Stanford University Mathematics Department
Algebra PhD Qualifying Exam: Fall 2013 (Morning Part)

There are 5 questions in the morning and 5 in the afternoon. Each question is worth ten points. Use a separate bluebook for each question.

1. Let G be a finite group. By a *G-module* we mean a $\mathbb{C}[G]$-module of finite $\mathbb{C}$-dimension.

(a) (5 points) Let $V_1, \cdots, V_h$ be pairwise non-isomorphic irreducible G-modules. Let

$$V = \bigoplus_{i=1}^{h} d_i V_i$$

be a module that contains V_i exactly d_i times. Describe the structure of the ring $\mathrm{End}_G(V)$. (Briefly justify your answer.)

(b) (5 points) Suppose that V is a nonzero G-module that is not irreducible. Prove that the ring $\mathrm{End}_G(V \otimes V)$ is not commutative.

2. (a) (5 points) Prove that if $\gamma \in \mathrm{GL}(3, \mathbb{Z})$ has finite order n then $n \leqslant 6$. (**Hint:** What can you say about the eigenvalues of γ?)

(b) (5 points) Exhibit $\gamma \in \mathrm{GL}(3, \mathbb{Z})$ of order 6.

3. A *Dedekind domain* is a commutative ring that is a Noetherian integrally closed integral domain that is not a field and in which every nonzero prime ideal is maximal. For example, a PID that is not a field is a Dedekind domain.

(a) (6 points) If A is a Dedekind domain and I, J are nonzero ideals then prove that the natural surjective map $I \otimes_A J \longrightarrow IJ$ defined by $a \otimes b \longmapsto ab$ is an isomorphism. (You may use without proof the fact that every local Dedekind domain is a discrete valuation ring.)

(b) (4 points) Prove that the conclusion of (a) is false $A = \mathbb{C}[x, y]$ and each I, J is equal to the maximal ideal $\mathfrak{m} = (x, y)$.

4. (a) (3 points) Let $A \subseteq B$ be commutative rings. Define what it means for B to be *integral* over A. Prove that if B is finitely generated as an A-module then it is integral over A.

(b) (3 points) Prove that if $A \subseteq B$ are domains with B integral over its subring A then A is a field if and only if B is a field.

(c) (4 points) Let A be a domain with field of fractions K. Let L a finite extension of K. Let B be the set of elements of L which are integral over A. Prove that B is a subring of L and that L is its field of fractions.

5. Let V be the $\mathbb{R}$-vector space $\mathbb{R}^{2n}$ with the bilinear form $B : V \times V \longrightarrow \mathbb{R}$ given by

$$B(x, y) = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i).$$

The form B is nondegenerate; that is, $B(x, V) = 0$ implies $x = 0$. It is alternating; that is, $B(x, x) = 0$. Since the characteristic is not 2, this is equivalent to saying that it is skew-symmetric; that is, $B(x, y) = -B(y, x)$. If U is a vector subspace of V, let $U^\perp$ be the orthogonal complement; that is, $U^\perp = \{x \in V | B(x, U) = 0\}$.

(a) (2 points) Explain briefly why $\dim(U) + \dim(U^\perp) = 2n$.

(b) (4 points) A subspace W of V is called *isotropic* if $B(W, W) = 0$; that is, if $W \subseteq W^\perp$. Show that if W is isotropic of dimension n then there exists another isotropic subspace W' such that $V = W \oplus W'$.

(c) (4 points) Let G be the group of $g \in \mathrm{GL}(V)$ such that $B(gx, gy) = B(x, y)$ for all $x, y \in V$. Show that G is transitive on the set of isotropic subspaces of dimension n.

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There are 5 questions in the morning and 5 in the afternoon. Each question is worth ten points. Use a separate bluebook for each question.

1. Let us say that a subgroup H of a group G is a *malnormal* subgroup if $gHg^{-1} \cap H = \{1\}$ for all $g \in G - H$.

Let G be a finite group acting transitively on a set S. We call G a *Frobenius group* if no nontrivial element $g \neq 1$ of G fixes more than one element of S.

(a) (5 points) Choose $x \in S$ and set $H = \mathrm{Stab}_G(x) = \{g \in G | gx = x\}$. Prove that G is a Frobenius group if and only if H is a malnormal subgroup of G.

(b) (5 points) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q) \right\}$ and $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q) \right\}$. Prove that H is a malnormal subgroup of G. (**Hint:** Let $S = \mathbb{F}_q$.)

2. Let K be a field.

(a) (1 point) Compute

$$g \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} g^{-1} \quad \text{with} \qquad g = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix}, \qquad x, y, z \in K, d_i \in K^\times.$$

(b) (4 points) Let V be an $\mathbb{R}$-vector space of dimension n, and let $A : V \longrightarrow V$ be a linear transformation. Prove that the following are equivalent:

(i) $A^n = 0$.

(ii) For every $\varepsilon > 0$ there is a basis of V such that every entry of the matrix of A has absolute value $< \varepsilon$.

(c) (5 points) Explain how find representatives of the equivalence classes of nilpotent $n \times n$ matrices over K under conjugation by $\mathrm{GL}(n, K)$. (**Hint:** rational canonical form.)

3. A map of commutative rings $f : A \longrightarrow B$ is *faithfully flat* if it is flat and $N \otimes_A B \neq 0$ for all nonzero A-modules N. (There are other equivalent definitions, but use this one for the questions below.)

(a) (3 points) Prove that if A and B are local rings and f is a flat map that is local (i.e., $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$) then f is faithfully flat.

(b) (4 points) Let $\mathrm{Spec}(R)$ be the set of prime ideals of a commutative ring R. Prove that f is faithfully flat if and only if f is flat and the map $\mathrm{Spec}(f) : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$ is surjective. (**Hint:** for $\Leftarrow$, apply (a) to a suitable localization.

(c) (3 points) Assume f is faithfully flat. For submodules M, M' of N prove $M \subset M'$ inside N if $M \otimes_A B \subset M' \otimes_A B$ inside $N \otimes_A B$. (**Hint:** Consider $M + M' \subset N$.) Deduce that if $a, a' \in A$ satisfy $f(a)|f(a')$ in B then $a|a'$ in A.

4. The *discriminant* of a polynomial $f(X) = \prod_{i=1}^{n}(X - r_i)$ is

$$\prod_{i<j}(r_i - r_j)^2.$$

Let $f(X) \in \mathbb{Q}[X]$ be a monic irreducible polynomial of degree 4 with roots $\alpha, \beta, \gamma, \delta$.

(a) (3 points) Prove that $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$, and $\alpha\delta + \beta\gamma$ are the roots of a monic cubic polynomial $g(X) \in \mathbb{Q}[X]$ whose discriminant is the same as the discriminant of f.

(b) (2 points) If $f \in \mathbb{Z}[X]$ prove that $g \in \mathbb{Z}[X]$.

(c) (5 points) Explain briefly why the Galois group of f over $\mathbb{Q}$ is one of the five groups S_4, A_4, Z_4, D_4, or $Z_2 \times Z_2$ (where Z_n is the cyclic group of order n). In which cases is g an irreducible polynomial?

5. Let k be an algebraically closed field.

(a) (3 points) State the Nullstellensatz for ideals in $k[x_1, \ldots, x_n]$, and explain why a pair of radical ideals $J, J' \subset k[x_1, \ldots, x_n]$ with associated zero loci $Z, Z' \subset k^n$ satisfy $J \supset J'$ if and only if $Z \subset Z'$. Use this to describe the maximal ideals in $k[x_1, \ldots, x_n]$.

(b) (3 points) Define the Zariski topology on k^n and check it is a topology.

(c) (4 points) Let $J \subset k[x_1, \ldots, x_n]$ be a radical proper ideal. Prove that the corresponding non-empty zero locus $Z \subset k^n$ is irreducible for the Zariski topology (i.e., not a union of two proper closed subsets) if and only if J is a prime ideal.