(2) Find an area-preserving transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(P,Q) = f(p,q)$, if its graph is given by the generating function $F(q,P) = (q+q^3)P$.

That is, the graph of the area-preserving map $f$ in $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq - dP \wedge dQ)$ is given by the generating function $F$ with respect to the polarization of $\mathbb{R}^4$ by the coordinate planes $(q,P)$ and $(p,Q)$.

*Solution:* Equip $\mathbb{R}^4$ with the symplectic form $\omega = dp \wedge dq - dP \wedge dQ$. Note that $\omega = d\alpha$, where $\alpha = p\,dq + Q\,dP$.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving map such that $\text{Graph}(f) \subset \mathbb{R}^4$ has generating function $F(q,P) = (q+q^3)P$. Since $f$ is area-preserving, so $\alpha|_{\text{Graph}(f)}$ is exact (why?). Saying that $F$ is a generating function for $\text{Graph}(f)$ means that

$$\alpha|_{\text{Graph}(f)} = dF.$$

Thus,

$$p\,dq + Q\,dP = \frac{\partial F}{\partial q}\,dq + \frac{\partial F}{\partial P}\,dP.$$

Since $\{dq, dP\}$ are assumed linearly independent on $\text{Graph}(f)$, this forces

$$\frac{\partial F}{\partial q} = p$$
$$\frac{\partial F}{\partial P} = Q.$$

Since $F(q,P) = qP + q^3P$, we thereby obtain

$$(1+3q^2)P = p \qquad \Longrightarrow \qquad P = \frac{p}{1+3q^2}$$
$$q + q^3 = Q \qquad \Longrightarrow \qquad Q = q + q^3.$$

Thus, the desired map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is

$$f(p,q) = \left( \frac{p}{1+3q^2},\ q+q^3 \right). \qquad \Diamond$$

*Remark:* We note that $f$ is area-preserving by construction. (And in fact, one can verify directly that our choice of $f$ does satisfy $f^*\omega = \omega$.)

(3) Verify the following properties of the Poisson bracket: (i) Skew-symmetry; (ii) Leibniz rule; (iii) Jacobi Identity.

*Solution:* Let $(M, \omega)$ be a symplectic manifold. The Poisson bracket $\{f, g\}$ of two functions $f, g \colon M \to \mathbb{R}$ is defined by

$$\{f, g\} = dg(X_f) = -df(X_g) = \omega(X_f, X_g) = X_f g = -X_g f.$$

Here, $X_f$ is the Hamiltonian vector field of the function $f$, i.e.: $X_f \lrcorner \omega = -df$.

(i) **Skew-symmetry.** This follows from $\{f, g\} = \omega(X_f, X_g) = -\omega(X_g, X_f) = \{g, f\}$.

(ii) **Leibniz rule.** This follows from

$$\begin{aligned} \{f, gh\} = d(gh)(X_f) = (h\, dg + g\, dg)(X_f) &= h\, dg(X_f) + g\, dh(X_f) \\ &= \{f, g\}h + \{f, h\}g. \end{aligned}$$

(iii) **Jacobi identity.** First, note that for any 2-form $\beta$, and any vector fields $X, Y, Z$:

$$\begin{aligned} d\beta(X, Y, Z) = X\beta(Y, Z) &- Y\beta(X, Z) + Z\beta(X, Y) \\ &- \beta([X, Y], Z) + \beta([X, Z], Y) - \beta([Y, Z], X). \end{aligned}$$

Since $\omega$ is a closed 2-form, we have

$$\begin{aligned} 0 = X_f\omega(X_g, X_h) &- X_g\omega(X_f, X_h) + X_h\omega(X_f, X_g) \\ &- \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f). \end{aligned} \tag{$\star$}$$

Now, note that

$$\begin{aligned} \{\{f, g\}, h\} &= -X_h\{f, g\} = -X_h\omega(X_f, X_g) \\ \{\{g, h\}, f\} &= -X_f\{g, h\} = -X_f\omega(X_g, X_h) \\ \{\{h, f\}, g\} &= -X_g\{h, f\} = -X_g\omega(X_h, X_f) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \omega([X_f, X_g], X_h) &= X_f X_g h - X_g X_f h \\ \omega([X_f, X_h], X_g) &= X_f X_h g - X_h X_f g \\ \omega([X_g, X_h], X_f) &= X_g X_h f - X_h X_g f. \end{aligned} \tag{2}$$

Inserting (1) and (2) into ($\star$), we obtain

$$\begin{aligned} 0 &= -\{\{f, g\}, h\} - \{\{g, h\}, f\} - \{\{h, f\}, g\} \\ &\quad + (X_g X_f h - X_f X_g h) + (X_f X_h g - X_h X_f g) + (X_h X_g f - X_g X_h f) \\ &= -\{\{f, g\}, h\} - \{\{g, h\}, f\} - \{\{h, f\}, g\} \\ &\quad + 2(X_h X_g f + X_f X_h g + X_g X_f h) \\ &= \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}, \end{aligned}$$

as desired. $\Diamond$

(4) Suppose that $\mathbb{R}^2$ is endowed with an area form $\omega = dp \wedge dq$. Let $H_t \colon \mathbb{R}^2 \to \mathbb{R}$, $t \in [0,1]$, be a family of smooth functions equal to 0 outside the unit disk $D$. Let $X_t := X_{H_t}$ be the Hamiltonian vector field generated by $H_t$, i.e.: $X_t \lrcorner \omega = -dH_t$. Let $f_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the flow of area-preserving transformations generated by $X_t$, i.e: $\frac{df_t}{dt}(x) = X_t|_{f_t(x)}$.

Let $z_0 \in \mathrm{Int}(D)$ be a fixed point of $f_1$, i.e: $f_1(z_0) = z_0$. Let $\gamma \colon [0,1] \to \mathbb{R}^2$ denote the loop defined by $\gamma(t) = f_t(z_0)$, $t \in [0,1]$. Then the integral $S(z_0) := \int_\gamma p\,dq - H_t\,dt$ is called the *action* of the fixed point $z_0$.

Prove that for any path $\delta \colon [0,1] \to \mathbb{R}^2$ such that $\delta(0) \in \mathbb{R}^2 - D$ and $\delta(1) = z_0$, one has

$$\int_{f_1(\delta)} p\,dq - \int_\delta p\,dq = S(z_0).$$

In particular, the integral on the left-hand side of the equation is independent of the choice of the path $\delta$, so that the action depends only on $f_1$, and not on the choice of the Hamiltonian $H_t$ which generates it.

*Solution:* Let $G(t) := \int_{f_t(\delta)} p\,dq$. Then

$$\begin{aligned}\int_{f_1(\delta)} p\,dq - \int_\delta p\,dq = G(1) - G(0) = \int_0^1 G'(t)\,dt &= \int_0^1 \frac{d}{dt}\int_{f_t(\delta)} p\,dq\,dt \\ &= \int_0^1 \frac{d}{dt}\int_\delta f_t^*(p\,dq)\,dt \\ &= \int_0^1 \int_\delta \frac{\partial}{\partial t} f_t^*(p\,dq)\,dt.\end{aligned}$$

We calculate

$$\begin{aligned}\frac{\partial}{\partial t} f_t^*(p\,dq) = f_t^*\mathcal{L}_{X_t}(p\,dq) &= f_t^*[d(X_t \lrcorner p\,dq) + X_t \lrcorner \omega] \\ &= f_t^*[d(X_t \lrcorner p\,dq) - dH_t] \\ &= d[f_t^*(X_t \lrcorner p\,dq - H_t)].\end{aligned}$$

So,

$$\begin{aligned}\int_\delta \frac{\partial}{\partial t} f_t^*(p\,dq) = \int_\delta d[f_t^*(X_t \lrcorner p\,dq - H_t)] &= f_t^*(X_t \lrcorner p\,dq - H_t)|_{\delta(0)}^{\delta(1)} \\ &= f_t^*(X_t \lrcorner p\,dq - H_t)(z_0) \\ &= (X_t \lrcorner p\,dq - H_t)(\gamma(t)) \\ &= \gamma^*(X_t \lrcorner p\,dq - H_t).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{f_1(\delta)} p\,dq - \int_\delta p\,dq &= \int_0^1 \gamma^*(X_t \lrcorner p\,dq - H_t) \\ &= \int_\gamma (X_t \lrcorner (p\,dq))\,dt - \int_\gamma H_t\,dt.\end{aligned}$$

We now note that

$$\begin{aligned}(X_t \lrcorner (p\,dq))\,dt &= \left(\frac{df_t}{dt} \lrcorner (p\,dq)\right) dt \\ &= \left(\frac{\partial}{\partial t} \lrcorner f_t^*(p\,dq)\right) dt \\ &= \frac{\partial}{\partial t} \lrcorner (f_t^*(p\,dq) \wedge dt) + f_t^*(p\,dq).\end{aligned}$$

But for any 2-form $\beta$ and any tangent vector $Y$ to the curve $\gamma$, we have $(Y \lrcorner \beta)|_\gamma = 0$. In particular, $\frac{\partial}{\partial t} \lrcorner (f_t^*(p\,dq) \wedge dt)\,|_\gamma = 0$. Therefore, we conclude that

$$\begin{aligned}\int_{f_1(\delta)} p\,dq - \int_\delta p\,dq &= \int_\gamma (X_t \lrcorner (p\,dq))\,dt - \int_\gamma H_t\,dt \\ &= \int_\gamma f_t^*(p\,dq) - \int_\gamma H_t\,dt \\ &= \int_\gamma p\,dq - \int_\gamma H\,dt \\ &= S(z_0),\end{aligned}$$

where the equality $\int_\gamma f_t^*(p\,dq) = \int_\gamma p\,dq$ follows from the fact that $f_t$ is area-preserving. $\Diamond$