, water is being pumped into the tank at a constant rate. The tank has height 6 meters, and the radius at the top is 2 meters.

If the water level is rising at a rate of 20 cm/min when the height of the water is 2 meters, find the rate at which water is being pumped into the tank.

- Volume of Cone: $V = \frac{1}{3}\pi r^2 h$