Ph.D. Qualifying Exam, Real Analysis

Spring 2013,   part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.

a. Show that if X is a Banach space and $X^* = X^{***}$ (under the natural inclusion) then $X = X^{**}$.

b. Show that if $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$ then there exists $\phi \in C_c^\infty(\mathbb{R}^n)$ (compactly supported infinitely differentiable function) such that $\phi(x_0) = 1$ and $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : |x - x_0| < \epsilon\}$.

2

a. Suppose f, g are positive measurable functions on $[0,1]$ and $f(x)g(x) \geq 1$ for $x \in [0,1]$. Show that

$$\int f(x)\,dx \int g(x)\,dx \geq 1.$$

b. Suppose that $(X, \mathcal{B}, \mu)$ is a σ-finite measure space, K is a measurable function on $X \times X$, and

$$\int |K(x,y)|\,d\mu(y) \leq C, \ \int |K(x,y)|\,d\mu(x) \leq C$$

μ-a.e. Show that the integral operator $A : L^2(X) \to L^2(X)$ defined by

$$(Af)(x) = \int K(x,y)\,f(y)\,d\mu(y)$$

is well-defined and bounded, and its norm is bounded by C.

3 Let X be a complex vector space. Suppose that $\{\rho_\alpha : \alpha \in A\}$ is a collection of seminorms on X such that for each $x \in X \setminus \{0\}$ there is $\alpha \in A$ such that $\rho_\alpha(x) \neq 0$, and $B : X \times X \to \mathbb{C}$ is a (jointly) continuous bilinear map in the locally convex topology generated by the ρ_α. Show that there exist $\alpha_1, \ldots, \alpha_n \in A$, $C > 0$, such that for all $x, y \in X$,

$$|B(x,y)| \leq C(\rho_{\alpha_1}(x) + \ldots + \rho_{\alpha_n}(x))(\rho_{\alpha_1}(y) + \ldots + \rho_{\alpha_n}(y)).$$

4 Suppose u is a distribution (an element of the dual of C^∞) on the circle $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Show that there exists a function $f \in C(\mathbb{T})$, $k \geq 0$ integer and $c \in \mathbb{C}$ such that $u = \frac{d^k}{dx^k} f + c$, where $\frac{d^k}{dx^k}$ is the kth distributional derivative. (As usual, $C(\mathbb{T})$ is regarded as a subset of the set $\mathcal{D}'(\mathbb{T})$ of distributions.)

5 For each of the following maps $f : \mathbb{R} \to X$, where X is a topological vector space, prove or disprove that the map is continuous, respectively differentiable. Here differentiability is the existence, for all $t \in \mathbb{R}$, of the limit $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ in the space X. We write $f(t) = f_t$ below.

a. $X = L^2(\mathbb{R})$, with standard norm, and $f_t(x) = \chi_{[t,t+1]}(x)$, $\chi_{[t,t+1]}$ the characteristic (or indicator) function of $[t, t+1]$.

b. $X = L^2(\mathbb{R})$, with standard norm, and $f_t(x) = \sin(x - t)$ if $t \leq x \leq t + \pi$, $f_t(x) = 0$ otherwise.

c. $X = \mathcal{S}'(\mathbb{R})$ (tempered distributions, the dual of Schwartz functions, $\mathcal{S}(\mathbb{R})$), with the weak-* topology, and $f_t = \delta_t$, the delta distribution at t.

Ph.D. Qualifying Exam, Real Analysis

Spring 2013, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose F, F_n, $n \geq 1$ integer, are increasing functions from the interval $[a,b]$, $a<b$, to $\mathbb{R}$ such that for all $x \in [a,b]$, $F(x) = \sum_{n=1}^{\infty} F_n(x)$. Prove that $F'(x) = \sum_{n=1}^{\infty} F_n'(x)$ almost everywhere with respect to the Lebesgue measure.

2 Suppose that $1 < p < \infty$, $f, f_n \in L^p([0,1])$, $n \in \mathbb{N}$, $\|f_n\|_{L^p} \leq 1$ for all n, and $f_n \to f$ almost everywhere. Show that $f_n \to f$ weakly and $\|f\|_{L^p} \leq 1$.

3 Suppose X is a separable Hilbert space.

a. Suppose $T \in \mathcal{L}(X)$ is compact and $T^* = T$. Show that there is a complete orthonormal set in X consisting of eigenvectors of T.

b. Give an example (with proof) of a non-selfadjoint $T \in \mathcal{L}(X)$ which is compact and which is such that the spectrum of T is $\{0\}$ but T has no eigenvectors.

4 Let X be an uncountable set equipped with the discrete topology. Let $\hat{X}$ be the one point compactification of X, and let $C(\hat{X})$ be the Banach space of real-valued continuous functions on $\hat{X}$.

a. Find (with proof) the σ-algebra of Baire sets (generated by compact G_δ sets) and the σ-algebra of Borel sets (generated by open sets).

b. Find a σ-subalgebra $\mathcal{B}$ of the Borel sets which contains the Baire sets and two distinct finite measures μ_1, μ_2 on $\mathcal{B}$ such that $\int f\,d\mu_1 = \int f\,d\mu_2$ for all $f \in C(\hat{X})$. Explain why the existence of these does not contradict the Riesz representation theorem concerning the dual of $C(\hat{X})$.

5 Suppose that $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$, $a_\alpha \in \mathbb{C}$, is a polynomial of degree m on $\mathbb{R}^n$; here for $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. Let $P(D)$ be the corresponding differential operator, $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, $D_j = -i\partial_j$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. We say that P is elliptic of order m if $\mathbb{R}^n \ni \xi \neq 0$ implies $\sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0$. Suppose that P is elliptic of order m.

Recall also that for $m \geq 0$, $H^m(\mathbb{T}^n)$ is the subset of $L^2(\mathbb{T}^n)$ consisting of functions whose Fourier coefficients satisfy $\sum_{k \in \mathbb{Z}^n} (1+|k|^2)^m |\hat{f}(k)|^2 < \infty$. Here $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\hat{f}(k) = (2\pi)^{-n/2} \int e^{-ix\cdot k} f(x)\,dx$, $k \in \mathbb{Z}^n$.

a. Show that with P considered as a map $P : H^m(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$, the nullspace of P is finite dimensional and is a subset of $C^\infty(\mathbb{T}^n)$.

b. Show that P is invertible as such a map if and only if it is injective.