# REAL ANALYSIS QUALIFYING EXAM SYLLABUS

**Remark:** This is only a rough guide for the qual topics. Some additional topics may show up on the exams, but it is expected that most problems will be on the listed topics. In particular, additional topics covered in 205AB in the preceding quarters will be covered.

## 1. Measure theory

(1) The basics of measure theory: outer measure, measure, measurability, $\sigma$-algebra, Lebesgue measure on $\mathbb{R}^n$, Borel and Radon measures, approximation by open and closed sets.
(2) Measurable functions: measurability of $\inf, \sup, \liminf, \limsup$ of sequences and algebraic operations, simple functions, Lusin's and Egorov's theorems, convergence in measure/probability.
(3) Integration: bounded convergence theorem, Fatou's lemma, monotone convergence theorem, absolute continuity of the integral.
(4) Differentiation and integration: the Vitali lemma, almost everywhere differentiability of monotone functions, integral of the derivative of a monotone function, functions of bounded variation, differentiation of the indefinite integral, the fundamental theorem of calculus (Newton-Leibniz) for absolutely continuous functions.
(5) Product meaures: Fubini and Tonelli theorems.
(6) Differentiation of measures: absolute continuity of measures, the Radon-Nykodim theorem, mutually singular measures, the Lebesgue decomposition, Lebesgue points, the Lebesgue-Besicovitch theorem, signed measures.
(7) The Riesz representation theorem on $L^p$, $1 \leq p < \infty$, the Riesz-Markov theorem (dual of $C_c$), the Riesz-Thorin interpolation theorem.

## 2. Functional analysis

(1) Hilbert spaces: direct sums, orthocomplements, Riesz' representation theorem, orthonormal bases in separable and non-separable cases, Fourier basis for $L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.
(2) Banach spaces: continuous linear maps between Banach spaces, duals, direct sums and quotients, Hahn-Banach theorem, Baire's theorem, Banach-Steinhaus (uniform boundedness principle), open mapping theorem.
(3) Topological spaces: (local) bases for topologies, convergence, continuity, compactness, construction of weak topologies, weak and weak-* topologies on Banach spaces, Stone-Weierstrass theorem, Banach-Alaoglu theorem, Riesz-Markov theorem.
(4) Locally convex spaces: Minkowski gauge, equivalence of convex-balanced-absorbing neighborhood and seminorm version, Hahn-Banach, metrizability, Fréchet spaces, Banach-Steinhaus, open mapping theorem.
(5) Distributions: $C^\infty(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and its dual, distributions $\mathcal{D}'(\mathbb{T})$, Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ and their dual, tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, differentiation of distributions, multiplication of distributions by smooth functions, convergence of sequences of distributions, the distributions $(. \pm i0)^{-1}$ and $\delta_0$. Support of distributions.

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(6) Bounded operators on Banach spaces: adjoints, the spectrum, the resolvent, spectrum of bounded self-adjoint operators on Hilbert spaces, integration and differentiation with values in Banach spaces, the spectral radius of operators, continuous functional calculus for bounded self-adjoint operators on Hilbert spaces and the spectral measure.

(7) Compact operators on Banach spaces: norm closure, composition and adjoint properties, approximability by finite rank operators in separable Hilbert spaces, analytic Fredholm theorem, Riesz-Schauder and Hilbert-Schmidt theorems.

## 3. Fourier analysis

(1) Fourier series on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$: orthogonality properties, $L^2$ completeness, uniform convergence or lack thereof, Fourier series for $C^\infty(\mathbb{T})$ and $\mathcal{D}'(\mathbb{T})$, interaction with differentiation and multiplication by $e^{i\theta}$, the Sobolev spaces $H^s(\mathbb{T})$, compactness of the inclusion $H^s(\mathbb{T}) \subset H^r(\mathbb{T})$ for $s > r$, the representation theorem for distributions as derivatives of continuous functions, the Schwartz kernel theorem.

(2) Fourier transform on $\mathbb{R}^n$: isomorphism on $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, isometry on $L^2(\mathbb{R}^n)$. Interaction with differentiation and multiplication by the coordinate functions $x_j$. Fourier transform of compactly supported distributions. Fourier transform of Gaussians and the heat equation. The Sobolev spaces $H^s(\mathbb{R}^n)$, and the inclusion $H^s(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$ for $s > n/2$.