Ph.D. Qualifying Exam, Real Analysis

Fall 2015, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose that A is a Borel set in $\mathbb{R}$ with the property that if $x \in A$ and if y is any other number such that the decimal expansions of x and y differ in at most finitely many places, then $y \in A$. Prove that the Lebesgue measure of either A or $\mathbb{R} \setminus A$ is 0.

2 Let T be a non-zero compact operator acting on a Hilbert space $\mathcal{H}$.

a. Give an example, with proof, of such an operator such that spec $(T) = \{0\}$.

b. Prove that this is impossible if T is self-adjoint.

3 Show that there exists a compactly supported C^∞ function ϕ on $\mathbb{R}$ such that $\phi \geq 0$, $\phi(0) > 0$, and the Fourier transform $\mathcal{F}\phi$ of ϕ is non-negative, where $(\mathcal{F}\phi)(\xi) = \int e^{-ix\xi}\phi(x)\,dx$. (Hint: when is the Fourier transform of a function real?)

4

a. Suppose $f \in C^2(\mathbb{R}^2)$. Show that the mixed partials (partial derivatives are defined as limits of difference quotients) are equal: $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$.

b. Given an example of $f \in C^1(\mathbb{R}^2)$ such that there is a point $x_0 \in \mathbb{R}^2$ at which the mixed partials $\frac{\partial^2 f}{\partial x\,\partial y}$, $\frac{\partial^2 f}{\partial y\,\partial x}$ exist, but are not equal.

c. Show that if $f \in L^1(\mathbb{R}^2)$ then $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$, where the partial derivatives are now defined in the distributional sense.

5 Let $f(x) = x^4 - 2x^2 + 1$, $x \in \mathbb{R}$. Prove that there exists an $\alpha \in \mathbb{R}$ (and find its value) such that

$$C^{-1}\lambda^{-\alpha} \leq \left| \int_{-\pi}^{\pi} e^{i\lambda f(x)} \sin^4 x\,dx \right| \leq C\lambda^{-\alpha}$$

as $\lambda \to +\infty$, for some $C > 0$.

Ph.D. Qualifying Exam, Real Analysis

Fall 2015, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Let $f \in L^1([0,1])$, and suppose that $\int_0^1 f\varphi^{(n)} = 0$ for every $\varphi \in C_c^\infty(0,1)$, where $\varphi^{(n)}$ is the n-th derivative. Show that f is a polynomial of degree at most $n-1$. (Hint: Approximate f by smooth functions using convolutions.)

2 Let ℓ^2 be the Hilbert space of square summable sequences, and $\mathcal{H}$ be the subspace consisting of sequences $\{x_n\}_{n=1}^\infty$ with $\sum_{n=1}^\infty n^2|x_n|^2 < \infty$. Show that $\mathcal{H}$ is of the first category in ℓ^2.

3

a. Give an example (with proof) of a sequence $\{A_n\}_{n=1}^\infty$ of bounded linear operators on ℓ^2 such that (i) $A_n \to 0$ in the strong operator topology, but not in norm, and (ii) $A_n \to 0$ in the weak operator topology, but not in the strong operator topology.

b. Show that the map $A \mapsto A^*$ is continuous in the norm topology and in the weak operator topology, but not in the strong operator topology.

4 Suppose that H_1 and H_2 are separable Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. Suppose that there exist $B \in \mathcal{L}(H_2, H_1)$ and compact operators E_j on H_j, $j = 1, 2$, such that $BA = I_1 - E_1$, $AB = I_2 - E_2$, where I_j is the identity operator on H_j. Show that the nullspace of A is finite dimensional, the range of A is closed in H_2, and its orthocomplement is finite dimensional.

5 Let $A \subset [0,1]$ be the middle third Cantor set: $A = \cap_n A_n$, where $A_0 = [0,1]$ and A_n is obtained from A_{n-1} by removing the middle third of each component interval. Show that if $f \in C([0,1])$, then $\lim_{n\to\infty} m(A_n)^{-1} \int_{A_n} f$ exists, where $m(A_n)$ is the Lebesgue measure. Show moreover that there is a Borel measure μ on $[0,1]$ such that $\lim_{n\to\infty} m(A_n)^{-1} \int_{A_n} f = \int_{[0,1]} f\, d\mu$.