1. MORNING (SPRING, 2012)

(1) Let R be a finite-dimensional algebra over a field k.

(a) Prove that if R is a commutative integral domain, then R is a field.

(b) Suppose R is not commutative. Prove that $rs = 1$ implies $sr = 1$.

(2) Suppose A is a (commutative) ring, and M is an A-module.

(a) If $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of A, must $M = 0$? Prove or disprove.

(b) If M is finitely generated and $M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of A, prove that $M = 0$. Give a counterexample if M is not finitely generated.

(3) Let E/k be a finite-degree extension of fields.

(a) Prove that $\mathrm{Aut}(E/k)$ has at most $[E : k]$ elements.

(b) If E is a finite field, prove that $\mathrm{Aut}(E/k)$ is cyclic and the norm $E^\times \to k^\times$ is surjective.

(c) Give an example of a finite cyclic extension such that the norm is not surjective.

(4) Let G be a finite group, F a field, and V a nonzero finite-dimensional F-linear representation of G.

(a) Give an example of G, F, and V such that V does not decompose as a direct sum of irreducible F-linear representations of G.

(b) Suppose that the order of G is not zero in F. Prove that V is a direct sum of irreducible F-linear representations of G.

(5) Let $C_\bullet$ be a complex of free abelian groups, with differential lowering degree by 1. Let A be an abelian group.

(a) Construct a short exact sequence

$$0 \to H_n(C_\bullet) \otimes_{\mathbf{Z}} A \to H_n(C_\bullet \otimes_{\mathbf{Z}} A) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{n-1}(C_\bullet), A) \to 0.$$

(Hint: use a short free abelian group resolution of A.)

(b) Suppose that $H_n(C_\bullet) = 0$ except for $n = 0$, and $H_0(C_\bullet) \cong \mathbf{Z}_{(5)} \oplus \mathbf{Z}/5\mathbf{Z}$. Here $\mathbf{Z}_{(5)}$ is the localization of $\mathbf{Z}$ at the prime (5). Compute the homology groups of $C_\bullet \otimes_{\mathbf{Z}} (\mathbf{Q}/\mathbf{Z})$.

(6) (a) State and prove the Hilbert basis theorem.

(b) Let A be a Noetherian ring, and J an ideal of A. Define the ring $G_J(A) = A \oplus J \oplus J^2 \oplus \dots$, in which the product of J^m and J^n is the usual product valued in the direct summand J^{n+m}. Prove that $G_J(A)$ is Noetherian.

(7) Let k be a field. Prove that for any $M \in \mathrm{Mat}_n(k)$ ($n \geq 1$), the transpose M^T of M is conjugate to M in $\mathrm{Mat}_n(k)$ (i.e., $M^T = gMg^{-1}$ for some $g \in \mathrm{GL}_n(k)$).

(8) Let $f : A \to B$ be a ring homomorphism.

(a) Define what it means to say that B is integral over A, and prove this holds when B is finitely generated as an A-module.

(b) If B is finitely generated as an A-module, prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed map. (Hint: reduce to the case where f is injective.)

(9) Let p be an odd prime. In this question, ζ_p denotes a primitive pth root of unity.

(a) Describe $\mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ and determine all primes p such that $\mathbf{Q}(\zeta_p)$ contains a subfield L whose Galois group over $\mathbf{Q}$ is isomorphic to $\mathbf{Z}/5\mathbf{Z}$.

(b) Prove that there exists a finite Galois extension E of $\mathbf{Q}$ such that $\mathrm{Gal}(E/\mathbf{Q}) \simeq \mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z}$ by constructing E as a subfield of an explicit Galois extension $F/\mathbf{Q}$, and explicitly describe the subgroup $\mathrm{Gal}(F/E) \subset \mathrm{Gal}(F/\mathbf{Q})$.

(10) (a) Let G be a group and H a subgroup of finite index $n > 0$. Prove that G contains a normal subgroup of index at most $n!$. (Hint: think about homomorphisms from G to S_n.)

(b) Let G be a group which is generated by two elements. Prove that G has at most 17 subgroups of index 3. (Hint: think about homomorphisms from G to S_3.)