Ph.D. Qualifying Exam, Real Analysis

Fall 2010, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose $(X, \|.\|_X)$, $(Y, \|.\|_Y)$ are Banach spaces, and Y is a subspace of X with the inclusion $\iota : Y \to X$ is continuous in the respective Banach space topologies. Suppose that $T_n \in \mathcal{L}(X)$ for $n \in \mathbb{N}$. Suppose moreover that for each $x \in X$ and $n \in \mathbb{N}$ one has $T_n x \in Y$, and in addition that for each $x \in X$ there exists C (independent of n) such that $\|T_n x\|_Y \leq C$. Show that for all n, $T_n \in \mathcal{L}(X, Y)$, and show that there exists C such that for all n one has $\|T_n\|_{\mathcal{L}(X,Y)} \leq C$.

2 Consider the spaces $L^p([0,1])$, $1 \leq p < \infty$. For which p is the unit ball, $\{f \in L^p : \|f\|_{L^p} \leq 1\}$, weakly sequentially compact, i.e. for which p is it true that if $\{f_n\}_{n=1}^{\infty}$ is a sequence in the unit ball in L^p then it has a weakly convergent subsequence? For each p, either prove or disprove weak sequential compactness.

3

a. Let f be a measurable real-valued function on a finite measure space $(X, \mathcal{B}, \mu)$. Define

$$m_n(f) = \mu\left(\{x : 2^n \leq |f(x)| < 2^{n+1}\}\right),$$

for $n \in \mathbb{Z}$. Give and prove a (non-trivial) upper and lower estimate of the L^p norm of f, $1 \leq p < \infty$, purely in terms of the quantities $m_n(f)$.

b. Suppose that $(X, \mathcal{B}, \mu)$ is a σ-finite measure space, K is a measurable function on $X \times X$, and

$$\int |K(x,y)|\, d\mu(y) \leq C, \quad \int |K(x,y)|\, d\mu(x) \leq C$$

μ-a.e. Show that the integral operator $A : L^2(X) \to L^2(X)$ defined by

$$(Af)(x) = \int K(x,y)\, f(y)\, d\mu(y)$$

is well-defined and bounded, and its norm is bounded by C.

4 Suppose that X is a complex Banach space and $\mathcal{T}$ is its weak topology.

a. Suppose that $(X, \mathcal{T})$ is first countable. Show that there are linear functionals $f_j \in X^*$, $j = 1, 2, \ldots$, such that every $f \in X^*$ is a *finite* linear combination of the f_j. That is, if $f \in X^*$ then there exists $N > 0$ and $a_j \in \mathbb{C}$, $j = 1, \ldots, N$, such that $f = \sum_{j=1}^{N} a_j f_j$.

b. Suppose that X is infinite dimensional. Show that $(X, \mathcal{T})$ is not metrizable.

5 We define a bounded operator $A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$(Ax)_k = x_{k-1} - 2x_k + x_{k+1}.$$

a. Show that A is a bounded symmetric operator.

b. Let $T : \ell^2(\mathbb{Z}) \to L^2([-\pi, \pi])$ be defined by

$$(Tx)(t) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} x_k\, e^{ikt}.$$

Show that the operator $TAT^{-1} : L^2([-\pi, \pi]) \to L^2([-\pi, \pi)]$ is a multiplication operator; that is,

$$(TAT^{-1} f)(t) = \mu(t)\, f(t)$$

for some function $\mu(t)$.

c. Determine the spectrum of A.

d. Find the eigenvalues of A.

Ph.D. Qualifying Exam, Real Analysis

Fall 2010, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Suppose that f is a compactly supported continuous function on $\mathbb{R}^n$ (i.e. f vanishes outside a compact set), and suppose that its Fourier transform $\hat{f}$, given by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$, vanishes on a non-empty open set. Show that f is identically 0.

b. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $1 < p < \infty$. Suppose that $h \in L^p(\mathbb{T})$, h is non-zero a.e., and let

$$V = \{Ph : \ P \text{ a trigonometric polynomial}\} \subset L^p(\mathbb{T}).$$

Show that V is dense in $L^p(\mathbb{T})$.

2 Let X denote the vector space of all sequences $\{a_n : n \in \mathbb{N}\}$ with $\sum_{n=1}^{\infty} n\,|a_n|^2 < \infty$.

a. Prove or disprove: the set X is a dense subset of $\ell^2(\mathbb{N})$.

b. Prove or disprove: the set X is a dense subset of $\ell^\infty(\mathbb{N})$.

3 Write a real number $x \in [0, 1)$ in the usual decimal expansion (pick the representation ending in 0's if there are two representations), $x = 0.x_1x_2x_3\ldots$. We let A be the set of $x \in [0, 1)$ with the property that there are infinitely many $n \in \mathbb{N}$ such that each of the digits $0, \ldots, 9$ appears among the first $10n$ digits (i.e. $x_1, ..., x_{10n}$) exactly n times. Prove that the set A is Lebesgue measurable and find its measure.

4 Suppose that $\mathcal{H}$ is a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$, and let T^* denote its adjoint.

a. Show that $\mathrm{Ker}(T) \oplus \overline{\mathrm{Ran}(T^*)} = \mathcal{H}$, where $\oplus$ is orthogonal direct sum.

b. Suppose that there exists $C > 0$ such that for all $x \in \mathcal{H}$, $\|x\| \leq C\|Tx\|$. Show that $\mathrm{Ran}(T)$ is a closed subspace of $\mathcal{H}$.

c. Show that if $TT^* = I = T^*T$, then $T - \lambda I \in \mathcal{L}(H)$ is invertible if $|\lambda| \neq 1$, and show that $\|(T - \lambda I)^{-1}\| \leq |1 - |\lambda||^{-1}$.

5 Let $\Omega_+ = \{z \in \mathbb{C} : \ 0 < \mathrm{Im}\, z < 1\}$, $\Omega_- = \{z \in \mathbb{C} : \ -1 < \mathrm{Im}\, z < 0\}$. Let $S(\mathbb{R})$ denote the space of Schwartz functions on $\mathbb{R}$, with seminorms $\rho_{k,l}(\phi) = \sup\{|x^l(\partial^k\phi)(x)| : \ x \in \mathbb{R}\}$, and $S'(\mathbb{R})$ its topological dual, tempered distributions.

a. Suppose that $u_+ : \Omega_+ \to \mathbb{C}$ is an analytic function with $|u_+(z)| \leq C(|\,\mathrm{Im}\, z|^{-k} + |\,\mathrm{Re}\, z|^{\ell} + 1)$ for some C, k, ℓ. For $\epsilon \in (0, 1)$, let $u_{+,\epsilon} \in S'(\mathbb{R})$ with $u_{+,\epsilon}(\phi) = \int_{\mathbb{R}} u(x + i\epsilon)\phi(x)\,dx$. Show that $u_{+,0} = \lim_{\epsilon\to 0+} u_\epsilon$ exists in $S'(\mathbb{R})$. (Hint: consider the indefinite integral of u from e.g. $z_0 = i/2$, and integrate first parallel to the real axis then to the imaginary axis and obtain an estimate for $\int_{z_0}^{z} u(w)\,dw$.) Define $u_{-,0}$ similarly, replacing Ω_+ by Ω_-.

b. For $u_\pm(z) = z^{-m}$, $z \in \Omega_\pm$, $m \geq 1$ integer, find $u_{+,0}(\phi) - u_{-,0}(\phi)$, $\phi \in S(\mathbb{R})$, in terms of $\partial^j\phi(0)$, $j \geq 0$.