

Competitive Insurance Markets with Unbounded Cost

Yehuda John Levy* & André Veiga[†]

(Draft: June 23rd, 2019)

Abstract

[Azevedo and Gottlieb \[2017\]](#) (AG) define a notion of equilibrium that always exists in the [Rothschild and Stiglitz \[1976\]](#) (RS) model of competitive insurance markets, provided costs are bounded. However, equilibrium predictions are sensitive to assumptions made about the upper bound of cost: introducing an infinitesimal mass of high cost individuals discretely increases all equilibrium prices and reduces coverage for all individuals. We measure model sensitivity to these assumptions by considering sequences of economies with increasing upper bounds of cost, and determining whether the sequence of their equilibria converges. We present sufficient conditions under which AG equilibrium exists when cost is unbounded. For simple insurance markets, we derive a condition which is necessary and sufficient for existence: surplus from insurance must increase faster than linearly with expected cost. This condition is empirically common. If this condition does not hold, a wider range of costs results in market unraveling because all prices increase without bound and, in the limit, an AG equilibrium does not exist. We use these results to show that the equilibrium for an insurance market with an unbounded continuum of types is characterized by a simple differential equation. We also provide examples of non-existence for lemons markets (where a single insurance product is available) with unbounded cost.

JEL: C62, G22, I13, D82

Keywords: Adverse Selection, Asymmetric Information, Insurance, Equilibrium Existence

*Adam Smith Business School, University of Glasgow, Glasgow, UK; john.levy@glasgow.ac.uk.

[†]Imperial College Business School, South Kensington Campus, London, UK; a.veiga@imperial.ac.uk.

Contents

1	Introduction	1
2	The AG Setting	3
2.1	Setup	3
2.2	Existence in AG	4
2.3	Existence with unbounded cost	5
3	Simple Insurance Markets	6
3.1	A model of insurance	7
3.2	Insurance Markets in AG	9
3.3	Equilibrium Characterization	9
3.4	Existence	11
3.5	Insurance Market with a Continuum of Types	12
3.6	Example: Discrete types	16
4	Market for Lemons	20
4.1	Existence	21
4.2	Non-existence	22
5	Generalization & Variations	23
5.1	General Insurance Setup	23
5.2	Equilibrium Properties in Insurance Markets	24
5.3	Unbounded Price in Insurance Markets	26
5.4	Non-Existence in Insurance Markets	28
5.5	Existence in Insurance Markets	29
5.6	Existence When the Limit Equilibrium is Not Know	30
6	Conclusion	30
A	Comparisons of Utilities	34
B	Unbounded Price	34
B.1	Price is Lipschitz in Weak Equilibrium	36
B.2	Price is Locally Lipschitz in Equilibrium	36
B.3	Full insurance is not a Heterogeneous Atom	37
B.4	Price is unbounded	38
C	Non-Existence	38
D	Simple Insurance: Continuous Types	39
D.1	Proposition 9	39
D.2	Proposition 10	40
D.3	Equilibrium with Bounded Risk	41
D.4	Equilibrium with Unbounded Risk	41
D.5	Non-Existence with Unbounded Risk	43
E	Simple Insurance: Discrete Types	43
E.1	Characterization of $\phi_k(\cdot)$	43
E.2	Uniqueness & Positivity of Coverage	44
E.3	Convergence of p^n and α^n	45
F	Existence via Approximations	45
F.1	Existence 1	46
F.2	Existence 2	48
F.3	Condition (*) holds in simple insurance	49
G	Equilibrium characterization in generalized Insurance	49
G.1	Coverage Weakly Increasing in Risk	50
G.2	Breaking Even	51
G.3	Coverage Strictly Increases with Risk	51
G.4	Coverage Continuous and Increasing in Risk	51
G.5	Use of Assumption 8	51
G.6	Full insurance is in the Support	52
G.7	Lipschitz-Type Property	53
G.8	Price as Approximate Upper Envelope	54
G.9	No Purchasing 0 Coverage	55
G.10	Weakening Assumption 8	56
H	Existence in Generalized Insurance	56

I	Generalized Equilibrium Properties	59
I.1	Continuity	59
I.2	Equilibrium is Weak Equilibrium	60
J	Discrete types: Direct Construction	62
J.1	Equilibrium in bounded economies	62
J.2	Convergence to equilibrium of unbounded economy	65

1 Introduction

The [Rothschild and Stiglitz \[1976\]](#) (RS) model of competitive insurance has (at least) two limitations. First, there need not exist a pure-strategies Nash equilibrium. Second, the equilibrium is sensitive to assumptions made about the upper bound of the cost distribution, because introducing an infinitesimal mass of high cost individuals discretely increases the equilibrium price of all insurance contracts. [Azevedo and Gottlieb \[2017\]](#) (AG) suggest a notion of equilibrium that always exists in the RS context. By tackling existence, the AG equilibrium concept allows us to focus on the second limitation: sensitivity to assumptions about cost. In fact, an AG equilibrium is only guaranteed to exist when the distribution of cost is bounded. Moreover, the predictions of AG equilibrium can be similarly sensitive to cost bounds. This can limit the practical usefulness of insurance models for two reasons. First, it is often unclear what constitutes a reasonable upper bound for cost (especially when costs exhibit “fat tails”). Second, it is unclear what settings feature the extreme sensitivity of the RS setting.

This article derives conditions under which markets with adverse selection (and, in particular, insurance markets) have AG equilibria which are robust to assumptions about the cost distribution. We do so by considering sequences of truncated economies where cost is bounded, and progressively relaxing this truncation. Our measure of robustness is whether the equilibria of the truncated economies converge, that is, whether an equilibrium exists for the limit economy with unbounded cost.

Our motivation is not that economies with unbounded costs are particularly relevant or realistic. Instead, we take existence of equilibria as our measure of whether a model’s predictions are sensitive to assumptions about the (support of the) cost distribution. If the equilibria of bounded economies converge to an equilibrium of the unbounded economy, then equilibrium predictions are robust. Conversely, if cost assumptions can have an unbounded impact on equilibrium predictions, this can be diagnosed by finding that the limit economy has no equilibrium.

We begin by considering the (quite general) setting described in AG. Intuitively, an AG equilibrium is a set of prices and choices such that a) individuals optimize, b) each contract breaks even and c) the choices and the prices of non-traded contracts are robust to small perturbations in the model’s fundamentals. In the RS model, there exists a unique AG equilibrium which predicts the same allocation as the pure strategies Nash equilibrium (when it exists). In that setting, we provide sufficient conditions under which an economy with unbounded cost has an equilibrium.

We then focus on the case of insurance markets, in the spirit of [Rothschild and Stiglitz \[1976\]](#). We allow individuals to differ in both risk and risk aversion (although a single parameter determines both), and assume that costlier types have higher marginal will-

ingness to pay for more generous insurance contracts. We begin by providing a novel characterization of the (unique) AG equilibrium for an arbitrary number of (possibly unbounded) discrete or continuous types. We also show that, if cost is unbounded, the price of full insurance is also unbounded in any equilibrium. Then, we derive a condition on model primitives which is both necessary and sufficient for existence of equilibrium when cost is unbounded. Intuitively, this condition requires that surplus from insurance increases faster than linearly with expected cost.

The condition divides insurance economies into “robust” and “fragile.” For fragile economies, where the condition does not hold, as cost becomes unbounded, the market unravels: the price of each alternative increases without bound and the chosen level of coverage of each type converges to zero. For fragile economies with unbounded cost, an equilibrium does not exist. Conversely, robust economies have an equilibrium no matter what the assumption is imposed on the support of cost.

This condition is intuitive and empirically relevant. For instance, if individual utilities are CARA and wealth shocks are Gaussian, our existence condition requires that the variance of wealth shocks increases (asymptotically) faster than linearly relative to the mean of these shocks. This condition is satisfied in the empirical findings of [Handel et al. \[2015\]](#) and, more broadly, it is empirically common that individuals with higher expected cost also experience higher variance in insurable wealth shocks ([Hendren \[2011\]](#), [Brown et al. \[2014\]](#)). Therefore, robust insurance economies seem empirically common.

Our results also emphasize that unidimensional heterogeneity necessarily implies that an economy is fragile. For instance, if all individuals have the same risk aversion (as in [Rothschild and Stiglitz \[1976\]](#)) and costs are unbounded, an AG equilibrium does not exist. Multidimensional heterogeneity, while rare in theoretical models, is commonly considered in applied work like [Cohen and Einav \[2007\]](#), [Einav et al. \[2010b, 2012, 2013\]](#), [Handel \[2013\]](#), [Handel et al. \[2015\]](#), [Veiga and Weyl \[2016\]](#). Such applied work (and, in fact, the calibration in AG) frequently assume unbounded cost distributions.

We use these results to characterize the equilibrium of an insurance market with an unbounded continuum of types, and show it is defined by a simple differential equation. We also characterize the equilibrium for an economy with unbounded discrete types.

We then briefly consider the setting of lemons markets, as in [Akerlof \[1970\]](#), [Einav et al. \[2010a\]](#). In these markets, there is a single non-zero insurance contract available, so individual choices are binary. We show that, even in such simple settings, unbounded costs (for instance, if cost has an exponential or Pareto distribution) can result in equilibrium non-existence.

We then generalize several of our results beyond simple insurance markets. For instance, unbounded costs imply unbounded prices even in more general settings where types are truly multidimensional and, therefore, there is pooling of multiple types in

each contract. This result allows us to identify a large class of economies in which costs are unbounded and equilibrium does not exist.

The usefulness of our results is two-fold. First, in insurance markets where model primitives are well known, we provide a novel characterization of equilibrium. This characterization shows under which conditions a wider distribution of cost types causes a market to unravel or not. Second, when model primitives are uncertain, we identify conditions under which assumptions about the support of the cost distribution have a large impact on equilibrium predictions.

The rest of the article is organized as follows. Section 2 summarizes the setting and results of AG. Section 3 specializes that model to the case of insurance markets, in the spirit of Rothschild and Stiglitz [1976]. Section 4 considers a market for lemons, as in Akerlof [1970], Einav et al. [2010a]. Section 5 generalizes several results. Section 6 concludes. All proofs are contained in the Appendix, although we provide some intuition for the proofs in the main text.

2 The AG Setting

In this section, we consider the model of a competitive market with adverse selection presented in AG, generalized to allow for unbounded costs. We summarize the existence results in AG, which require cost to be bounded. Then, we provide sufficient conditions for existence in economies with unbounded cost.

2.1 Setup

A consumer type is a vector $\theta \in \Theta$, where Θ is a Polish space with measure P .¹ The type θ can describe each individual's risk, risk aversion, wealth, etc. An alternative is a vector $x \in X$, where X is a locally compact Polish space. Alternatives can be characterized by deductibles, co-insurance rates, etc. Price is $p \in \mathbb{R}_+$. A contract is a pair $(x, p) \in X \times \mathbb{R}_+$. We consider Borel-measurable *price functions* $p : X \rightarrow \mathbb{R}_+$ where $p(x)$ is the price of alternative x . An economy \mathcal{E} is a triple $\mathcal{E} = [\Theta, P, X]$.

Utility is a continuous function $u(\theta, x, p) = u_\theta(x, p)$, where $u : \Theta \times X \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly decreasing in p . Cost is a continuous function $c(\theta, x) \geq 0$, where $c : \Theta \times X \rightarrow \mathbb{R}_+$. Cost depends on type θ which creates the possibility of adverse selection.

An allocation is a distribution α on $\Theta \times X$ with marginal P on Θ , and marginal α_X on X , such that $\int_{\Theta \times X} c(\theta, x) d\alpha < \infty$. One can think of $\alpha(\{\theta, x\})$ as the mass or density of types θ purchasing alternative x under allocation α .

¹A Polish space is a complete, separable metrizable space.

AG assume that Θ, X are compact. Since $c(\theta, x)$ is continuous, this implies that cost is bounded. In this article, we will allow Θ , and therefore cost, to be unbounded. We assume that, for each compact sub-set of alternatives $K \subseteq X$, if all agents choose alternatives in K , expected cost is finite: formally, $\int_{\Theta} \sup_{x \in K} c(x, \theta) dP(\theta) < \infty$.

2.2 Existence in AG

AG define a “weak equilibrium” as a price function and an allocation (p, α) where individuals are maximizing and each contract breaks even (for each x , price equals the average cost of its buyers under α).

Definition 1. A pair (p, α) , consisting of a price function $p : X \rightarrow \mathbb{R}$ and an allocation α is a *weak equilibrium* of an economy $\mathcal{E} = [\Theta, X, P]$ if, for α -a.e. $(\theta, x) \in \Theta \times X$, the following two conditions hold. First, individuals maximize: for α -a.e. (θ, x) , we have $\sup_{x' \in X} u_{\theta}(p(x'), x') = u_{\theta}(p(x), x)$. Second, contracts break even: for α -a.e. x , we have $p(x) = \mathbb{E}_{\alpha}[c \mid x]$.²

Typically, there exists a large multiplicity of weak equilibria because, for alternatives x which are not traded, $p(x)$ is arbitrary. This motivates AG’s definition of *equilibrium*. Intuitively, an equilibrium is a weak equilibrium which is also robust to the introduction of a small mass of zero-cost “behavioral” types who purchase every alternative $x \in X$.³ More precisely, an equilibrium is the limit of a sequence of weak equilibria of perturbed economies \mathcal{E}_j where the mass of behavioral types vanishes as $j \rightarrow \infty$. This requirement pins down the prices of alternatives which are not purchased in equilibrium, and implies that there typically exist much fewer equilibria than weak equilibria. This article uses “equilibrium” exclusively to refer to the notion of equilibrium described by AG and formalized below.

Definition 2. Consider the economy $\mathcal{E} = [\Theta, X, P]$, and the sequence of perturbed economies $\mathcal{E}_j = [\Theta \cup \bar{X}_j, \bar{X}_j, P + \eta_j]$. Let $(\bar{X}_j)_{j \in \mathbb{N}}$ be a sequence of finite subsets of \bar{X} which converge to \bar{X} in the sense of Hausdorff,⁴ where \bar{X} is a Polish space such that X is dense in \bar{X} .⁵ Let $(\eta_j)_{j \in \mathbb{N}}$ be a sequence of measures, with η_j supported on \bar{X}_j , strictly positive on \bar{X}_j , and $\eta_j(\bar{X}_j) \rightarrow 0$. Suppose there exists a sequence of pairs $(p_j, \alpha_j)_{j \in \mathbb{N}}$, such that:

²The conditional expectation is well-defined, as we have assumed that X is locally compact and for each compact set $K \subseteq X$, $\int_{\Theta} \sup_{x \in K} c(x, \theta) dP(\theta) < \infty$.

³A similar construction is used by [Dubey and Geanakoplos \[2002\]](#). AG prove (do not assume) that every equilibrium is a weak equilibrium.

⁴I.e., for each $x \in \bar{X}$, there is $(x_n)_{n \in \mathbb{N}}$ converging to x with $x_n \in \bar{X}^n$ for each $n \in \mathbb{N}$.

⁵Formally, X embeds to a dense subset of \bar{X} , but we disregard such technicalities for the sake of brevity at no cost to the generality. In insurance markets, we often take $X = [0, 1]$ which naturally embeds in $\bar{X} = [0, 1]$.

- (p_j, α_j) is a weak equilibrium \mathcal{E}_j , where type $x \in \overline{X}_j$ has zero cost and prefers x to any other alternative (regardless of price); the types $x \in \overline{X}_j$ are known as *behavioral types*.
- $\alpha_j \rightarrow \alpha$ weakly.⁶
- Whenever $(x_j)_{j \in \mathbb{N}}$ converges to $x \in X$ with $x_j \in \overline{X}_j$, then $p_j(x_j) \rightarrow p(x)$.

Then, the pair (p, α) is an *equilibrium* of $\mathcal{E} = [\Theta, X, P]$.

AG prove that every economy satisfying certain boundedness conditions has an equilibrium, as formalized by Theorem 1. More specifically, existence requires certain technical assumptions on utilities and cost and, importantly, requires that the space of types and alternatives (Θ and X) are both compact.

Theorem 1. *Suppose that X, Θ are compact metric spaces. Suppose also that u obeys a form of Lipschitz-ness in X uniformly over types,⁷ and that c is continuous (which implies c is bounded). Then, an equilibrium exists.*

AG also derive additional properties of equilibrium (their Proposition 1). An equilibrium is also a weak equilibrium, and hence in equilibrium a.e. agent is optimizing (we use this conclusion implicitly throughout this article). The price function $p(\cdot)$ is Lipschitz and continuous. Every alternative that is not traded in equilibrium has a price low enough that some individual is indifferent between buying it and not, and the cost of that individual at the non-traded alternative is at least as high as the price of the alternative. We generalize these results for environments with unbounded types in Proposition 12 and Lemma 18.

2.3 Existence with unbounded cost

To assess robustness to assumptions on the support of costs, we will consider sequences of economies with truncated cost distributions, which progressively approximate an economy with unbounded cost. We then consider existence of equilibrium of this limit unbounded economy.

The truncated economy $\mathcal{E}^n = [\Theta^n, \overline{X}, P(\cdot | \Theta^n)]$ has a bounded type space Θ^n . More specifically, we consider a sequence of compact subsets $\Theta^1 \subseteq \Theta^2 \subseteq \dots \subseteq \Theta$ with $\cup_n \Theta^n = \Theta$. We assume that for each $n \in \mathbb{N}$, each $\theta \in \Theta^n$ and each $\theta' \in \Theta \setminus \Theta_n$, we

⁶i.e., for each $f : \Theta \times X \rightarrow \mathbb{R}$ continuous and bounded, $\int f d\alpha^n \rightarrow \int f d\alpha$.

⁷Formally, there exists L , such that for any $p \leq p'$ in the image of c , any $x, x' \in X$, and any type $\theta \in \Theta$, if $u(x, p, \theta) \leq u(x', p', \theta)$, then $p' - p \leq Ld(x, x')$, where $d(\cdot, \cdot)$ is a metric. This is Assumption 2 in AG. If utility is of the form $u(\theta, x, p) = v(\theta, x) - p$, this amounts to $u(\theta, x, p)$ being Lipschitz in x , uniformly in θ (same Lipschitz constant for all θ).

have $c(\theta, \cdot) \leq c(\theta', \cdot)$. That is, types not included in Θ^n have costs higher than those in Θ^n . The alternative space \bar{X} is the compactification of X (the union of X with its limit points). We assume that $c(\cdot, \cdot)$ and $u(\cdot, \cdot, \cdot)$ extend continuously to $\bar{X} \times \Theta$ and $\bar{X} \times \Theta \times \mathbb{R}_+$. Conditional distributions in each \mathcal{E}^n are defined in the standard way: $P(\cdot | \Theta^n) = P(\cdot \cap \Theta^n) / P(\Theta^n)$.

By Theorem 1, each truncated economy \mathcal{E}^n has an equilibrium (p^n, α^n) . Our first result considers the existence of an equilibrium (p, α) of the limit unbounded economy \mathcal{E} .

Proposition 1. *For each $n \in \mathbb{N}$, let (p^n, α^n) be the equilibrium of the truncated economy \mathcal{E}^n , such that:*

- *There is a function $p : X \rightarrow \mathbb{R}_+$ s.t. $p^n \rightarrow p$ uniformly on compact subsets of X .*
- *There is a distribution α on $\Theta \times X \subseteq \Theta \times \bar{X}$ s.t. $\alpha^n \rightarrow \alpha$ weakly.*
- *(*) There is $c_0 > 0$ such that for large enough k and $n > k$, types in $\Theta^n \setminus \Theta^k$ purchase only options x with cost $p^n(x) \geq c_0$. I.e., $\alpha^n(\{p^n(x) \geq c_0\} | \alpha \in \Theta_n \setminus \Theta_k) = 1$.*

Then (p, α) is an equilibrium of the unbounded economy $\mathcal{E} = [\Theta, X, P]$.

Intuitively, condition (*) requires that costly types do not purchase cheap contracts. This condition holds naturally in the insurance models we will consider.

Proof. The proof uses a diagonalization argument. For each truncated economy \mathcal{E}^n , its equilibrium (p^n, α^n) is the limit of the weak equilibria (p_j^n, α_j^n) of a sequence of perturbed economies \mathcal{E}_j^n which have a vanishing mass of behavioral types, as described in AG. We then consider the sequence \mathcal{E}^n and show that an appropriate diagonal of weak equilibria $(p_{j_n}^n, \alpha_{j_n}^n)$ converge to an AG equilibrium of \mathcal{E} when $n \rightarrow \infty$. Finally, we modify the equilibria on this diagonal to include all types, as $\alpha_{j_n}^n$ only allocates types in Θ^n . Due to the behavioral types, this can be done without changing the price $p_{j_n}^n$. For details, see Appendix F1. □

One limitation of Proposition 1 is that it does not impose requirements directly on model primitives. This is a limitation of our results in this general setting. However, in Section 3, where we consider insurance markets with additional structure, we are able to derive a condition on model primitives which is necessary and sufficient for Proposition 1 to hold in that setting.

3 Simple Insurance Markets

We now specialize the model of Section 2 to insurance settings, in the spirit of [Rothschild and Stiglitz \[1976\]](#). This section contains the bulk of our contribution. First, we

provide a novel characterization of equilibrium with arbitrarily many (and possibly unbounded) types, thereby extending AG’s analysis of RS. Then, we derive a condition on model primitives which is necessary and sufficient for equilibrium existence in insurance markets with unbounded cost. Lastly, we characterize equilibrium in insurance markets with an unboundedly continuous or discrete types.

Intuitively, the existence condition we derive requires that surplus from insurance increases sufficiently fast with risk. When this condition does not hold, economies are said to be “fragile”: relaxing assumptions on the bound of the cost distribution has a large (unbounded) effect on equilibrium predictions. We also show that, for fragile economies, expanding the support of the cost distribution results in market unraveling (the price of every alternative diverges, and every type’s allocation converges to zero insurance coverage). When the condition holds, economies are “robust.”

3.1 A model of insurance

An individual of type θ is exposed to a stochastic wealth loss described by the random variable Z_θ . Let $\mu : \Theta \rightarrow \mathbb{R}_{++}$ be a continuous map assigning to type θ her expected loss, denoted $\mu = \mu(\theta) = \mathbb{E}[Z_\theta]$. We assume $\mu > 0$ P -a.s. We allow the marginal distribution of μ to *not* be compactly supported: $\forall M \geq 0, P(\{\theta \mid \mu \geq M\}) > 0$. Denote the marginal probability of μ by $P_\mu = P \circ \mu^{-1}$, and let $\underline{\mu} = \min(\text{supp}(P_\mu))$.

We assume the following parameterization of alternatives. An individual who purchases alternative $x \in [0, 1]$ is only exposed to the random shock $(1 - x)Z_\theta$, with the remaining xZ_θ being absorbed by the insurer. Full insurance corresponds to $x = 1$, and zero insurance to $x = 0$. Hence, the alternative space, unless otherwise specified, will always be either $X = [0, 1]$ or $X = [0, 1)$, i.e., we may remove the option of purchasing full coverage, because, as we will show in Section 5.3, any equilibrium with unbounded cost must have $\lim_{x \rightarrow 1} p(x) = \infty$. In fact, all the results in Section 3 hold if the upper bound of coverage is replaced with $x_{max} \in (0, 1)$. We will proceed by considering $X = [0, 1]$ or $X = [0, 1)$ for notational and expositional simplicity.

The cost to a risk neutral insurer of alternative x sold to a type θ is $c(\theta, x) = \mu x$. We assume $\int_\Theta \mu(\theta) dP < \infty$: even if each individual chooses full insurance ($x = 1$), the average cost is finite.

We make the following assumption regarding utilities.

Assumption 1. *Utility (certainty equivalent) is*

$$u(\theta, x, p) = x\mu_\theta + g(x)\nu_\theta - p,$$

for some continuous function $\nu_\theta : \Theta \rightarrow \mathbb{R}_{++}$. We assume $g : [0, 1] \rightarrow \mathbb{R}$ is twice continu-

ously differentiable, $g' > 0$ in $x \in [0, 1)$, $g'' \leq 0$, $g'(1) = 0$ and $g''(1) < 0$.

Utility from alternative x has three components. First, $x\mu_\theta$ is the individual's expected cost, which is passed on to the insurer. Second, $g(x)\nu_\theta$ captures the individual's surplus from insurance, where ν_θ is the individual's "insurance value." Notice that $g(x)$ is common to all individuals, while ν_θ is heterogeneous. If $\nu_\theta > 0$, individuals are willing to pay for insurance above their expected cost (for instance, due to risk aversion). There are decreasing marginal returns from insurance and, at full insurance, insurance has zero marginal value ($g'(1) = 0$). We also assume certainty equivalents are quasilinear in prices. This assumption is not innocuous, but is common in models with constant absolute risk aversion (CARA).⁸ These assumptions imply that types are effectively two-dimensional, since only (μ_θ, ν_θ) matter for decisions and costs.

The marginal willingness to pay for additional insurance is

$$w_\theta(x) = \frac{\partial u_\theta}{\partial x}(x, p) = \mu_\theta + g'(x)\nu_\theta \geq \mu_\theta.$$

with equality iff $x = 1$. By Assumption 1, w_θ is independent of price.

We also make the following assumptions on the distribution of types.

Assumption 2. $\nu_\theta = \nu(\mu)$ with $\nu(\cdot)$ a weakly increasing function, locally Lipschitz and $\nu(\mu) > 0$ for $\mu > \underline{\mu}$.

Assumption 3. P_μ is either absolutely continuous with a.e. positive density on a (bounded or unbounded) interval, or purely atomic with finitely many atoms in each bounded interval.

Assumption 2 implies that the marginal willingness to pay for insurance w_θ is strictly increasing in cost μ . The assumption of w_θ monotonic in μ is true in RS, Riley [1979], in AG's treatment of insurance markets, and in all models of insurance with one-dimensional types that we are aware of.⁹ Under Assumption 2, utility is completely determined by risk μ : two types with the same risk also have the same marginal willingnesses to pay, so types are effectively one dimensional: $\theta = (\mu, \nu(\mu))$.¹⁰ This nonetheless generalizes the analysis of insurance markets by RS and AG, where $\nu(\mu)$ was assumed to be constant.¹¹

⁸Notice that quasi-linearity implies that utility is 1-Lipschitz in price.

⁹This assumption is not present in models with multidimensional types, like Wambach [2000], Picard [2017], Veiga and Weyl [2016], Smart [2000], Villeneuve [2003]. Villeneuve [2003] and Smart [2000] consider, as we do, settings where individuals differ in risk and risk aversion, but costs are bounded. Wambach [2000], Crocker and Snow [2011] and Snow [2009] consider other forms of multidimensional heterogeneity.

¹⁰We write μ_θ, ν_θ when the type space Θ is generic. We write μ, ν when the type space is a subset of \mathbb{R}_+ .

¹¹For this reason, we sometimes write, for instance, w_μ instead of w_θ .

Assumption 3 imposes some regularity on the marginal distribution of risk μ . The condition is mild, and satisfied in all insurance models that we are aware of.¹²

Example 1 illustrates a setting where Assumptions 1 and 2 are satisfied. This CARA-Gaussian parameterization is used, for instance, in Veiga and Weyl [2016], Levy and Veiga [2017].

Example 1. Suppose wealth shocks are Gaussian: $Z_\theta \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$. Suppose utility is CARA: $U_\theta(y) = e^{-a_\theta y}$ where y is wealth and a_θ is risk aversion. Each type has initial wealth w_θ . Then, insurance value is $\nu_\theta = a_\theta \sigma_\theta^2$ and certainty equivalents are $u(\theta, x, p) = x\mu_\theta + \frac{1}{2} (1 - (1 - x)^2) \nu_\theta - p$. Marginal willingness to pay for insurance is $w_\theta(x) = \mu_\theta + (1 - x)\nu_\theta$ and Assumption 1 is satisfied. Assumption 2 is satisfied if $\nu_\theta = \nu(\mu_\theta)$ for a weakly increasing function $\nu(\cdot)$.

3.2 Insurance Markets in AG

AG consider a specialization of their general model to insurance markets with two cost types. In insurance markets, since $c(\theta, x) = \mu x$, so the AG equilibrium break-even condition becomes

$$p(x) = x \cdot \mathbb{E}_\alpha[\mu \mid x], \quad \alpha - a.e. x. \quad (1)$$

In that setting, AG derive several properties of equilibrium (AG's Corollary 1). An equilibrium always exists and is unique. In equilibrium, the high-cost individual obtains full insurance, and is indifferent between her choice and the contract chosen by the low-cost individual.¹³ Moreover, the equilibrium price function $p(x)$, is the upper envelope of the line $p = 0$ and each type's indifference curve at her chosen contract. Notice that only 2 contracts are purchased in equilibrium but $p(x)$ defines a price for all contracts $x \in X$. This is illustrated in Figure 1.

3.3 Equilibrium Characterization

We generalize AG by characterizing equilibrium in insurance markets with arbitrarily many (and possibly unbounded) types. Recall that $X = [0, 1]$ or $X = [0, 1)$.

¹²It is not conceptually difficult, but practically cumbersome, to extend Theorem 2 using the techniques in this paper to the class of distributions P_μ with the following properties: P_μ has finitely many atoms in each bounded interval, and there are a finite (possibly empty) or infinite sequence $a_1 < b_1 < a_2 < b_2 < \dots$ s.t. the non-atomic part of P_μ is concentrated on $\cup_n [a_n, b_n]$ and distributes on this set absolutely continuously with a.e. positive density.

¹³In the RS model, the AG equilibrium has the same allocation as the Riley [1979] equilibrium, which is also the Nash equilibrium, when the latter exists. However, a Riley [1979] equilibrium need not always exist, as shown by Azevedo and Gottlieb [2016].

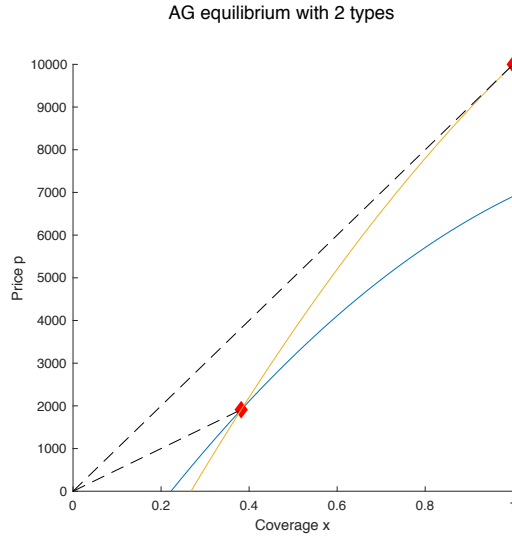


Figure 1: Equilibrium in an insurance market with 2 types, as described in AG. Dashed black lines represent zero-profit lines for each type. Solid lines represent each type's indifference curves at chosen contracts. The equilibrium price $p(x)$ is the upper envelope of the two indifference curves, above the line $p = 0$. Red diamonds represent the combination (p, x) chosen by each type.

Theorem 2. *Suppose Assumptions 1, 2 and 3 hold for a bounded or unbounded insurance economy $\mathcal{E} = [\Theta, X, P]$. There is at most a single equilibrium (p, α) for \mathcal{E} , and when it exists, it satisfies:*

1. *Price $p(x)$ is continuous, and it is strictly increasing for $\{x \mid p(x) > 0\}$.*
2. *There is a continuous and strictly increasing mapping $\sigma : \text{supp}(P_\mu) \rightarrow X$ that assigns to each type μ , the alternative $\sigma(\mu)$ that she chooses α -a.s.*
3. *Each contract breaks even: P -a.s., $p(\sigma(\mu)) = \mu \cdot \sigma(\mu)$.*
4. *Full insurance ($x = 1$) is in the support of the equilibrium and zero insurance ($x = 0$) is purchased by a set of individuals with measure zero.*
5. *Price is Lipschitz in any interval bounded away from full insurance ($x = 1$); if μ is bounded P -a.s., price is Lipschitz.*
6. *If P_μ is discrete, each type is indifferent between the contract she chooses α -a.s. and the next highest coverage purchased in α .¹⁴*

Proof. Most of Theorem 2, except for the uniqueness of equilibrium, follow immediately from the more general Proposition 6, proved in Appendix G. The uniqueness (if equilibrium exists) follows from Corollary 2 (continuous types, bounded costs), Corollary 4

¹⁴Formally: Let $\mu_1 < \mu_2$ be two atoms of P_μ with no atom between them. Suppose type μ_1 purchases x_1 and type μ_2 purchases x_2 α -a.s.. Then, type μ_2 is indifferent between $(x_2, p(x_2))$ and $(x_1, p(x_1))$.

(continuous types, unbounded costs), Proposition 2 (discrete types, bounded costs), or Proposition 4 (discrete types, unbounded costs). \square

When risk μ is essentially bounded, existence follows from AG, but Theorem 2 establishes uniqueness. When μ is unbounded, Theorem 2 shows that uniqueness still holds (when equilibrium exists). Moreover, the equilibrium has several properties familiar from AG and RS: higher cost types purchase more generous insurance, full insurance is purchased by some type in equilibrium, and incentive compatibility binds “downwards.”¹⁵

Theorem 2 has an additional implication for unbounded economies. In the limit where cost becomes unbounded, the equilibrium price of the most generous contract also becomes unbounded.

Corollary 1. *Under Assumptions 1, 2 and 3, if P_μ is not compactly supported, then in any AG equilibrium*

$$\lim_{x \rightarrow 1} p(x) = \infty.$$

Proof. From Theorem 2, the allocation $\sigma(\mu)$ is strictly increasing, so there is full separation of types and the costliest types purchase the maximum available insurance coverage. If types are unbounded, this cost is arbitrarily high and therefore $\lim_{x \rightarrow 1} p(x)$ diverges. For details, see Appendix B. \square

This condition will play a key role in our analysis of existence. For equilibrium to exist, preferences must be such that $p(x)$ can diverge at full insurance ($x = 1$) but must converge for all $x < 1$, while still preserving the incentive compatibility required by Proposition 6. Corollary 1 is why, for economies with unbounded cost, we consider the (non-compact) alternative space to be $X = [0, 1)$.

Our interpretation of Corollary (1) is not that that we should expect infinite prices in any insurance markets. Our goal in analyzing the equilibrium of unbounded economies is to determine the robustness of the model to assumptions about cost. We take Corollary (1) as a tool that will aid us in that analysis.

3.4 Existence

In this section we consider, as in Section 2, sequences of truncated bounded economies \mathcal{E}^n which approximate an economy \mathcal{E} with unbounded cost. We will derive

¹⁵We remark that, although Theorem 2 is used to prove Corollary 2, Corollary 4, Proposition 2 and Proposition 4, these proofs do not rely on the uniqueness of equilibrium. Instead, those proofs rely only on the other properties listed in Theorem 2 (and hence we have not fallen trap to circular logic). We present the uniqueness as part of Theorem 2 for expositional simplicity, instead of leaving this conclusion for a later corollary.

a necessary and sufficient condition on model primitives for the equilibria of \mathcal{E}^n to converge to the equilibrium of \mathcal{E} , for unbounded continuous types (Subsection 3.5) and unbounded discrete types (Subsection 3.6).

We briefly recall our construction of truncated economies. We consider the unbounded economy $\mathcal{E} = [\Theta, X, P]$ with the unbounded type space Θ , the measure P on Θ , and the alternative space $X = [0, 1)$ as discussed in Corollary 1. Recall that $\underline{\mu} = \min(\text{supp}(P_\mu))$. We also consider a sequence of truncated economies $\mathcal{E}^n = [\Theta^n, \bar{X}, P(\cdot | \Theta^n)]$, with bounded type space $\Theta^n = \{\mu \in [\underline{\mu}, \bar{\mu}_n] \cap \text{supp}(P_\mu)\}$ for some $\bar{\mu}_n$, bounded alternative space $\bar{X} = [0, 1]$, and conditional distribution $P(\cdot | \Theta^n) = P(\cdot \cap \Theta^n) / P(\Theta^n)$. Since each Θ^n, \bar{X} is compact, \mathcal{E}^n has an equilibrium (p^n, α^n) . We will consider a sequence of economies \mathcal{E}^n with $\bar{\mu}_n \rightarrow \infty$.

When the equilibrium of \mathcal{E} exists, we say the insurance economies \mathcal{E}^n are “robust.” Since the equilibria (p^n, α^n) of \mathcal{E}^n converge as cost becomes unbounded, the model’s predictions are robust to assumptions about the support of cost. Conversely, when \mathcal{E} does not have an AG equilibrium, and we say that the economies \mathcal{E}^n are “fragile.” For these economies, assumptions about the distribution of risk have large effects on equilibrium predictions.

We will apply the existence result in Proposition 1 to the setting of insurance markets. To do so, Lemma 7 (Appendix E1) shows that the technical condition (*) holds in insurance markets. While Proposition 1 requires knowledge of the limit equilibrium (p, α) , this result is enough for the setting of simple insurance markets because the structure of equilibrium is known from Proposition 6.

We will illustrate how to use Proposition 1 by means of two of examples. In each case, we will derive a condition on model primitives which is necessary and sufficient for the remaining conditions of Proposition 1 to hold. First, we consider a competitive insurance market with an unbounded continuum of cost types. This setting is appealing because the equilibrium (when it exists) is particularly tractable and characterized by a simple differential equation. Our second example considers unbounded discrete types, which is a setting more similar to traditional analyses of insurance markets like RS and AG.

3.5 Insurance Market with a Continuum of Types

We now consider an insurance market with a continuum of cost types. Riley [1979] showed that a Nash equilibrium (in pure strategies) do not exist in in this setting. AG equilibrium exist (uniquely) if costs are bounded. We will obtain a simple condition which is necessary and sufficient for existence of equilibrium when cost is unbounded. When this condition does not hold, an economy is “fragile.” In this case, assumptions

about the support of costs can have large effects on equilibrium predictions. Moreover, expanding the support of the cost distribution results in market unravelling: the equilibrium allocation to each type converges to zero.

We assume that P_μ has support $[\underline{\mu}, \infty)$ and Lebesgue-a.e. positive density (so Assumption 3 holds). From Proposition 6, any equilibrium (p, α) is unique, has p strictly increasing, locally Lipschitz, and features a continuous and strictly increasing choice rule $x = \sigma(\mu)$.

We will first derive necessary conditions, and then show that these are also sufficient. In the main text, we assume differentiability and that all types maximize utility (instead of P -a.e. type). For general proofs, see Appendix D.

First, we consider a bounded economy \mathcal{E}^n where cost is $\mu \in [\underline{\mu}, \bar{\mu}_n]$. From Theorem 2, each \mathcal{E}^n has an equilibrium (p^n, α^n) with a continuous increasing allocation rule $\sigma^n : [\underline{\mu}, \bar{\mu}_n] \rightarrow [0, 1]$. We omit superscript n for clarity. For every type $\mu \in (\underline{\mu}, \bar{\mu}_n)$, the optimal choice is characterized by the First Order Condition

$$\frac{\partial u}{\partial x} \Big|_{x=\sigma(\mu)} = \mu + g'(\sigma(\mu))v(\mu) - p'(\sigma(\mu)) = 0.$$

Since $\sigma(\mu)$ is strictly increasing, it admits an inverse $\tau = \sigma^{-1}$, so we can re-write this as

$$\tau(x) + g'(x)v(\tau(x)) - p'(x) = 0.$$

Actuarially fair prices imply that, for every x in the interior of the support of α_X ,

$$p(x) = \tau(x) \cdot x \Rightarrow p'(x) = \tau(x) + x \cdot \tau'(x).$$

Summing these gives

$$\frac{\tau'(x)}{\nu(\tau(x))} = \frac{g'(x)}{x}. \quad (2)$$

Then $\sigma^n(\mu)$ can be recovered by integrating both sides of (2) over x and using the change of variables $\tau(x) = \mu$. Proposition 6 provides the boundary condition $\sigma^n(\bar{\mu}) = 1$. We therefore obtain the following result, which applies to any bounded insurance economy.

Corollary 2. *The bounded economy \mathcal{E}^n has a unique equilibrium (p^n, α^n) , where the choice rule $\sigma^n(\mu)$ satisfies, $\forall \mu \in [\underline{\mu}, \bar{\mu}_n]$,*

$$\int_{\mu}^{\bar{\mu}_n} \frac{1}{\nu(\mu')} d\mu' = \int_{\sigma^n(\mu)}^1 \frac{g'(x)}{x} dx \quad (3)$$

and, in particular, $\sigma^n(\mu) > 0$ for $\mu > \underline{\mu}$.¹⁶

¹⁶If $\nu(\underline{\mu}) > 0$, $\sigma(\underline{\mu}) > 0$.

Proof. Since $\frac{g'(x)}{x} \geq 0$, there is a unique $\sigma^n(\mu)$ that solves (3) for each μ , so integration establishes uniqueness and the (3). For details, see Appendix D.3. \square

We now consider the sequence of economies \mathcal{E}^n as $\bar{\mu}_n \rightarrow \infty$. Corollary 2 suggests that a necessary condition for the equilibria (p^n, α^n) to converge is, for some $\mu \in [\underline{\mu}, \infty)$,

$$\int_{\mu}^{\infty} \frac{1}{\nu(\mu')} d\mu' < \infty. \quad (4)$$

Corollary (3) formalizes the result.

Corollary 3. *Suppose (4) does not hold. Then:*

- For each $0 < x$, price diverges: $\lim_{n \rightarrow \infty} p^n(x) = \infty$.
- For each μ , coverage converges to zero: $\lim_{n \rightarrow \infty} \sigma^n(\mu) = 0$
- \mathcal{E} does not have an equilibrium.

Proof. From Corollary 1, as $\bar{\mu}_n \rightarrow \infty$, we must have $\lim_{x \rightarrow 1} p^n(x) = \infty$. However, the curves $p^n(x)$ must also satisfy incentive compatibility throughout. If the integrability condition fails, the curves $p^n(x)$ cannot be sufficiently convex to diverge at full insurance without also diverging at every other alternative. In that case, increasing $\bar{\mu}_n$ causes all prices to rise without bound and the allocation of every individual approaches $x = 0$. In the limit, equilibrium does not exist. For details, see Appendix D.5. \square

For “fragile” economies (where (4) does not hold) relaxing the truncation of the cost distribution results in the market progressively unravelling. The price of each positive coverage alternative increases without bound and, as a consequence, the levels of coverage chosen by each type μ approaches zero. Moreover, for fragile economies, the limit economy \mathcal{E} does not have an equilibrium. When (4) does not, assumptions about the cost distribution have an unbounded effect on the predictions regarding equilibrium prices. Notice that, if $\nu(\mu) = \nu_0$ is a constant (as in Rothschild and Stiglitz [1976]), the economy is fragile. While it is intuitive that the RS market would unravel if types with higher expected cost were added, we are not aware of any article that makes this point formally.

This result is illustrated in Figure 2. The left panel depicts a numerical simulation of an insurance economy where ν is constant and therefore (4) does not hold. The different curves correspond to the price function $p^n(x)$ in the equilibrium of several truncated economies \mathcal{E}^n with increasing values of $\bar{\mu}_n$, showing that $p^n(x)$ diverges for each x .¹⁷

¹⁷A graph of $\mu = \sigma^{-1}(x)$ would look similar (since each price p is associated to a single cost μ). This implies that, as $n \rightarrow \infty$, each type μ obtains progressively lower coverage in equilibrium.

Condition (4) is not just necessary for equilibrium existence, but also sufficient. When (4) is satisfied, changes in the support of the type distribution have a bounded effect on equilibrium predictions. Therefore, (4) creates a sharp distinction between fragile and robust economies. This is formalized by Corollary 4.

Corollary 4. *Suppose (4) holds. Then there exists a unique equilibrium (p, α) of \mathcal{E} , and the associated choice rule $\sigma : [\underline{\mu}, \infty] \rightarrow [0, 1)$ is defined by*

$$\int_{\underline{\mu}}^{\infty} \frac{1}{\nu(\mu')} d\mu' = \int_{\sigma(\underline{\mu})}^1 \frac{g'(x)}{x} dx. \quad (5)$$

and, in particular, $\sigma(\mu) > 0$ for $\mu > \underline{\mu}$.¹⁸

Proof. Condition (4) ensures that Proposition 1 applies ($p^n \rightarrow p$ uniformly on compact subsets of $X = [0, 1)$ and $\alpha^n \rightarrow \alpha$ weakly), hence equilibrium exists. Uniqueness and (5) follow since for any $\mu' > \mu > \underline{\mu}$

$$\int_{\underline{\mu}}^{\mu'} \frac{1}{\nu(\mu')} d\mu' = \int_{\sigma(\underline{\mu})}^{\sigma(\mu')} \frac{g'(x)}{x} dx$$

so taking $\mu' \rightarrow \infty$ gives the result, as $\lim_{\mu \rightarrow \infty} \sigma(\mu) = 1$ by Theorem 2. For details, see Appendix D.4. \square

The result is illustrated in Figure 2, where the right panel shows a setting where $\nu = \mu^{1.2}$, so the integrability condition (4) is satisfied. As the support of μ expands, the functions $p^n(x)$ converge to $p(x)$, and each type's choice $\sigma^n(\mu)$ converges to $\sigma(\mu)$.

How restrictive is (4)? It is satisfied, for instance, if P -a.s., ν grows asymptotically at least as fast as $C\mu^\alpha$ for some $C > 0$ and $\alpha > 1$.¹⁹ In the CARA-Gaussian framework of Example 1, $\nu_\theta = a_\theta \sigma_\theta^2$ is the product of the CARA risk aversion coefficient a_θ and the variance of shocks σ_θ^2 . In this case, (4) holds if a_θ is constant and σ_θ^2 increases more than linearly with the expected cost μ_θ . On the other hand, the result also shows that models where ν_θ is constant necessarily describe “fragile” economies.

Condition (4) seems empirically reasonable for markets like health and auto insurance where individuals with higher expected risk tend to have larger variance in outcomes (Hendren [2011], Brown et al. [2014]). For instance, Handel et al. [2015] estimate the distribution of healthcare expenditures conditional on individual covariates and based on an empirical model with CARA utility. Those authors find (Table III of their article) that, as age increases, both the variance σ_θ^2 and the mean μ_θ increase, but the former increases faster than linearly w.r.t. the latter.

¹⁸If $\nu(\underline{\mu}) > 0$, $\sigma(\underline{\mu}) > 0$.

¹⁹Formally, $\liminf_{\mu \rightarrow \infty} \frac{\nu(\mu)}{\mu^\alpha} > 0$, where the limit is taken along the support of P_μ .

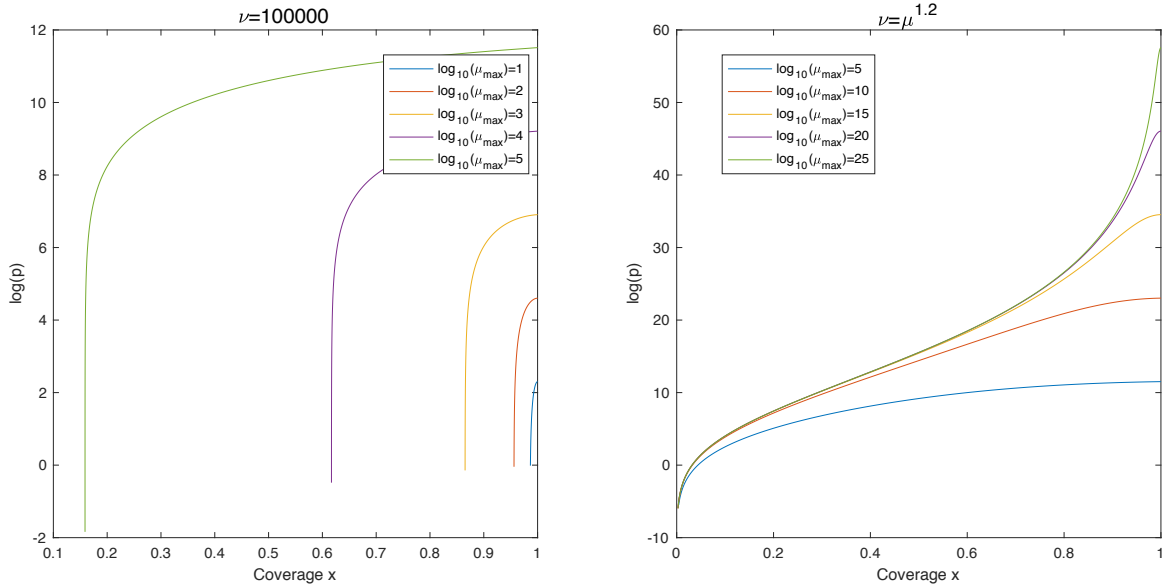


Figure 2: Left panel: equilibrium does not exist when ν is constant. Right panel: equilibrium does exist when $\nu = \mu^{1.2}$. The different curves correspond to the prices $p^n(x)$ of simulated economies where $\log_{10}(\bar{\mu}_n)$ took the values $\{1, 2, 3, 4, 5\}$ when ν constant and values $\{5, 10, 15, 20, 25\}$ when $\nu = \mu^{1.2}$.

Condition (4) will not hold, for instance, if $\nu \leq C\mu^\alpha + D$ with $\alpha \leq 1$ and $C, D \in \mathbb{R}$. Section 5.3 provides an example where this condition implies equilibrium non-existence, in a more general setting where type $\theta = (\mu_\theta, \nu_\theta)$ may be truly two-dimensional.

3.6 Example: Discrete types

We now consider an insurance market with unbounded discrete types. Our goal is to illustrate features of equilibrium familiar from RS and AG. For instance, $p(x)$ is the upper envelope of indifference curves (as in AG) and incentive compatibility constraints bind “downwards” (as in RS). We present heuristic arguments in the main text, and proofs in Appendix E.

The unbounded economy $\mathcal{E} = [\Theta, X, P]$ has types $\Theta = \{(\mu_k, \nu_k)_{k=1}^\infty\}$ where μ_k, ν_k are strictly increasing in k , with $\mu_k \rightarrow \infty, \nu_k \rightarrow \infty$. The alternative space is $X = [0, 1)$. The truncated economy $\mathcal{E}^n = [\Theta^n, \bar{X}, P(\cdot | \Theta^n)]$ has types $\Theta^n = \{(\mu_k, \nu_k)_{k \leq n}\}$, alternative space $\bar{X} = [0, 1]$ and distribution $P(\cdot | \Theta^n) = P(\cdot \cap \Theta^n) / P(\Theta^n)$.

We first construct the unique equilibrium of each \mathcal{E}^n . Type θ_k chooses (x_k, p_k) , where x_k, p_k are strictly increasing in k . Contracts break even: $p_k = x_k \mu_k, \forall k$. Type μ_k is indifferent between (x_k, p_k) and (x_{k-1}, p_{k-1}) . Combining these implies

$$\frac{\mu_k - \mu_{k-1}}{\nu_k} = \frac{g(x_k) - g(x_{k-1})}{x_{k-1}} \quad (6)$$

which is the discrete analogue of (2).

We now determine the equilibrium choices x_k^n of economy \mathcal{E}^n . The highest-cost type μ_n^n obtains the maximum available insurance contract $(1, \mu_n^n)$. Then, (6) pins down the value of x_{n-1}^n , then of x_{n-2}^n , and so forth. In fact, (6) implies that there exists a function $\phi_k(\cdot)$ that determines any $x_k^n = \phi_k(x_{k+1}^n)$ based on knowledge of x_{k+1}^n , so equilibrium choices can be defined recursively.²⁰

We now build the price function $p^n(x)$. Let $\mathcal{I}_k^n : [0, 1] \rightarrow \mathbb{R}$ be the indifference curve of type μ_k^n through her chosen contract, (x_k^n, p_k^n) . \mathcal{I}_k^n is expressed algebraically by (15) in Appendix A. We define $p^n(x) = \mathcal{I}_k^n(x)$ if $x \in [x_{k-1}^n, x_k^n]$.²¹ Incentive compatibility requires $\mathcal{I}_k^n(x_k^n) = \mathcal{I}_{k+1}^n(x_k^n)$, so $p^n(\cdot)$ is continuous.²² Moreover, $\mathcal{I}_k^n(x_k^n) = \mu_n x_k^n$ so the break-even condition is satisfied. Notice that $p^n(x)$ is the upper envelope of indifference curves at each individual's chosen contract, as in AG.

Proposition 2 below uses these results to characterize the (unique) equilibrium of \mathcal{E}^n , thereby generalizing AG's Corollary 1. Figure 3 provides a visual illustration of the equilibrium for \mathcal{E}^4 .

Proposition 2. *The truncated economy \mathcal{E}^n has a unique equilibrium where price is $p^n(\cdot)$ and the allocation α^n is concentrated on $(\mu_k, \nu_k, x_k^n)_{k \leq n}$, with $\alpha^n(\{(\mu_k, \nu_k, x_k^n)\}) = \frac{P(\mu_k)}{P(\Theta^n)}$.*

Proof. Follows from Theorem 2.²³ □

We now consider the unbounded economy \mathcal{E} . Appendix E.2 shows that $(x_k^n)_{n \geq k}^\infty$ is monotonically decreasing for each k , and hence $\lim_{n \rightarrow \infty} x_k^n = x_k \in [0, 1)$ exists, which will be the choice of type k in the limit economy \mathcal{E} . Moreover, continuity of $\phi_k(\cdot)$ also implies incentive compatibility in the limit economy, so $x_k = \phi_k(x_{k+1})$ as desired. The equilibrium allocation is $\alpha(\{(\mu_k, x_k)\}) = P(\mu_k)$. Moreover, we define $\mathcal{I}_k : [0, 1] \rightarrow \mathbb{R}$ as the indifference curve of μ_k through (p_k, x_k) , and piece these together as above to form the price function $p(x) = \mathcal{I}_k(x)$ if $x \in [x_{k-1}, x_k]$.²⁴

Since (6) is the discrete analogue of (2), it seems likely that, in this setting, a necessary condition for existence will be a discrete analogue of (4):

²⁰We have $x_k^n := \phi_k(\phi_{k+2}(\dots(\phi_{n-1}(1))\dots))$. In the CARA-Gaussian framework of Example 1, $\phi_k(\cdot)$ is the positive solution of a second degree equation.

²¹Let $x_0^n = \min[x \mid \mathcal{I}_0^n(x) \geq 0]$, and for convenience, set $x_{-1}^n = 0$ and $\mathcal{I}_0^n = 0$.

²²Notice also that, for $x \leq x_k^n$, the indifference curve is below the break even line ($\mathcal{I}_k^n(x) \leq \mu_n x_k^n$), as shown in Figures 1 and 3.

²³

Proof. As remarked there, to avoid circular logic, the uniqueness stated in that theorem should not be relied on here, but the specific allocations follows immediately from the other properties, as the highest type purchases full insurance and the other types allocations are determined inductively □

²⁴Again, $x_0 = \min[x \mid \mathcal{I}_0(x) \geq 0]$, and set $x_{-1} = 0$ and $\mathcal{I}_0 = 0$. Again $p(\cdot)$ is well-defined and continuous, and $p(x) = \max_{k \in \mathbb{N}} g_k(x)$. Note that p involves 'infinitely many pieces' and hence is not defined at $x = 1$.

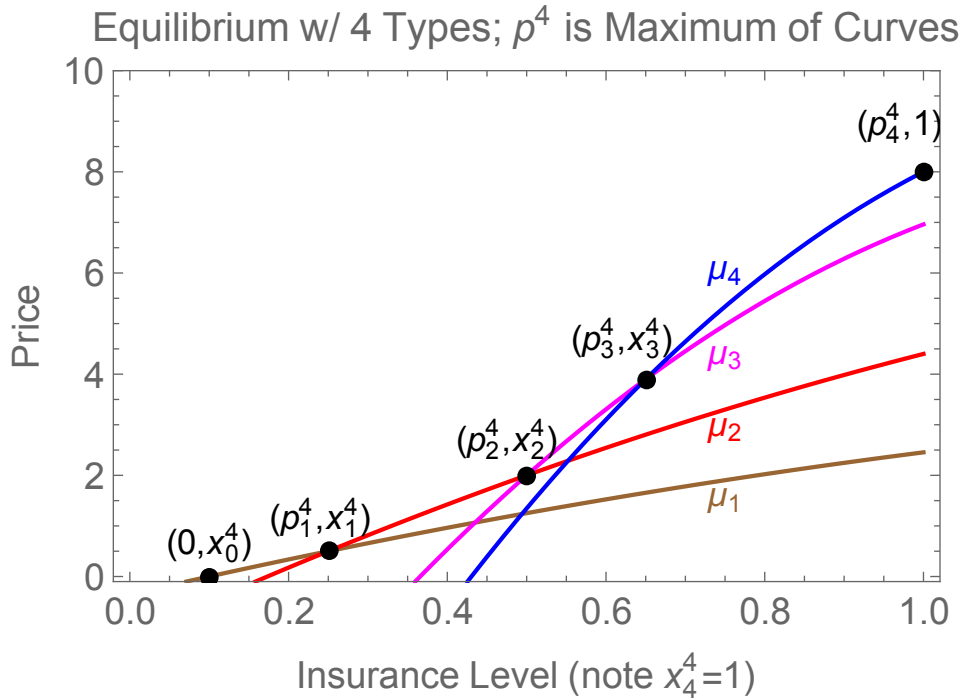


Figure 3: Construction of the Price Function $p^4(x)$ for \mathcal{E}^4 . For each interval $x \in [x_{k-1}^4, x_k^4]$, prices are given by $p^4(x) = \mathcal{I}_k^4(x)$ where $\mathcal{I}_k^4(x)$ is the indifference curve of type k , who chooses (x_k, p_k) . As in AG, the price $p^4(x)$ corresponds to the upper envelope of the indifference curves of buyers.

$$\sum_{k=1}^{\infty} \frac{\mu_{k+1} - \mu_k}{\nu_{k+1}} < \infty. \quad (7)$$

Proposition 3 below shows that this condition is indeed necessary, in parallel to Corollary 3.

Proposition 3. *Suppose (7) does not hold. Then:*

- For each $x > 0$, price diverges: $\lim_{n \rightarrow \infty} p^n(x) = \infty$
- For each $k \in \mathbb{N}$, coverage converges to zero: $\lim_{n \rightarrow \infty} x_k^n = 0$.
- \mathcal{E} does not possess an equilibrium.

Proof. See Appendix E.2. □

If (7) does not hold, as we relax the truncation on economy \mathcal{E} , the price of each contract increases without bound and each type's chosen coverage converges to zero. In the limit, the AG equilibrium does not exist.

In parallel to your analysis of economies with continuous types, (7) is also sufficient for equilibrium existence, as we now show.

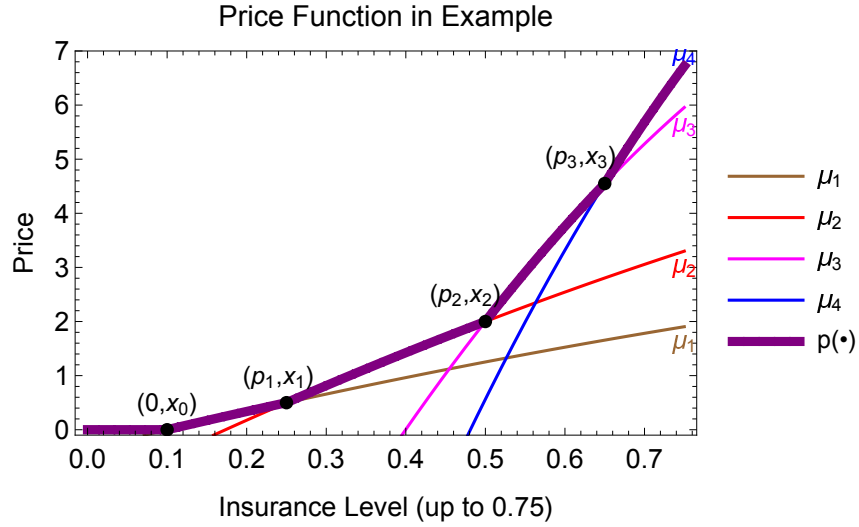


Figure 4: Construction of Price Function for the unbounded economy with discrete types. We show only the domain $x \in (0, \frac{3}{4})$, since $\lim_{x \rightarrow 1} p(x) = \infty$.

Proposition 4. *Suppose that (7) holds. Then:*

- $p_n \rightarrow p$ uniformly on compact subsets of $[0, 1)$, and $\alpha^n \rightarrow \alpha$ weakly. Therefore, by Proposition 1, (p, α) is an equilibrium.
- The equilibrium is unique.

Proof. Convergence follows from the definitions of α^n , α , p^n , p in terms of the x_k^n , since $x_k^n \rightarrow x_k$ for each k , and since each compact subset of $[0, 1)$ contains only finitely many of the points x_n . Details are given in Section E.3. Uniqueness is shown in Appendix E.2; essentially, it shows that in any equilibrium, for each $k \in \mathbb{N}$, the contact x_k purchased by type (μ_k, ν_k) must satisfy $x_k := \lim_{n \rightarrow \infty} \phi_k(\phi_{k+2}(\cdots(\phi_n(1))\cdots))$ and, in particular, that the limit is well-defined. \square

The intuition for the summability condition (7) is similar to that of (4). In this case, $p(x)$ is the upper envelope of indifference curves. The slope of these indifference curves is $w_\theta(x) = \mu_\theta + (1 - x)\nu_\theta$. For unbounded economies, $\lim_{x \rightarrow 1} p(x) = \infty$ by Proposition 3, so w_θ must increase sufficiently fast as $x \rightarrow 1$. Since the choice of x increases with μ , then ν must increase sufficiently fast to allow for $p(x) \xrightarrow{x \rightarrow 1} \infty$ without also arbitrarily raising prices for all $x < 1$ (which would result in non-existence). The function $p(\cdot)$ of the unbounded economy \mathcal{E} is illustrated in Figure 4.

We now give a heuristic argument as to why (7) guarantees existence of an equilibrium. In particular, if in equilibrium type (μ_k, ν_k) purchases covering x_k , the (x_k) are such that type (μ_{k+1}, ν_{k+1}) is indifferent between $(x_{k+1}, \mu_{k+1}x_{k+1})$ and $(x_k, \mu_k x_k)$. This implies

$$\mu_{k+1}x_k + g(x_k)\nu_{k+1} - \mu_k x_k = \mu_{k+1}x_{k+1} + g(x_{k+1})\nu_{k+1} - \mu_{k+1}x_{k+1} = g(x_{k+1})\nu_{k+1}$$

Denote $\Delta x_k = x_{k+1} - x_k$, $\Delta \mu_k = \mu_{k+1} - \mu_k$ and $\Delta g_k = g(x_{k+1}) - g(x_k)$. The expression above becomes

$$\Delta \mu_k x_k = \Delta g_k \nu_{k+1} \quad (8)$$

For such an equilibrium, we must have many (x_k) close to full coverage; fix T small and a range of values s.t. for $K_1 \leq k \leq K_2$, $x_k \approx 1 - T$. Now suppose $x_{k+1} > x_k$ are in this range; since $g'(1) = 0$, the second order expansion of Δg_k around $x = 1$ gives

$$\begin{aligned} \Delta g_k &= g(x_{k+1}) - g(x_k) \approx \left[g(1) + \frac{g''(1)}{2} (1 - x_{k+1})^2 \right] - \left[g(1) + \frac{g''(1)}{2} (1 - x_k)^2 \right] = \\ &= -\frac{g''(1)}{2} [2 - x_{k+1} - x_k] \Delta x_k \approx -g''(1) \cdot T \cdot \Delta x_k \end{aligned}$$

where the first \approx refers to a second-order approximation. Denote $t_k = \frac{\Delta \mu_k}{\nu_{k+1}}$ and $G = -g''(1) > 0$. Plugging this into (8) and

$$\Delta x_k \approx \frac{\Delta g_k}{G \cdot T} = \frac{t_k \cdot x_k}{G \cdot T} \approx \frac{(1 - T)}{G \cdot T} \cdot t_k \quad (9)$$

Hence,

$$\sum_k \Delta x_k < \infty \iff \sum_k t_k < \infty$$

A formal argument of the necessity of (7) could be made from the heuristic one above. An argument for sufficiency is somewhat more delicate but is similar to the argument given in H, which proves a very similar result in a more general setting.²⁵

Appendix J also contains an alternative proof that (p, α) is an equilibrium of \mathcal{E} . This is a “direct” construction of the equilibrium which does not use Proposition 1.

4 Market for Lemons

We now briefly consider a further simplification of insurance markets to the classic case of lemons markets, as in Akerlof [1970], Einav et al. [2010a], where individuals make a binary choice. Our goal is chiefly to illustrate the possibility of equilibrium non-existence when cost is unbounded, even in these very simple settings.

We maintain the insurance framework described in Section 3, in particular Assumption 1. We further assume that individuals make a binary choice, captured by the prod-

²⁵The difference is the following: The proof Lemma 17 begins by fixing types θ'', θ' and denoting $\mu'' := \mu(\theta'') > \mu' := \mu(\theta')$, $x'' := \sigma(\mu'') > x' := \sigma(\mu') \geq 1 - \delta$, and $\nu'' := \underline{\nu}_\delta(\theta'')$, $\nu' := \underline{\nu}_\delta(\theta')$, these last terms being defined in Section 5.1; for our simpler model, $\nu' = \nu(\theta')$, $\nu'' = \nu(\theta'')$. Equation (27) in that proof gives an inequality relating the differences Δx , Δp between the contracts purchased between these two types and the difference in prices, using μ' , μ'' , ν' but not ν'' . In our case, this inequality becomes equality, and - more importantly - with ν'' in place of ν' , which gives (when $\nu'' > \nu'$) a tighter bound.

uct space consisting of a single non-zero product, so $X = \{0, \bar{x}\}$ for some $\bar{x} > 0$. For simplicity, we assume that the insurance value $\nu = \nu(\cdot)$ is constant in this case, and assume μ distributes with positive PDF on some interval $(\underline{\mu}, \infty)$. Note that the utility from purchasing non-zero coverage at price p for an agent of riskiness μ is $\bar{x}\mu + g(\bar{x})\nu - p$.²⁶

4.1 Existence

AG show that an equilibrium exists in lemons markets, assuming risk μ is bounded. We extend this discussion to certain unbounded cases. Suppose an equilibrium exists. The equilibrium must prescribe $p(0) = 0$ and $p(\bar{x})$ equals the average cost of buyers. Henceforth, let $p = p(\bar{x})$ denote the price of the non-zero alternation. If $\bar{x} \cdot \underline{\mu} + g(\bar{x})\nu > \bar{x} \cdot \mathbb{E}[\mu]$, then charging price $p = \bar{x} \cdot \mathbb{E}[\mu]$ and having all agents purchase is an equilibrium. Otherwise, for $p = p(\bar{x})$, let the marginal type be μ^* defined by $\bar{x}\mu^* + g(\bar{x})\nu = p$. Equilibrium must occur at a price p where

$$p = \mathbb{E}[\bar{x}\mu \mid \mu \geq \mu^*].$$

If types are bounded, such a price p always exists. (The existence of such a p is necessary also for the existence of a pure-strategies Nash equilibrium, e.g. in the RS model in which firms set contracts and consumers respond.) Hence we can state:

Proposition 5. *If μ distributes with positive PDF on some interval $(\underline{\mu}, \infty)$ s.t.*

$$\mathbb{E}[\mu \mid \mu \geq M] - M \xrightarrow{M \rightarrow \infty} 0 \quad (10)$$

then an equilibrium exists for any $\bar{x} \in (0, 1]$ and any $\nu > 0$. In particular, (10) holds if μ distributes with PDF ϕ and CDF Φ s.t.

$$\lim_{M \rightarrow \infty} \phi(M) = \lim_{M \rightarrow \infty} \frac{1 - \Phi(M)}{\phi(M)} = 0.$$

Proof. The condition $p = \mathbb{E}[\bar{x} \cdot \mu \mid \mu \geq \mu^*]$ is equivalent to

$$\bar{x}\mu^* + g(\bar{x})\nu = \bar{x}\mathbb{E}[\mu \mid \mu \geq \mu^*] = \bar{x}\mu^* + \bar{x}\zeta(\mu^*).$$

That is, we define $\frac{g(\bar{x})}{\bar{x}}\nu = \zeta(\mu^*)$ where $\zeta(\mu^*) \xrightarrow{\mu^* \rightarrow \infty} 0$ is a continuous function. Either there exists such a point, or $\zeta(\underline{\mu}) < \frac{g(\bar{x})}{\bar{x}}\nu$, in which case, as remarked above, $p = \bar{x} \cdot \mathbb{E}[\mu]$ is an equilibrium in which all types purchase coverage.

²⁶We do not consider in this article existence when the space of alternatives is a fixed finite subset of $[0, 1]$. That analysis is outside the scope of this article and left for future research.

For the second part,²⁷ let $P(\cdot)$ denote probability. Then,

$$\begin{aligned} E[\mu \mid \mu \geq M] - M &= E[\mu - M \mid \mu - M \geq 0] = \frac{E[(\mu - M)^+]}{P(\mu \geq M)} = \frac{\int_0^\infty P((\mu - M)^+ \geq x) dx}{P(\mu \geq M)} \\ &= \frac{\int_0^\infty P((\mu - M)^+ \geq x) dx}{P(\mu \geq M)} = \frac{\int_M^\infty P(\mu \geq x) dx}{P(\mu \geq M)} = \frac{\int_M^\infty (1 - \Phi(x)) dx}{\Phi(M)} \end{aligned}$$

where $Y^+ = \max[Y, 0]$. Hence, by L'Hospital's law, since $\phi = \Phi'$,

$$\lim_{M \rightarrow \infty} E[\mu \mid \mu \geq M] - M = \lim_{M \rightarrow \infty} \frac{\int_M^\infty (1 - \Phi(x)) dx}{\Phi(M)} = \lim_{M \rightarrow \infty} \frac{1 - \Phi(M)}{\phi(M)} = 0$$

□

For example, suppose μ distributes with a half-Gaussian distribution. That is, $\mu \sim |\tilde{\mu}|$ for $\tilde{\mu} \sim N(0, 1)$. Let Φ denote the CDF of $N(0, 1)$ and $\phi(x) = \sqrt{\frac{2}{\pi}} \cdot e^{-\frac{1}{2}x^2}$ the PDF of μ in \mathbb{R}_+ . Notice that $\phi'(x) = (-x) \cdot \phi(x)$, so by L'Hopital's law, $\lim_{M \rightarrow \infty} \frac{1 - \Phi(M)}{\phi(M)} = \lim_{M \rightarrow \infty} \frac{-\phi(M)}{\phi'(M)} = \lim_{M \rightarrow \infty} \frac{1}{M} = 0$.

4.2 Non-existence

If μ is unbounded, it is possible that an equilibrium does not exist. This will occur if every price is lower than the average cost of those who would purchase at that price, so firms are unable to break even at any price. We now provide two simple examples.

Suppose that μ has an exponential distribution with parameter $\frac{1}{\lambda}$, so that $\mathbb{E}[\mu] = \lambda$ and $\mathbb{E}[\mu \mid \mu \geq \mu^*] = \mu^* + \lambda$. The condition for non-existence of equilibrium $p < \mathbb{E}[\bar{x}\mu \mid \mu \geq \mu^*]$ is equivalent to

$$\bar{x}\mu^* + g(\bar{x})\nu < \bar{x}(\mu^* + \lambda) \Leftrightarrow \frac{g(\bar{x})}{\bar{x}}\nu < \lambda.$$

Therefore, if average cost (λ) is large and risk aversion (ν) is low, equilibrium does not exist.

Suppose now that μ has a Pareto distribution²⁸ with parameters μ_0 and $\lambda > 1$. That is, μ is concentrated on (μ_0, ∞) with PDF $f(\mu) = \lambda\mu_0^\lambda\mu^{-\lambda-1}$. The conditional mean is $\mathbb{E}[\mu \mid \mu \geq \mu^*] = \frac{\lambda}{\lambda-1}\mu^*$ for any $\mu^* \geq \mu_0$. In this case, equilibrium does not exist if

$$\bar{x}\mu^* + g(\bar{x})\nu < \bar{x}\frac{\lambda}{\lambda-1}\mu^* \Leftrightarrow \frac{g(\bar{x})}{\bar{x}}\nu < \frac{1}{\lambda-1}.$$

Again, if risk aversion is sufficiently low, equilibrium will not exist.

²⁷The authors are grateful to Ilan Nehama for pointing out this argument.

²⁸That is, a distribution with CDF given by $F(\mu) = 1 - (\frac{\mu_0}{\mu})^\lambda$ for $\mu > \mu_0$.

5 Generalization & Variations

In this section, we generalize several of the results presented above.

5.1 General Insurance Setup

We now describe a more general insurance setting, where we relax Assumptions 1 and 2 and 3. We maintain the notation $\mu = \mu(\theta) = \mathbb{E}[Z_\theta]$, and the assumption that an alternative x covers a share x of the individual's cost. Therefore, cost is $c(x, \theta) = x\mu$. Unless otherwise specified, we take the alternative space to be either $X = [0, 1]$ or $X = [0, 1)$. We require the utility functions u_θ to be defined for all alternatives $[0, 1]$ (and all prices in \mathbb{R}_+).

We emphasize that, in this section, we do assume that the most generous contract is full insurance (unlike in Section 3).

We now list several assumptions, which will replace Assumptions 1 and 2. Not all of these assumptions are used simultaneously in the results below. We will be explicit about which are necessary for each of the results.

Assumption 4. *Utility is quasilinear in price: $u_\theta(x, p) = x\mu_\theta + g_\theta(x) - p$, with $g_\theta : [0, 1] \rightarrow \mathbb{R}$ smooth, strictly increasing and concave ($\frac{\partial g_\theta}{\partial x} > 0$, $\frac{\partial^2 g_\theta}{\partial x^2} < 0$ in $(0, 1)$), with $g'_\theta(1) = 0$.*

Assumption 4 generalizes Assumption 1, where it was required that $g_\theta(x) = g(x)\nu_\theta$. The more demanding aspect of Assumption 4 is quasi-linearity of utility. Notice that Assumption 4 guarantees the condition on utilities required by Proposition 8.

The marginal willingness to pay for additional insurance is

$$w_\theta(x) = \frac{\partial u_\theta}{\partial x}(x, p) = \mu_\theta + g'_\theta(x) \geq \mu_\theta$$

with equality if and only if $x = 1$. By quasi-linearity, w_θ is independent of price.

Notice that $-\frac{\partial w_\theta}{\partial x}(x) = -\frac{\partial^2 u_\theta}{\partial x^2}(x) > 0$ is the curvature of indifference curves in (x, p) -space for type θ . We will denote, for a given $\delta > 0$,

$$\bar{\nu}_\delta(\theta) := \max\left\{-\frac{\partial w_\theta}{\partial x}(x) \mid x \in [1 - \delta, 1]\right\} \quad (11)$$

$$\underline{\nu}_\delta(\theta) := \min\left\{-\frac{\partial w_\theta}{\partial x}(x) \mid x \in [1 - \delta, 1]\right\} \quad (12)$$

The quantity $\bar{\nu}_\delta(\theta)$ is the highest level of curvature, in (x, p) -space, that type θ exhibits among contracts with generosity greater than $1 - \delta$. Similarly, $\underline{\nu}_\delta(\theta)$ captures the lowest such curvature.²⁹

²⁹For instance, in the CARA-Gaussian framework of Example 1, $\bar{\nu}_\delta(\theta) = \underline{\nu}_\delta(\theta) = \nu_\theta$ is independent of δ .

The following assumption constrains the joint distribution of μ and ν .

Assumption 5. For each μ , there is a $\bar{\delta} > 0$ such that $\bar{\nu}_{\bar{\delta}}$ can be bounded as a function of μ . Formally, for each $\bar{\mu}$ s.t. $P(\mu \leq \bar{\mu}) > 0$, there is $\bar{\rho} = \bar{\rho}_{\bar{\delta}}(\bar{\mu})$ s.t. $P(\bar{\nu}_{\bar{\delta}}(\theta) \leq \bar{\rho} \mid \mu \leq \bar{\mu}) = 1$.

WLOG, we can take $\bar{\rho}_{\bar{\delta}}(\cdot)$ increasing and right-continuous.³⁰ This assumption, while allowing for unbounded μ , does not allow the distribution of (μ, ν) to be, for instance, bivariate Gaussian or lognormal, although these distributions can be approximated arbitrarily closely.

The following assumption relaxes Assumption 2.

Assumption 6. $w_{\theta_2} \geq w_{\theta_1}$ iff $\mu(\theta_2) \geq \mu(\theta_1)$.

If Assumption 6 holds, all types with the same riskiness have the same utility function, so we may view $\underline{\nu}_{\delta}$ as a function of μ , defined on the support of P_{μ} . That is, types are effectively one-dimensional. Notice that assumption 6 implies Assumption 5.

Recall $\underline{\mu} = \min(\text{supp}(P_{\mu}))$. If (a, b) is a maximal open interval not intersecting the support of P_{μ} ,³¹ we interpret $\underline{\nu}_{\delta}(\mu) = \underline{\nu}_{\delta}(a)$ in (a, b) . That is, we complete the function $\underline{\nu}_{\delta}$ from the domain $\text{supp}(P_{\mu})$ to $[\underline{\mu}, \infty)$ such that

$$\underline{\nu}_{\delta}(\mu) = \nu_{\delta}(\max\{\zeta \in \text{supp}(P_{\mu}) \mid \zeta \leq \mu\}).$$

5.2 Equilibrium Properties in Insurance Markets

We now show that the insurance market equilibrium properties of Proposition 6 hold more generally than previously stated. The result does not require Assumptions 5, nor Assumption 4, using instead a generalization (Assumption 7), described below.

Assumption 7. Utility is $u : \Theta \times [0, 1] \times \mathbb{R}_+$ is continuous, $u_{\theta}(\cdot, \cdot)$ is twice differentiable for all $\theta \in \Theta$ and the second derivatives are continuous in (θ, x, p) , with $\frac{\partial u_{\theta}}{\partial x} > 0$, $\frac{\partial u_{\theta}}{\partial p} < 0$, and

$$w_{\theta}(x, p) = -\frac{\partial u_{\theta}}{\partial x}(x, p) / \frac{\partial u_{\theta}}{\partial p}(x, p)$$

satisfies $w_{\theta} \geq \mu_{\theta}$ with equality iff $x = 1$, $\frac{\partial w_{\theta}}{\partial x} \leq 0$ in $(0, 1)$.

Note that, now, w_{θ} depends on both coverage and price.

We will need the following technical assumption:

Assumption 8. For every $\mu^* > \underline{\mu}$, the set $\{\theta \mid \mu(\theta) \leq \mu^*\}$ is compact in Θ .

³⁰If $\bar{\rho}_{\bar{\delta}}(\cdot)$ is not monotonic, it can just be replaced with its 'monotonic closure', $x \rightarrow \sup_{0 \leq y \leq x} \bar{\rho}_{\bar{\delta}}(y)$. If it is not right-continuous, it can be replaced with $x \rightarrow \lim_{y \rightarrow x^+} \bar{\rho}_{\bar{\delta}}(y)$. Both of these operations preserve linear growth, which is what we require in Proposition 7.

³¹That is, there are no types with riskiness μ in (a, b) , but $a, b \in \text{supp}(P_{\mu})$.

That is, for each upper bound on cost, only a compact set of types have cost below this bound. Like Assumption 6, Assumption 8 rules out e.g. log-normal distributions. We know of no case of interest in which the former assumption holds and the latter does not, so Assumption 8 is with a small additional loss of generality.

The following result generalized Theorem 2 and thereby generalizes the treatment of RS in AG.

Proposition 6. *Suppose that Assumptions 7, 6, and 8 hold. Any equilibrium (p, α) of the bounded or unbounded insurance economy $\mathcal{E} = [\Theta, X, P]$, satisfies the following properties:*

1. Price $p(x)$ is continuous and strictly increasing on the domain on which it is non-zero. (Lemma 3)
2. There is a continuous mapping $\sigma : \text{supp}(P_\mu) \rightarrow [0, 1]$, strictly increasing on the domain in which it is non-zero, that assigns to each type μ her chosen contract. Formally, $\alpha\{(\mu, x) \mid x = \sigma(\mu)\} = 1$.³² (Lemma 11)
3. Contracts are actuarially fair. Formally, $P - a.s., p(\sigma(\mu)) = \mu \cdot \sigma(\mu)$. (Lemma 10)
4. Full insurance is in the support of the equilibrium.³³ (Corollary 13.)
5. Let $x_0 < 1$. If $L \geq w_\theta$ in $\{x \leq x_0\}$ for a.e. θ which chooses coverage up to x_0 (i.e., $\alpha(\cdot \mid x \leq x_0) - a.s.$) then $p(\cdot)$ is L -Lipshitz in $[0, x_0]$.³⁴ Formally, $L = \text{ess} - \text{sup}\{\max_{x \in [0, 1]} (w_\mu \mid \sigma(\mu) \leq x_0)\}$.³⁵
6. Let $x \in (0, 1]$ and $p(x) > 0$. Let $\eta(\mu, x)$ be the price at which type μ is indifferent between his purchased contract $(\sigma(\mu), p(\sigma(\mu)))$ and $(x, \eta(\mu, x))$. Then, for every $y > x$ for which $\alpha_X((x, y]) > 0$,

$$p(x) = \text{ess} - \text{sup}\{\eta(\mu, x) \mid x < \sigma(\mu) \leq y\} \quad (13)$$

That is, price is an envelope of indifference curves, and in fact one can take just those indifference curves of types who buy the 'next alternatives above it'.³⁶

³²Under Assumptions 1, 2, and 3, we show that $x = 0$ is never chosen in equilibrium (see Sections 3.5 and 3.6). However, we have not been able to show this in more general settings. Therefore, we only assert that σ is strictly increasing on $\text{supp}(P_\mu) \setminus \sigma^{-1}(\{0\})$.

³³The support of a positive measure is the smallest closed non-null set. Therefore, we mean that every neighborhood of $x = 1$ has strictly positive measure under α .

³⁴If Θ is compact, this implies that the price function is Lipshitz.

³⁵Recall that the essential supremum ("*ess - sup*") of a random variable X with distribution P is the supremum over all $x \in \mathbb{R}$ s.t. $P(X > x) > 0$; intuitively, this is the 'supremum up to measure zero'.

³⁶In particular, if $\mu_1 < \mu_2$ are atoms of P_μ but there are α -a.s. no types with riskiness between μ_1, μ_2 , then type μ_2 is indifferent between $(\sigma(\mu_2), p(\sigma(\mu_2)))$ and the alternative "just below", $(\sigma(\mu_1), p(\sigma(\mu_1)))$. This follows by taking $x = \sigma(\mu_1), y = \sigma(\mu_2)$ in the first part.

Proposition 1, Part (6)
 $\sigma(\mu)=y, \sigma(\mu')=y', \sigma(\mu'')=y''$

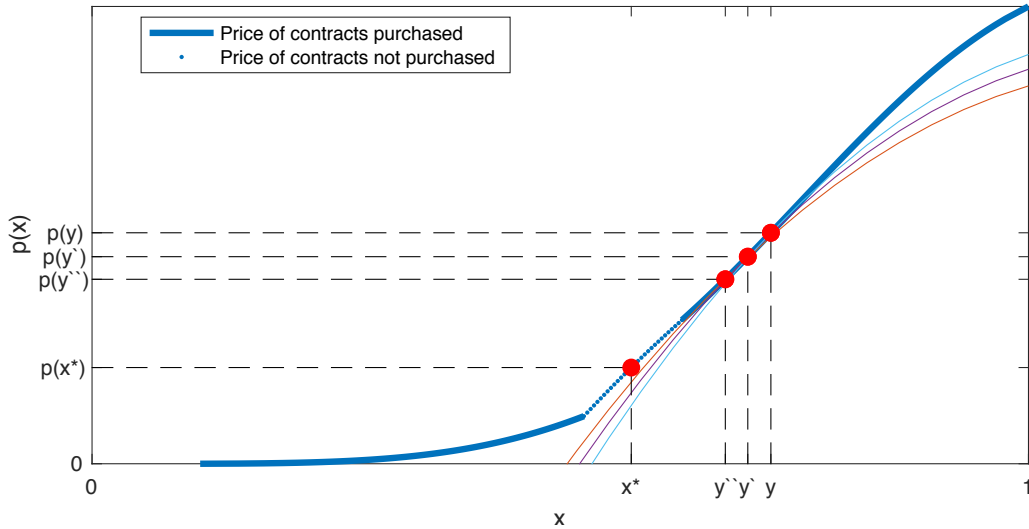


Figure 5: Suppose an alternative x^* is not purchased in equilibrium. Consider a sequence of types $\mu, \mu', \mu'' \dots$ who purchase coverages $y, y', y'' \dots$, respectively, which converge to the infimum of purchased alternatives strictly above x^* . Consider the indifference curves of these types. The price at x^* is the supremum of the indifference curves of these types at x^* . A similar figure holds when x^* is in the support of the equilibrium but no alternative in a right neighborhood of x^* is purchased (e.g., when α_X is purely atomic).

7. If in addition Assumption 3 holds, then 0 is α -a.s. never purchased, and in particular $\sigma : \text{supp}(P_\mu) \rightarrow [0, 1]$ is strictly increasing.

Proof. See Appendix G. □

Proposition 6 shows that several features of competitive insurance markets, derived by RS and AG, remain true in unbounded settings (if an equilibrium exists). The allocation $x = \sigma(\mu)$ is increasing in risk μ , each contract breaks even, and incentive compatibility constraints bind “downward.” Notice that the equilibrium price $p(x)$ is the upper envelope of the indifference curves, which is illustrated in Figure 5.

5.3 Unbounded Price in Insurance Markets

We now show that, more generally in insurance markets with unbounded cost, price is unbounded. The proof uses Assumptions 4 and 5. However, the result does not require $X = [0, 1]$ or $X = [0, 1)$, instead requiring only that $X \subseteq [0, 1)$ be Borel³⁷ such that

³⁷We require this to be able to define a Borel measure on X . As the example of Section 3.4 hints, a countable discrete set is often appropriate. An obvious initial objection to a non-compact space of contracts is that agents may not have a best contract, even when prices are continuous. However, when utilities extend continuously to $[0, 1] \times \mathbb{R}_+$ and $\lim_{x \rightarrow 1} p(x) = \infty$, agents do attain their maximal utility contract.

full insurance is a limit point of X .³⁸ Moreover, the proof does not require that ν is a function of μ (Assumption 6) and therefore applies to settings where types are truly multi-dimensional. Let $\overline{X \setminus \{1\}}$ be the closure of $X \setminus \{1\}$.

Theorem 3. *Suppose that Assumptions 4 and 5 hold, and that P_μ is not compactly supported. Then, in any equilibrium,*

$$\lim_{x \rightarrow 1} p(x) = \infty$$

In particular, $1 \notin X$, i.e., full insurance is not a possible contract.

Proof. First, we use Assumption 1 to establish bounds on the slope of $p(x)$ in equilibrium. Then, we show that full insurance cannot be an atom of an equilibrium (unless $x = 1$ is bought by a single type). Then, we show that smaller and smaller neighborhoods of full insurance attract arbitrarily large risks. Finally, we use this to show that the price cannot be bounded in equilibrium. For details, see Appendix B. \square

The use of Assumption 5 should not be seen as imposing significant restrictions on the result's implications (even though Assumption 5 excludes common distribution such as bivariate Gaussian or lognormal distributions). First, it is not clear if this assumption is needed, as we have not found any example of a distribution with unbounded costs but with an equilibrium having bounded prices. Second, any distribution is arbitrarily well approximated by a distribution satisfying Assumption 5. Suppose, for instance, that the distribution of types $f^*(\theta)$, for which 5 did not hold, resulted in a bounded equilibrium price $p^*(\cdot)$. Approximating $f^*(\theta)$ arbitrarily closely with a distribution $f(\theta)$ for which 5 holds, would result in some unbounded equilibrium prices $p(\cdot)$ which would be arbitrarily different from $p^*(\cdot)$. Therefore, the result should be understood to say that no economy satisfying Assumption 4 and P_μ is not compactly supported, can possess an equilibrium with bounded prices which is robust in any reasonable sense.

Figure 6 illustrates graphically one of the steps of the proof of Theorem 3. If μ is unbounded, full insurance cannot be an atom, purchased by multiple types, of the equilibrium (Lemma 4) even though $x = 1$ must be in the support of the equilibrium (this follows from full insurance being a limit point of X and the unbounded willingness to purchase insurance). At full insurance, $g'_\theta(1) = 0$, so indifference curves are (approximately) straight lines with slope determined by μ . If multiple types bought full insurance, then there would be some type μ^* which is the cheapest (lowest μ) buyer of $x = 1$. Necessarily, $\mu^* < p(1)$. However, since $x = 1$ is a limit point of X , there is some type

³⁸I.e., there is a sequence (x_n) in X with $x_n \rightarrow 1$ but $\forall n, x_n < 1$. We require that $x = 1$ is a limit point of X in order to prove that full insurance is not an atom of the equilibrium, in Proposition 3.

purchasing a slightly lower alternative $(x, p(x))$ who has riskiness $\tilde{\mu} \geq \frac{p(x)}{x} > \mu^*$. This type $\tilde{\mu}$ would then prefer to purchase coverage greater than x . Therefore, this cannot be an equilibrium. A slightly technical argument but with a similar geometric flavor then shows that price could not be bounded.

Contradiction to Atom at Full Insurance:
 $(p^*, 1)$ Should Be Above Red Dashed Line

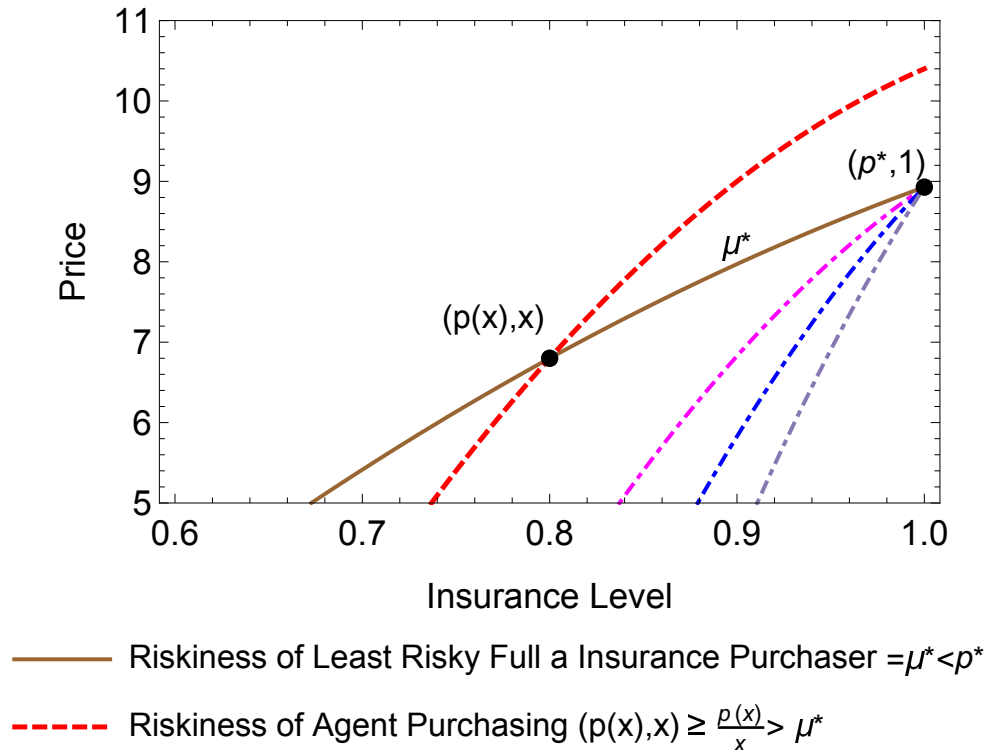


Figure 6: Proof of Lemma 4.

5.4 Non-Existence in Insurance Markets

In Section 3 we mentioned that the condition (4) required for existence of equilibrium there would not hold, for instance, if $\nu \leq C\mu^\alpha + D$ with $\alpha \leq 1$. Theorem 3 describes a more general class of economies where equilibrium cannot be bounded, and we now show that under these more general conditions, such a bound will imply non-existence of equilibrium. The result requires only Assumption 4 and 5, so types are allowed to be “truly” multidimensional.

Proposition 7. *Under Assumption 4 and 5, if P_μ is not compactly supported, and for some $\delta, C, D > 0$ P -a.s., $\bar{\nu}_\delta(\theta) \leq C\mu(\theta) + D$, there exists no AG equilibrium.*

Proof. We give here a heuristic argument under the simple setup of Assumption 1, if equilibrium p is continuously differentiable, $X = [0, 1)$, and the equilibrium distribution, marginal on the alternatives $[0, 1]$, has full support at least in a neighborhood $(\delta, 1]$.

The first-order condition of utility maximization of a type θ purchasing insurance in $(\delta, 1)$ is $p'(x) = \mu_\theta + g'(x)\nu_\theta$. In particular, the break-even condition $p(x) = \mathbb{E}_\alpha[c | x], \alpha - a.e. x$ suggests for any such x , there is a type θ purchasing x such that $p(x) \geq x\mu_\theta$. Hence,

$$\begin{aligned} p'(x) &\leq \mu_\theta + g'(x)\nu_\theta \leq \frac{p(x)}{x} + g'(x) \cdot (C\frac{p(x)}{x} + D). \\ &= (1 + Cg'(x))\frac{p(x)}{x} + g'(x)D \leq A \cdot p(x) + B \end{aligned}$$

where $A = \frac{1}{\delta}(1 + \sup_{[\delta,1]} g'(x))$, $B = D \sup_{[\delta,1]} g'(x)$. This will imply, for $x \geq \delta$,

$$(e^{-Ax}p(x))' \leq e^{-Ax} \cdot B \longrightarrow e^{-Ax}p(x) \leq e^{-A\delta}p(\delta) - \frac{B}{A}[e^{-Ax} - e^{-A\delta}]$$

Therefore $p(x)$ is bounded, which results in a contradiction to Theorem 3: no AG equilibrium exists. For a full proof, see Appendix C. \square

In the framework of Section 3.1, Proposition 7 applies if, for some $C, D > 0$, P -a.s. it holds that $\nu_\theta \leq C\mu_\theta + D$. That is, if ν_θ is bounded by a linear function of μ_θ . This includes the case of consumers homogeneous in risk aversion (as in RS) but also applies in settings where types are truly multidimensional.

5.5 Existence in Insurance Markets

We now generalize the existence results for insurance markets of Sections 3.5 and 3.6. The result uses Assumptions 4 and 6 and therefore requires that types be effectively one-dimensional.

Theorem 4. *Suppose that Assumptions 4 and 6 hold. Denote $X = [0, 1)$. Suppose that for some $\delta > 0$,*

$$\int_{\underline{\mu}}^{\infty} \frac{1}{\underline{\nu}_\delta(\mu)} d\mu < \infty \quad (14)$$

Then the unbounded economy \mathcal{E} possesses an AG equilibrium.

Proof. See Appendix H. \square

When P_μ is purely atomic, supported on $\mu_1 < \mu_2 < \mu_3 < \dots$, (14) should be written as

$$\sum_{n=1}^{\infty} \frac{\mu_{n+1} - \mu_n}{\underline{\nu}_\delta(\mu_n)} d\mu < \infty$$

5.6 Existence When the Limit Equilibrium is Not Known

As mentioned, a limitation of Proposition 1 is that it requires knowledge of the equilibrium of the limit economy, (p, α) , which is often not available. Proposition 8 is a useful variant which does not require such knowledge, but requires stronger assumptions.

Proposition 8. *Assume that for every two alternatives $x, y \in X$, price $p \geq 0$, and type $\theta \in \Theta$, there is price q high enough s.t. $u(\theta, x, p) > u(\theta, y, q)$. For each n , let (p^n, α^n) be an equilibrium of \mathcal{E}^n such that:*

- *The collection $(p^n)_{n=1}^\infty$ is point-wise bounded and equicontinuous in X .³⁹*
- *For every $M \in \mathbb{R}$, there is a compact subset K of X , s.t. $\inf_{x \notin K} p^n(x) \geq M$ for all n large enough.*
- *Condition (*) of Proposition 1.*

Then there exists an equilibrium (p, α) of the unbounded economy \mathcal{E} , which is a limit of a subsequence of the equilibria $(p^n, \alpha^n)_{n=1}^\infty$ in the sense of Proposition 1.

Proof. See Appendix F2. □

We use Proposition 8 in Section 5.1 to derive existence results in insurance markets which are more general than those considered in Section 3. Note that the first condition required in Proposition 8 (one option is preferred over another if the latter's price is high enough) holds when utility is quasi-linear in price, as in Section 3.

6 Conclusion

The analysis of insurance markets in RS and AG is sensitive to assumptions on the bounds of the cost distribution, which can significantly reduce the predictive power of such models. We show that, when assumptions on these bounds are fully relaxed, AG equilibria need not exist. However, we also provide sufficient conditions for equilibrium to exist when costs are unbounded.

We apply our results to insurance markets. In a simplified setting, we derive new properties of equilibrium that generalize AG and RS. We then present a condition which is necessary and sufficient for the existence of a unique equilibrium in these markets. In particular, we are able to derive a characterization of equilibrium for insurance markets with an unbounded continuum of types which is particularly tractable and based on a

³⁹A collection of real-valued functions \mathcal{F} on a metric space (X, d) is point-wise bounded if $\forall x \in X$, $\sup_{f \in \mathcal{F}} |f(x)| < \infty$, and \mathcal{F} is equicontinuous if for each $\varepsilon > 0$ and each $x \in X$, there is $\delta > 0$ such that if $y \in X$ with $d(x, y) < \delta$, then $|f(y) - f(x)| < \varepsilon$ for all $f \in \mathcal{F}$.

simple differential equation. We also show that, when it exists, the equilibrium features unbounded price at full insurance.

Our characterizations of equilibrium for simple insurance markets required a number of assumptions on utilities, contracts, and the distribution of types. In particular, several of our results require that types effectively be one-dimensional. Relaxing these assumptions would be an interesting avenue for future research.

Moreover, we consider only the AG equilibrium concept, since it exists in a wide range of screening markets. It would be interesting to replicate the exercise in this paper to other equilibrium concepts, such as those of [Miyazaki \[1977\]](#), [Wilson \[1977\]](#), [Spence \[1978\]](#).⁴⁰

We also do not consider in this article existence of equilibrium when the space of alternatives is a fixed finite subset of $[0, 1]$. This analysis is outside the scope of this article and also left for future research.

⁴⁰[Gemmo et al. \[2018\]](#) shows existence of the MWS equilibrium for continuous type distributions.

References

- George A. Akerlof. The market for “lemons”: Quality uncertainty and the market mechanism. *Quarterly Journal of Economics*, 84(3):488–500, 1970.
- Eduardo M Azevedo and Daniel Gottlieb. An example of non-existence of riley equilibrium in markets with adverse selection. 2016.
- Eduardo M. Azevedo and Daniel Gottlieb. Perfect competition in markets with adverse selection. *Econometrica*, 85(1):67–105, 2017. ISSN 1468-0262. doi: 10.3982/ECTA13434. URL <http://dx.doi.org/10.3982/ECTA13434>.
- Jason Brown, Mark Duggan, Ilyana Kuziemko, and William Woolston. How does risk selection respond to risk adjustment? new evidence from the medicare advantage program. *The American Economic Review*, 104(10):3335–3364, 2014.
- Alma Cohen and Liran Einav. Estimating risk preferences from deductible choice. *American Economic Review*, 97(3):745–788, 2007.
- Keith J Crocker and Arthur Snow. Multidimensional screening in insurance markets with adverse selection. *Journal of Risk and Insurance*, 78(2):287–307, 2011.
- Pradeep Dubey and John Geanakoplos. Competitive pooling: Rothschild-stiglitz reconsidered. *quarterly Journal of economics*, pages 1529–1570, 2002.
- Liran Einav, Amy Finkelstein, and Mark R. Cullen. Estimating welfare in insurance markets using variation in prices. *Quarterly Journal of Economics*, 125(3):877–921, 2010a.
- Liran Einav, Amy Finkelstein, and Paul Schrimpf. Optimal mandates and the welfare cost of asymmetric information: Evidence from the uk annuity market. *Econometrica*, 78(3):1031–1092, 2010b.
- Liran Einav, Mark Jenkins, and Jonathan Levin. Contract pricing in consumer credit markets. *Econometrica*, 80(4):1387–1432, 2012.
- Liran Einav, Amy Finkelstein, Stephen P. Ryan, Paul Schrimpf, and Mark R. Cullen. Selection on moral hazard in insurance markets. *American Economic Review*, 103(1): 178–219, 2013.
- Irina Gemmo, Christian Kubitza, and Casey Rothschild. The existence of the miyazaki-wilson-spence equilibrium with continuous type distributions. *Available at SSRN 3285728*, 2018.

- Ben Handel, Igal Hendel, and Michael D. Whinston. Equilibria in health exchanges: Adverse selection vs. re-classification risk. https://www.econ.berkeley.edu/%7Ebhandel/wp/ASRC_2013.pdf, 2015.
- Benjamin R. Handel. Adverse selection and inertia in health insurance markets: When nudging hurts. *American Economic Review*, 103(7):2643–2682, 2013.
- Nathaniel Hendren. Private information and insurance rejections. <http://econ-www.mit.edu/files/7488>, 2011.
- Yehuda Levy and Andre Veiga. Frictions and equilibria in insurance markets. *Working Paper*, 2017.
- Hajime Miyazaki. The rat race and internal labor markets. *The Bell Journal of Economics*, 8(2):394–418, 1977.
- Pierre Picard. Splitting risks in insurance markets with adverse selection. 2017.
- John G. Riley. Informational equilibrium. *Econometrica*, 47(2):331–359, 1979.
- Michael Rothschild and Joseph E. Stiglitz. Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *Quarterly Journal of Economics*, 90(4):629–649, 1976.
- Walter Rudin. *Real and complex analysis*. Tata McGraw-Hill Education, 1987.
- Michael Smart. Competitive insurance markets with two unobservables. *International Economic Review*, 41(1):153–170, 2000.
- Arthur Snow. On the possibility of profitable self-selection contracts in competitive insurance markets. *Journal of Risk and Insurance*, 76(2):249–259, 2009.
- Michael Spence. Product differentiation and performance in insurance markets. *Journal of Public Economics*, 10(3):427–447, 1978.
- Andre Veiga and E Glen Weyl. Product design in selection markets. *Quarterly Journal of Economics*, 2016.
- Bertrand Villeneuve. Concurrence et antisélection multidimensionnelle en assurance. *Annales d'Economie et de Statistique*, pages 119–142, 2003.
- Achim Wambach. Introducing heterogeneity in the rothschild-stiglitz model. *Journal of Risk and Insurance*, 67(4):579–591, 2000.
- Charles Wilson. A model of insurance markets with incomplete information. *Journal of Economic Theory*, 16(2):167–207, 1977.

A Comparisons of Utilities

Assumption 4 implies Lemma 1 below, which is later used to obtain upper and lower bounds on the slopes of the price in equilibrium.

Lemma 1. *Under Assumption 4, in any equilibrium (p, α) , if $1 \geq x_2 \geq x_1 \geq \delta \geq 0$, then for a given type $\theta \in \Theta$:*

$$u_\theta(p_2, x_2) \geq u_\theta(p_1, x_1) \Rightarrow \frac{p_2 - p_1}{x_2 - x_1} \leq \mu_\theta + \bar{\nu}_\delta(\theta) \cdot \left(1 - \frac{x_1 + x_2}{2}\right) \quad (15)$$

and

$$u_\theta(p_2, x_2) \leq u_\theta(p_1, x_1) \Rightarrow \frac{p_2 - p_1}{x_2 - x_1} \geq \mu_\theta + \underline{\nu}_\delta(\theta) \cdot \left(1 - \frac{x_1 + x_2}{2}\right) \quad (16)$$

Proof. By Assumption 4, $u_\theta(x, p) = g_\theta(x) - p$ and $w_\theta(1) = \mu_\theta + \frac{\partial g_\theta}{\partial x}|_{x=1} = \mu_\theta$, so

$$\frac{\partial g_\theta}{\partial x}(s) = \mu_\theta - \int_s^1 \frac{\partial^2 g_\theta}{\partial^2 x} dx.$$

Now, for any $0 \leq x_1 < x_2 \leq 1$,

$$u_\theta(x_2, p_2) - u_\theta(x_1, p_1) = p_1 - p_2 + \int_{x_1}^{x_2} \frac{\partial g_\theta}{\partial x} dx = p_1 - p_2 + \mu_\theta(x_2 - x_1) + \int_{x_1}^{x_2} \int_s^1 \left(-\frac{\partial^2 g_\theta}{\partial^2 x}\right) dx.$$

Denote

$$\Delta(x_1, x_2) = \int_{x_1}^{x_2} \int_s^1 1 dx = \frac{1}{2}(1 - x_1)^2 - \frac{1}{2}(1 - x_2)^2 = (x_2 - x_1)\left(1 - \frac{x_1 + x_2}{2}\right).$$

If $x_2 > x_1 \geq \delta$, $\underline{\nu}_\delta(\theta) \cdot \Delta(x_1, x_2) \leq \int_{x_1}^{x_2} \int_s^1 \left(-\frac{\partial^2 g_\theta}{\partial^2 x}\right) dx \leq \bar{\nu}_\delta(\theta) \cdot \Delta(x_1, x_2)$, and hence

$$\begin{aligned} p_1 - p_2 + \mu_\theta(x_2 - x_1) + \Delta(x_1, x_2) \cdot \underline{\nu}_\delta(\theta) &\leq u_\theta(x_2, p_2) - u_\theta(x_1, p_1) \leq \\ &\leq p_1 - p_2 + \mu_\theta(x_2 - x_1) + \Delta(x_1, x_2) \cdot \bar{\nu}_\delta(\theta) \end{aligned}$$

Dividing by $x_2 - x_1$ gives the lemma. □

B Unbounded Price

In this section prove Theorem 3: when μ is unbounded, any AG equilibrium has $p(\cdot)$ unbounded. The result requires Assumptions 4 and 5. However, it does not require Assumption 2 or its weaker analogue, Assumption 6.

Throughout this section, fix $\bar{\delta} > 0$ for which Assumption 5 holds. Suppose that (p, α) constitute an AG-equilibrium. We denote the marginal distribution of α on Θ by P , and denote the marginal of α on X by α_X . Consider any $x \in \text{supp}(\alpha_X)$.

We now define analogues of the maximum and minimum risk μ which purchases each alternative x , which will also be of use in later sections. To eliminate the influence of zero-measure types μ purchasing x , we use a variation of the essential supremum and infimum, defined for $x \in \text{supp}(\alpha_X)$ as⁴¹

$$\psi^+(x) = \lim_{\delta \rightarrow 0^+} \left[\sup \left\{ \mu \mid \alpha(\{\theta \mid \mu_\theta \geq \mu\} \times (x - \delta, x + \delta)) > 0 \right\} \right] \quad (17)$$

$$\psi^-(x) = \lim_{\delta \rightarrow 0^+} \left[\inf \left\{ \mu \mid \alpha(\{\theta \mid \mu_\theta \leq \mu\} \times (x - \delta, x + \delta)) > 0 \right\} \right]. \quad (18)$$

Intuitively, $\psi^+(x)$ captures the largest value of μ which purchases x under α , and $\psi^-(x)$ as the lowest such value of μ .

Note that either of these quantities can, a priori, be infinite. Moreover, note that for each $A \subseteq \text{supp}(\alpha_X)$,

$$\sup \left\{ \mu \mid \alpha(\{\theta \mid \mu_\theta \geq \mu\} \times A) > 0 \right\} \leq \sup_{x \in A} \psi^+(x) \quad (19)$$

$$\inf \left\{ \mu \mid \alpha(\{\theta \mid \mu_\theta \leq \mu\} \times A) > 0 \right\} \geq \inf_{x \in A} \psi^-(x) \quad (20)$$

Therefore, by comparing the left-hand terms of the following expressions to the left-hand terms of the previous expressions for shrinking neighborhoods A around x ,

$$\limsup_{y \rightarrow x} \psi^+(y) \leq \psi^+(x) \quad (21)$$

$$\liminf_{y \rightarrow x} \psi^-(y) \geq \psi^-(x) \quad (22)$$

where the limits are taken along $\text{supp}(\alpha_X)$. These hold with equality when x is not an atom of $\text{supp}(\alpha_X)$: If $x \in \text{supp}(\alpha_X)$, then for each $\mu < \psi^+(x)$ and $\delta > 0$, $\alpha(\{\theta \mid \mu_\theta \geq \mu - \varepsilon\} \times (x - \delta, x + \delta)) > 0$. If x is not an atom of α_X , then for some $y \in (x - \delta, x + \delta) \setminus \{x\}$, $\psi^+(y) \geq \mu$ by (19), and this was for any $\delta > 0$ and any $\mu < \psi^+(x)$.

The remainder of this section proves Theorem 3 in five steps.

⁴¹Observe that, if α' is the marginal of α on the variables (μ, x) - i.e., $\alpha' = P \circ (\mu, id)^{-1}$ - and $x \rightarrow \alpha'(\cdot \mid x)$ is a decomposition of α' conditional on x , then for α -a.e. $x \in X$, $\psi^+(x)$ is a supremum of the support of $\alpha'(\cdot \mid x)$. Similarly, $\psi^-(x)$ is an infimum of this support. The limits exist as the terms they are taken over are monotonic.

B.1 Price is Lipschitz in Weak Equilibrium

Our first auxiliary results establishes bounds on prices in a weak equilibrium. (We defined “weak equilibrium” in Section 2.2).

Lemma 2. *Assume (p, α) is a weak equilibrium s.t. p is continuous on $\text{supp}(\alpha_X)$.⁴² Let $0 < 1 - \bar{\delta} < x_1 < x_2$ be two points in $\text{supp}(\alpha_X)$. Then,*

$$\begin{aligned} \frac{p(x_1)}{x_1} &\leq \psi^+(x_1) \leq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \\ &\leq \psi^-(x_2) + \left(1 - \frac{x_1 + x_2}{2}\right) \bar{\rho}_{\bar{\delta}}(\psi^-(x_2)) \leq \frac{p(x_2)}{x_2} + \left(1 - \frac{x_1 + x_2}{2}\right) \bar{\rho}_{\bar{\delta}}\left(\frac{p(x_2)}{x_2}\right) \end{aligned}$$

In particular, $\psi^+(x)$ must be finite for each $x \in \text{supp}(\alpha_X)$ with $x < 1$.

Proof. The break-even condition $(p(x) = \mathbb{E}_\alpha[c \mid x], \alpha - a.e. x)$ requires that for α_X -a.e. $x > 0$ in $\text{supp}(\alpha_X)$, $\psi^-(x) \leq \frac{p(x)}{x} \leq \psi^+(x)$. This, together with the monotonicity of $\bar{\rho}_{\bar{\delta}}$, implies the first and last inequalities. To show the third inequality, notice that for each $x_1 \in [0, 1]$, utility maximization implies

$$\alpha(\{u(\theta, p(x), x) \geq u(\theta, p(x_1), x_1)\}) = 1$$

In particular, by the definition of ψ^- , there is a sequence $y_n \rightarrow x_2$ in $\text{supp}(\alpha_X)$ and types (θ_n) with $\mu(\theta_n) \rightarrow \psi^-(x_2)$ (if x_2 is an atom of α_X , take $y_n \equiv x_2$), such that for all n , $u(\theta_n, p(y_n), y_n) \geq u(\theta_n, p_1, x_1)$. Recall that $\bar{\rho}_{\bar{\delta}}(\cdot)$ is monotonically increasing and right-continuous. Hence also $(\bar{\rho}_{\bar{\delta}}(\theta_n))$ satisfies $\limsup_{n \rightarrow \infty} \bar{\nu}_{\bar{\delta}}(\theta_n) \leq \bar{\rho}_{\bar{\delta}}(\psi^-(x_2))$. An application of Lemma 1, the fact that $\liminf_{y \rightarrow x_2} \psi^-(y) = \psi^-(x_2)$ if x_2 is not an atom of α_X , the right-continuity of $\bar{\rho}_{\bar{\delta}}$ and the continuity of p on $\text{supp}(\alpha_X)$ completes the proof. The second inequality follows similarly, by using the fact that that $\bar{\nu}_{\bar{\delta}}(\theta_n) \geq 0$. \square

B.2 Price is Locally Lipschitz in Equilibrium

The second auxiliary result shows the outer-most bounds hold for equilibria everywhere (not just on the support of α_X).

Lemma 3. *Assume (p, α) is an equilibrium. If $0 < 1 - \bar{\delta} < x_1 < x_2$ with $p(x_2) > 0$ (x_1, x_2 are not necessarily in the support of α_X), then*

$$\frac{p(x_1)}{x_1} \leq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \leq \frac{p(x_2)}{x_2} + \left(1 - \frac{x_1 + x_2}{2}\right) \bar{\rho}_{\bar{\delta}}\left(\frac{p(x_2)}{x_2}\right)$$

⁴²In particular, this is true if $\text{supp}(\alpha_X)$ is finite.

In particular, p is non-decreasing.

Proof. If $p(x_1) = 0$, the first inequality is trivial, so assume $p(x_1) > 0$. Since $p(\cdot)$ is part of an equilibrium, there are finite subsets $\bar{X}^n \subseteq X$, prices $p^n : \bar{X}^n \rightarrow \mathbb{R}_+$ and associated distributions α^n on $(\Theta \cup \bar{X}^n) \times \bar{X}^n$ as described in Section 2.2 which witness that (p, α) is an equilibrium. Let $y^n \rightarrow x_1$ and $z^n \rightarrow x_2$ with $y^n, z^n \in \bar{X}^n$. Since each (p_n, α_n) is a weak equilibrium whose projection to X is finitely supported, it follows from Lemma 2 that, at each n ,

$$\frac{p_n(y_n)}{y_n} \leq \frac{p_n(z_n) - p(y_n)}{z_n - y_n} \leq \frac{p_n(z_n)}{z_n} + \left(1 - \frac{y_n + z_n}{2}\right) \bar{\rho}_{\bar{\delta}} \left(\frac{p(z_n)}{z_n}\right)$$

Taking $n \rightarrow \infty$, and recalling that $\bar{\rho}_{\bar{\delta}}$ is monotonically increasing and right-continuous, completes the proof. \square

B.3 Full insurance is not a Heterogeneous Atom

We now show that full insurance ($x = 1$) cannot be an atom of α_X purchased by multiple types if μ is not essentially bounded w.r.t. P . I.e., it is not possible to 'stuff all the highest types' there. Of course, this proposition is only relevant when $1 \in X$.

Lemma 4. *If $x = 1$ is an atom of α_X , it must be that $\alpha(\cdot \mid \{x = 1\})$ is concentrated on one riskiness, i.e., there must be $\tilde{\mu} \in \mathbb{R}_+$ s.t. $\alpha(\mu(\theta) = \tilde{\mu} \mid x = 1) = 1$.*

Proof. Suppose, by way of contradiction, that $x = 1$ is an atom of α_X but not concentrated on a single riskiness. Then, the break-even condition (1) requires that there are some types buying $x = 1$ who are less risky than the average buyers of that contract, with $p(1)$ being determined by these average buyers. Define $\mu^* = \psi^-(1)$ and $p^* := p(1)$; then $\mu^* < E_\alpha[\mu \mid x = 1] = p(1) = p^*$.

Lemma 2 implies that, for any $x \geq 1 - \bar{\delta}$,

$$\frac{p(x)}{x} \leq \frac{p^* - p(x)}{1 - x} \leq \mu^* + \frac{1}{2} \bar{\rho}_{\bar{\delta}}(\mu^*)(1 - x).$$

In turn, this implies

$$p^* x \geq p(x) \geq p^* - \mu^*(1 - x) - \frac{1}{2} \bar{\rho}_{\bar{\delta}}(\mu^*)(1 - x)^2.$$

It then follows that

$$\frac{1}{2} \bar{\rho}_{\bar{\delta}}(\mu^*)(1 - x) \geq p^* - \mu^*.$$

However, this last condition cannot hold for x close enough to 1 since $p^* > \mu^*$, a contradiction. \square

The proof of Lemma 4 is illustrated graphically in Figure 6.

It follows then that:

Lemma 5. $\sup_{x < 1} \psi^+(x) = \infty$, that is, types with arbitrarily large cost μ purchase less than full insurance.

Proof. Since full insurance cannot be an atom with unbounded support of riskiness of α , the projection of $\alpha(\cdot \mid [0, 1))$ to riskiness μ cannot be compactly supported. Hence, the lemma follows from applying (19) with $A = X \setminus \{1\}$. \square

B.4 Price is unbounded

We now prove Theorem 3.

Proof. Suppose not. Denote $p^* = \lim_{x \rightarrow 1} p(x) < \infty$. Fix some $0 < \bar{\delta} \leq \delta < 1$. Recall that $p^* = p(1)$. Then, from Lemma 3 and the facts that p and $\bar{\rho}_{\bar{\delta}}$ are non-decreasing, for all $1 - \delta \leq x_1 < x_2$ in X ,

$$\psi^+(x_1) \leq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \leq \frac{p(x_2)}{x_2} + \bar{\nu}\left(\frac{p(x_2)}{x_2}\right) \leq M := \frac{p^*}{1 - \delta} + \bar{\rho}_{\bar{\delta}}\left(\frac{p^*}{1 - \delta}\right),$$

while for $x_1 \leq 1 - \delta$ in X ,

$$\psi^+(x_1) \leq \lim_{x_2 \rightarrow 1^-} \frac{p(x_2) - p(x_1)}{x_2 - x_1} = \frac{p^* - p(x_1)}{1 - x_1} \leq \frac{p^*}{\delta}$$

Hence, $\sup_{x < 1} \psi^+(x) \leq \max[M, \frac{p^*}{\delta}]$, contradicting Lemma 5. \square

C Non-Existence

We now prove Proposition 7: equilibrium does not exist for insurance markets with unbounded cost, if $\bar{\nu}_{\delta}(\theta)$ is bounded by a linear function of μ . Fix $\bar{\delta} > 0$ for which Assumption 5 holds.

Proof. Let C, D be such that $\bar{\nu}_{\delta}(\theta) \leq C\mu(\theta) + D$ P -a.s.. Let $x_n \rightarrow 1$ strictly monotonically in X . WLOG, $x_1 > 1 - \bar{\delta}$. Denote $B = \max[\frac{1+C}{x_1}, D]$ and $p_n = p(x_n)$. Then, $\bar{\rho}_{\bar{\delta}}(\frac{p_{n+1}}{x_{n+1}}) \leq C\frac{p_{n+1}}{x_{n+1}} + D$, so by Lemma 3,

$$\frac{p_{n+1} - p_n}{x_{n+1} - x_n} \leq \frac{p_{n+1}}{x_{n+1}} + (1 - x_n)\bar{\rho}_{\bar{\delta}}\left(\frac{p_{n+1}}{x_{n+1}}\right) \leq \frac{p_{n+1}}{x_{n+1}} + C\frac{p_{n+1}}{x_{n+1}} + D \leq B(p_{n+1} + 1)$$

Denote $q_n = p_n + 1$ and $\delta_n = B(x_{n+1} - x_n)$; w.l.o.g. $\delta_n < 1$ for all n , and hence

$$q_{n+1} \leq \frac{q_n}{1 - \delta_n} \Rightarrow q_n \leq q_1 \cdot \prod_{j < n} \frac{1}{1 - \delta_j}$$

by induction. Then, $\sum \delta_n = \sum B(x_{n+1} - x_n) = B(1 - x_1) < \infty$ which implies $\prod_{j < \infty} \frac{1}{1 - \delta_j} < \infty$. This, together with the monotonicity of p , shows that

$$\sup_{x \in X} p(x) = \lim_{n \rightarrow \infty} p_n < \infty$$

which contradicts Proposition 7. □

D Simple Insurance: Continuous Types

This section proves the results for simple insurance markets with a continuum of types (Section 3.5). We will make use of Assumptions 1, 2, and 3 throughout, which in particular guarantee by Theorem 2 a coverage function σ from riskiness to coverage satisfying the assumptions given there. Under the latter assumption, since the riskiness μ entirely determines the type, we will write w_μ instead of w_θ when $\mu = \mu(\theta)$. We will use a similar convention for u_μ . Recall that α_X denotes the marginal distribution of α on X .

D.1 Proposition 9

Proposition 9. *Suppose P_μ , conditional on some interval $I = (\underline{\mu}, \bar{\mu})$, has full support with a.e. strictly positive density w.r.t. the Lebesgue measure, and (p, α) is an equilibrium with associated coverage function $\sigma(\mu)$. Suppose $\sigma(\underline{\mu}) > 0$ (i.e., a.e. type with riskiness in I purchases positive coverage). Then, denoting $J = (\sigma(\underline{\mu}), \sigma(\bar{\mu}))$ and letting α_J be the projection of α to alternatives and conditional on J , α_J is equivalent to the Lebesgue measure on J , i.e., α_J and the Lebesgue measure on J are absolutely continuous w.r.t. each other. In other words, α_J and the Lebesgue measure on J have the same null sets or, equivalently, α_J has Lebesgue-a.e. positive density.*

Proof. Note that $\sigma : I \rightarrow J$ is a strictly increasing bijection (and in particular, continuous), and $\alpha_J = P_\mu(\cdot | J) \circ \sigma^{-1}$. Hence, it suffices to show that both σ and $\tau = \sigma^{-1} : J \rightarrow I$ are locally Lipschitz in the interior of I, J respectively. By Theorem 2, p is L -Lipschitz in J for some $L > 0$. Now,

$$p(\sigma(\mu)) = \mu \cdot \sigma(\mu), \quad P_\mu - a.s$$

Since σ is continuous and P_μ has full support in I , this implies

$$p(\sigma(\mu)) = \mu \cdot \sigma(\mu), \quad \forall \mu \in I$$

and hence

$$\tau(x) = \frac{p(x)}{x}, \quad \forall x \in J$$

Since p is Lipschitz in J and $\sigma(\underline{\mu}) > 0$, τ is Lipschitz. Observe that for fixed z , the mapping $w_{\tau(x)}(z) = \tau(x) + g'(z) \cdot \nu(\tau(x))$ is well-defined α_J a.e.. Note that $\text{supp}(\alpha_J) = J$, as σ is continuous. So, for α_J a.e. every $x_1 \in J$ and every $x_2 \in J$,

$$\frac{p(x_2) - p(x_1)}{x_2 - x_1} \geq \int_{x_1}^{x_2} w_{\tau(x_1)}(x) dx$$

and hence $p'(x) \geq w_{\tau(x)}$ Lebesgue a.e.. Therefore, Lebesgue-a.s.,

$$\tau'(x) = \frac{1}{x} \left[p'(x) - \frac{p(x)}{x} \right] \geq \frac{1}{x} [w_{\tau(x)}(x) - \tau(x)] = \frac{1}{x} g'(x) \cdot \nu(\tau(x)) \geq g'(x) \cdot \nu(\underline{\mu})$$

Hence, in each sub-interval of J which is bounded away from full insurance, $\tau' > 0$ is bounded away from 0 and hence $\sigma = \tau^{-1}$ is locally Lipschitz in the interior of $\tau(J)$. \square

D.2 Proposition 10

We now prove the necessary condition for equilibrium with continuous types.

Proposition 10. *If P_{μ} , conditional on some interval $I = (\underline{\mu}, \bar{\mu})$, has full support with a.e. strictly positive density w.r.t. the Lebesgue measure, and (p, α) is an equilibrium with associated coverage function σ with $\sigma(\underline{\mu}) > 0$, then*

$$\int_{\underline{\mu}}^{\bar{\mu}} \frac{1}{\nu(\mu)} d\mu = \int_{\sigma(\underline{\mu})}^{\sigma(\bar{\mu})} \frac{g'(x)}{x} dx \quad (23)$$

Proof. Let (p, α) be an equilibrium, with associate $\sigma : I \rightarrow [0, 1]$ as above. Denoting $J = [\sigma(\underline{\mu}), \sigma(\bar{\mu})]$, we know from Proposition 9 that α_J is equivalent to the Lebesgue measure. Given the differentiability of the price function Lebesgue a.s. in J (equivalently, α_J -a.s.), utility maximization implies

$$\frac{\partial}{\partial x} (u_{\mu}(x, p(x))) \Big|_{x=\sigma(\mu)} = 0 \Leftrightarrow \frac{\partial u_{\tau(x)}}{\partial x} (x, p(x)) + \frac{\partial u_{\tau(x)}}{\partial p} (x, p(x)) \cdot p'(x) = 0, \quad a.e. x \in J \quad (24)$$

where $\tau(x) = \frac{p(x)}{x} = \sigma^{-1}(x)$. Explicitly: For this to hold at some x we need two conditions: p to be differentiable at x , and for the type with riskiness $\mu = \sigma^{-1}(x)$ to be a utility maximizer. Both of these properties hold Lebesgue a.s. in J , equivalently, α_J -a.s. Hence,

$$p'(x) = - \frac{\frac{\partial u_{\tau(x)}}{\partial x} (x, p(x))}{\frac{\partial u_{\tau(x)}}{\partial p} (x, p(x))} = \frac{p(x)}{x} + g'(x) \cdot \nu\left(\frac{p(x)}{x}\right), \quad a.e. x \in J$$

Denote $\tau = \sigma^{-1}$ on J . The equilibrium property of actuarially fairness (and continuity of prices and τ) shows that

$$p(x) = \tau(x) \cdot x, \quad \forall x \in J$$

and hence σ^{-1} is differentiable a.e. and

$$p'(x) = \tau(x) + x \cdot \tau'(x), \quad a.e. \ x \in J$$

Rewriting (24) gives

$$\tau(x) + g'(x) \nu(\tau(x)) - p'(x) = 0, \quad a.e. \ x \in J.$$

Summing these gives

$$g'(x) \cdot \nu(\tau(x)) = x \cdot \tau'(x), \quad a.e. \ x \in J$$

or

$$\frac{g'(x)}{x} = \frac{1}{\nu(\tau(x))} \tau'(x) = \frac{1}{\nu(\tau(x))} \tau'(x), \quad a.e. \ x \in J$$

and hence, by change of variable (note that τ is strictly increasing),

$$\int_{\sigma(\underline{\mu})}^{\sigma(\bar{\mu})} \frac{g'(x)}{x} dx = \int_{\sigma(\underline{\mu})}^{\sigma(\bar{\mu})} \frac{1}{\nu(\tau(x))} \tau'(x) dx = \int_{\underline{\mu}}^{\bar{\mu}} \frac{1}{\nu(\mu)} d\mu$$

as required. Note that we have made use of a change of variable formula for functions which are differentiable a.e., see e.g. Theorem 7.26 of [Rudin \[1987\]](#) (3rd Ed). \square

D.3 Equilibrium with Bounded Risk

We now prove the equilibrium characterization when μ is bounded (Corollary 2).

Proof. Follows immediately from Proposition 10, as one of the properties of equilibrium of Proposition 6 under these assumptions is $\sigma(\bar{\mu}) = 1$. The right-hand side defines σ uniquely as $g' > 0$ except possibly at 1. The last conclusion follows since $g'(0) > 0$ so $\int_0^1 \frac{g'(x)}{x} dx = \infty$. \square

D.4 Equilibrium with Unbounded Risk

We now characterize equilibrium when μ is unbounded (Corollary 4).

Proof. We prove the last part first: WLOG, $(\bar{\mu}_n)_n$ is increasing. Define the function σ using (5). Since $g' > 0$ in $(0, 1)$ and $\int_0^1 \frac{g'(x)}{x} dx = \infty > \int_{\mu}^{\infty} \frac{1}{\nu(\mu)} d\mu$ for all $\mu > \underline{\mu}$, this is

well-defined, continuous, and its image is an interval. Let $x_{min} = \text{ess} - \text{inf}(\sigma(\mu(\theta)))$, and $x_{min}^n = \text{ess} - \text{inf}(\sigma^n(\mu(\theta)))$.⁴³ Define $\tau = \sigma^{-1} : [x_{min}, 1) \rightarrow \mathbb{R}$, $p(x) = x \cdot \tau(x)$ for $x \in [\sigma(\underline{\mu}), 1)$, and extend p in $[0, x_{min})$ in the following way: For each $x \in [0, x_{min})$, set

$$p(x) = \max[0, \text{ess} - \text{sup}_\theta \phi(\theta, x)]$$

where $\phi(\theta, \cdot)$ is the indifference curve of type θ through his purchased contract $(\sigma(\mu, \theta), p(\sigma(\mu(\theta))))$, i.e.,

$$\phi(\theta, x) = p(\sigma(\mu(\theta))) - \int_x^{\sigma(\mu(\theta))} w_\theta(t) dt$$

i.e., $p(\cdot)$ is the upper envelope of indifference curves. Finally, define for $\tilde{\Theta} \subseteq \Theta$,

$$\alpha(\theta \in \tilde{\Theta}, x \in (a, b)) = P(\theta \in \tilde{\Theta}, \mu(\theta) \in (\tau(a), \tau(b)))$$

By (3), we obtain

$$\int_{\underline{\mu}}^{\bar{\mu}_n} \frac{1}{\nu(\mu)} d\mu = \int_{\sigma^n(\underline{\mu})}^1 \frac{g'(x)}{x} dx.$$

Comparing with (5), it follows that $\sigma^n \rightarrow \sigma$ point-wise, and in particular $x_{min}^n \rightarrow x_{min}$, and (x_{min}^n) is monotonically decreasing. Letting $\tau_n = \sigma_n^{-1} : [x_{min}^n, 1)$, we have $\tau_n \rightarrow \tau$ point-wise. Therefore, $p^n \rightarrow p$ point-wise on $[x_{min}^n, 1]$, as $p^n(x) = x \cdot \tau^n(x)$ on $[x_{min}^n, 1]$. Let $x' < x_{min}$; for large enough n , $x < x_{min}^n$. Recall from Proposition 6,

$$p^n(x) = \max[0, \text{ess} - \text{sup}_\theta \phi^n(\theta, x)]$$

where

$$\phi^n(\theta, x) = p(\sigma^n(\mu(\theta))) - \int_x^{\sigma^n(\mu(\theta))} w_\theta(t) dt$$

is the indifference curve of type θ through his purchase contract under (p^n, σ^n) . Observe that $n > k \rightarrow \bar{\mu}_n \geq \bar{\mu}_k \rightarrow \sigma_n \leq \sigma_k \rightarrow \tau_n \geq \tau_k$ and $\phi^n \geq \phi^k \rightarrow p_n \geq p_k$ - intuitively, the indifference curve of each type moves to the left, so prices go up. Hence $p^n \rightarrow p$ point-wise in $[0, x_{min}]$ as well, as $(\phi^n(\theta, \cdot))$ is increasing with n .⁴⁴ Hence, we have $p_n \rightarrow p$ point-wise. Since $n > k \rightarrow p_n \geq p_k$, Dini's theorem implies that the $p_n \rightarrow p$ uniformly in every compact subset of $[0, 1)$.

Finally, for any continuous bounded $f : [\underline{\mu}, \infty) \times [0, 1] \rightarrow \mathbb{R}$,

$$\int f(\mu, x) \cdot d\alpha = \int f(\mu, \sigma(\mu)) dP_\mu = \lim_{n \rightarrow \infty} \int f(\mu, \sigma_n(\mu)) dP_\mu = \lim \int f(\mu, x) \cdot d\alpha_n$$

⁴³The essential infimum is defined similar to the essential supremum.

⁴⁴Lemma: Let (f_n) be a sequence of bounded Borel functions on a measurable space, $f_1 \leq f_2 \leq f_3 \leq \dots$ which converge point-wise to f . Then $\text{ess} - \text{sup}(f_n) \rightarrow \text{ess} - \text{sup}(f)$.

by the bounded convergence theorem, and hence $\alpha_n \rightarrow \alpha$ weakly.

We now prove the first point of Corollary 4. Existence now follows from the approximation result Proposition 1. Uniqueness can be proved as follows. Given any equilibrium with associated choice function σ , for each $\bar{\mu} > \underline{\mu}$, (23) holds. Take $\bar{\mu} \rightarrow \infty$. Since full coverage is in the support of the equilibrium and σ is increasing, $\lim_{\mu \rightarrow \infty} \sigma(\mu) = 1$. The last part of the proposition follows again as $\int_0^1 \frac{g'(x)}{x} dx = \infty$. \square

D.5 Non-Existence with Unbounded Risk

We now show that the integrability condition is indeed necessary for equilibrium (Corollary 3).

Proof. For the first part: We can find intervals $I = [a_n, b_n]$ with $a_n, b_n \rightarrow \infty$ for which $\int_{a_n}^{b_n} \frac{1}{\nu(\mu)} d\mu \rightarrow \infty$ but since $\sigma(a_n), \sigma(b_n) \rightarrow 1$, $\int_{\sigma(a_n)}^{\sigma(b_n)} \frac{g'(x)}{x} dx \rightarrow 0$, which contradicts (23). For the second part: Fixing $\mu \in \mu(\Theta)$,

$$\int_{\mu}^{\bar{\mu}_n} \frac{1}{\nu(\mu')} d\mu' = \int_{\sigma^n(\mu)}^1 \frac{g'(x)}{x} dx$$

And hence

$$\infty = \int_{\mu}^{\infty} \frac{1}{\nu(\mu')} d\mu' = \lim_{n \rightarrow \infty} \int_{\sigma^n(\mu)}^1 \frac{g'(x)}{x} dx$$

However,

$$\int_{\sigma^n(\mu)}^1 \frac{g'(x)}{x} dx \leq \max_{[0,1]} g' \cdot \int_{\sigma^n(\mu)}^1 \frac{1}{x} dx = \ln(\sigma^n(\mu)) \cdot \max_{[0,1]} g'$$

For this to converge to ∞ , we must have $\sigma^n(\mu) \rightarrow 0$. \square

E Simple Insurance: Discrete Types

This section proves some of the results for simple insurance markets with discrete types (Section 3.6).

E.1 Characterization of $\phi_k(\cdot)$

For each $k \in \mathbb{N}$ and each coverage $u \in [0, 1]$, let $v = \phi_k(u)$ denote the unique coverage $v < u$ such that type $\theta_{k+1} = (\mu_{k+1}, \nu_{k+1})$ is indifferent between contracts $(p, x) = (\mu_{k+1} \cdot u, u)$ and $(p, x) = (\mu_k \cdot v, v)$, where the second contract is actuarially fair for type θ_k . Such unique v exists as $w_{\theta_{k+1}}(x) > \mu_{k+1} > \mu_k$ for all $x \in (0, 1)$.

We now show $\phi_k(\cdot)$ is continuous, strictly increasing, and $\phi_k(u) \leq u$ and $\phi_k(u) = u$ if and only if $u = 0$.

Lemma 6. $0 < \phi'_k \leq 1$ in $(0, 1)$. Therefore, ϕ_k is strictly increasing (in particular, $\phi_k(u) = 0$ if and only if $u = 0$). Moreover, ϕ_k is non-expansive (i.e., 1-Lipschitz).

Proof. The definition of $\phi_k(\cdot)$ requires that, for each $z \in (0, 1]$, type $k + 1$ is indifferent between $(\mu_{k+1} \cdot z, z)$ and $(\mu_k \cdot \phi_k(z), \phi_k(z))$. This implies

$$\mu_{k+1} \cdot \phi_k(z) + g(\phi_k(z)) \cdot \nu_{k+1} - \mu_k \cdot \phi_k(z) = \mu_{k+1} \cdot z + g(z) \cdot \nu_{k+1} - \mu_{k+1} \cdot z = g(z) \cdot \nu_{k+1} \quad (25)$$

where $\nu_{k+1} = \nu(\mu_{k+1})$.

Differentiation gives

$$\phi'_k(z) \cdot [\mu_{k+1} - \mu_k + g'(\phi_k(z)) \cdot \nu_{k+1}] = g'(z) \cdot \nu_{k+1}$$

We then obtain

$$\phi'_k(z) = \frac{g'(z) \cdot \nu_{k+1}}{\mu_{k+1} - \mu_k + g'(\phi_k(z)) \cdot \nu_{k+1}} \leq \frac{g'(z) \cdot \nu_{k+1}}{g'(\phi_k(z)) \cdot \nu_{k+1}} = \frac{g'(z)}{g'(\phi_k(z))}.$$

Since we assumed g differentiable, then ϕ_k is differentiable and therefore continuous. Since g is concave, g' is decreasing. Moreover, we have shown that $\phi_k(z) < z$. Therefore, $g'(\phi_k(z)) > g'(z)$, which implies $\phi'_k \leq 1$. \square

By the strict monotonicity of the $\phi_k(\cdot)$, for each given k , $(x_k^n)_{n=1}^\infty$ is a strictly decreasing sequence in n and, since it is bounded below by 0, it converges to $x_k \in [0, 1)$ which is the choice of type k in the limit economy \mathcal{E} in the equilibrium we will construct below.

E.2 Uniqueness & Positivity of Coverage

We now prove that, if equilibrium exists on a discrete unbounded type space, equilibrium is unique and all types purchase positive levels of coverage (Proposition 4).

Proof. Suppose an equilibrium in which type (μ_k, ν_k) purchases coverage $x_k = \sigma(\mu_k)$ then $x_1 < x_2 < \dots$. By the equilibrium properties, $\sigma(\mu_k) = \phi_k(\sigma(\mu_{k+1}))$, i.e., $x_k = \phi_k(x_{k+1})$. Inductively, for any $n > k$

$$x_k := \phi_k(\phi_{k+2}(\dots(\phi_{n-1}(x_n))\dots))$$

Lemma 6 shows that ϕ_k is non-expansive (i.e., 1-Lipschitz), so it follows that

$$x_k := \lim_{n \rightarrow \infty} \phi_k(\phi_{k+2}(\dots(\phi_n(1))\dots))$$

which defines x_k uniquely (note that since each ϕ_k is monotonic, the expression the limit is taken over is decreasing with k but non-negative, so the limit exists).

To show that all types purchase positive coverage, it suffices to show $x_1 > 0$. By Proposition 6, full insurance is in the support of the equilibrium distribution, and hence for some N , $x_N > 0$. But $x_1 = \phi_1(x_2) = \dots = \phi_1(\phi_2(\dots(\phi_{N-1}(x_N))\dots))$, and for each $u > 0$ and each $k \in \mathbb{N}$, $\phi_k(u) > 0$. Therefore, $x_1 > 0$. \square

E.3 Convergence of p^n and α^n

Suppose, as in Section 3.6, that in the truncated economy with types $\{(\mu_k, \nu_k)\}$, the unique equilibrium is (p^n, α^n) , where under the allocation α^n type (μ_k, ν_k) purchases coverage x_k^n . Section 3.6 showed that the limit $x_k := \lim_{n \rightarrow \infty} x_k^n$ exists since $(x_k^n)_{n \geq k}$ is monotonically decreasing and $x_k > 0$ for each k . We have given a heuristic argument under the summability condition (7) for this but do not formally prove it, as it resembles the argument in Section H.

Let (p, α) be the candidate equilibrium, as described in Section 3.6. We prove, in the context of discrete types, that $p^n \rightarrow p$ uniformly on compact subsets of $[0, 1)$, and $\alpha^n \rightarrow \alpha$ weakly.

Proof. Let $f : [0, 1] \times \mathbb{R}_+$ be any continuous and bounded function. Let P_n be the mass of the n -th type, and $Q_j^n = \frac{P_j}{\sum_{j \leq n} P_j}$ be the conditional mass in the n -th economy. The conditional distributions on types converges in norm to the distribution on the infinite type space. f is continuous, so for each k , $f(x_k^n) \rightarrow f(x_k)$. Hence, by the bounded convergence theorem,

$$\int_{\Theta \times [0,1]} f d\alpha^n = \sum_{k=1}^n f(x_k^n) \cdot Q_k \rightarrow \sum_{k=1}^{\infty} f(x_k) \cdot P_k = \int_{\Theta \times [0,1]} f d\alpha$$

and hence $\alpha^n \rightarrow \alpha$ weakly.

To show $p^n \rightarrow p$ uniformly on compact subsets of $[0, 1)$, it suffices to show that for each interval of the form $I_k = [x_k, x_{k+1}]$, $p^n \rightarrow p$ uniformly in I_k . The (p^n) are uniformly Lipschitz in I^k , with Lipschitz constant $L = \mu_{k+1} + \nu_{k+1} \cdot \sup_{[0,1]} g'$, as $x_{k+1}^n \geq x_{k+1}$. Furthermore, since g_{k+1}^n (as the indifference curve of type $k+1$ through x_{k+1}^n) converges point-wise to g_{k+1} (as the indifference curve of type k through x_k^n), p coincides with g_{k+1} in I_k , and p^n coincides with g_{k+1}^n in $[x_k^n, x_{k+1}^n]$, we have $p^n \rightarrow p$ point-wise. But the combination of point-wise convergence and uniform Lipschitz implies uniform convergence. \square

F Existence via Approximations

This Section provides the most general conditions under which the existence results of AG extend to settings where cost is unbounded (Section 5). Proposition 1 requires

knowledge of the equilibrium in the unbounded economy. Proposition 8 does not require such knowledge but requires additional assumptions on utilities.

F.1 Existence 1

We now prove Proposition 1.

Recall that, given the definition of AG equilibrium in Section 2, we say that $((\bar{X}_j), (p_j, \alpha_j), (\eta_j))$ witnesses that (p, α) is an equilibrium.

For each truncated economy \mathcal{E}^n , its equilibrium (p^n, α^n) is the limit of the weak equilibria (p_j^n, α_j^n) of a sequence of perturbed economies \mathcal{E}_j^n which have a vanishing mass of behavioral types, as described in AG. We then consider the sequence of economies \mathcal{E}^n and show that an appropriate diagonal of weak equilibria $(p_{j_n}^n, \alpha_{j_n}^n)$ converge to an AG equilibrium of \mathcal{E} when $n \rightarrow \infty$. Finally, we modify the equilibria on this diagonal to include all types, as α_j^n only allocates types of Θ^n ; due to the behavioral types, this can be done without changing the price p_j^n .

Proof. For each n , let $(\bar{X}_n), (Y_n^k)_{k \in \mathbb{N}}, (\zeta_n^k)_{k \in \mathbb{N}}, (q_n^k, \beta_n^k)_{k \in \mathbb{N}}$ be sequences of Polish spaces that X is dense in \bar{X}_n , finite sets of alternatives, of behavioral types, and of weak equilibria which witness that (p_n, α_n) is an equilibrium of the restricted economy n . Note that q_n^k refers to the price function while β_n^k refers to the distribution over types and contracts. Note also that Y_n^k will in general include points of \bar{X} which are not in X . Then, for each $n \in \mathbb{N}$, $\bar{X}_n \subseteq \bar{X}$, where \bar{X} is a fixed compactification of X .⁴⁵

Let $(Z_j)_{j \in \mathbb{N}}$ be a sequence of compact subsets of X with $X = \cup_j Z_j$, and for each $j \in \mathbb{N}$, $Z_j \subseteq Z_{j+1}^\circ$; such exists as X is locally compact and separable. Note that since each $p_n(\cdot)$ is continuous by Lemma 18 of Section I, X is locally compact, and $p_n \rightarrow p$ uniformly on compact sets, it follows that p is continuous. Hence, by passing to a subsequence of (p_n, α_n) , we may assume that:

- For all $n \in \mathbb{N}$ and all $x \in Z_n$, $|p_n(x) - p(x)| < \frac{1}{n}$.

W.l.o.g., since each Y_n^n is finite and $p_n \rightarrow p$ uniformly on compact sets in X , we may assume by Lemma 19 of Section I there are indices $(k_n)_n$ such that (after passing possibly to a sub-sequence of (p_n, α_n)):

- For all $n \in \mathbb{N}$ and all $x \in Y_n^{k_n}$, $|q_n^{k_n}(x) - p_n(x)| < \frac{1}{n}$.
- For all $n \in \mathbb{N}$, $d(\beta_n^{k_n}, \alpha^n) < \frac{1}{n}$, where $d(\cdot, \cdot)$ is a metric for the weak topology.

Now, denote $\gamma_n = \beta_n^{k_n}$, $r_n = q_n^{k_n}$, $\bar{W}_n = Y_n^{k_n}$. Then γ_n is concentrated on $(\Theta \cup \bar{W}_n) \times \bar{W}_n$, $\gamma_n \rightarrow \alpha$, and for all $n \in \mathbb{N}$, $|r_n(x) - p(x)| < \frac{2}{n}$ for all $x \in \bar{W}_n \cap Z_n$. We contend that for

⁴⁵Formally, for each $n \in \mathbb{N}$ there is an embedding $\phi_n : \bar{X}_n \rightarrow \bar{X}$, which is identity on X .

each $x \in X$, and each sequence (x_n) in X with $x_n \rightarrow x$ and $x_n \in \overline{W}_n$ for each $n \in \mathbb{N}$, $r_n(x_n) \rightarrow p(x)$. Indeed, since $x \in X$ there is N s.t. for all $n \geq N$, $x_n \in \text{int}(Z_n)$; hence $x_n \in \text{int}(Z_n) \cap \overline{W}_n$, so $|r_n(x_n) - p(x_n)| < \frac{2}{n}$, and since p is continuous at x , $p(x_n) \rightarrow p(x)$; hence, $r_n(x_n) \rightarrow p(x)$.

The problem, however, is that the marginal of γ_n on Θ is $P(\cdot \mid \Theta_n)$, not P ; hence we must modify it. The idea is after adding the types in $\Theta \setminus \Theta_n$ to the distribution γ_n - where each type is choosing a utility-maximizing option in \overline{W}_n - price can only go up, as elements outside Θ_n are more costly than those in it. Hence, by adding some behavioral types - who have cost 0 - the added cost cancels out. The formal treatment is as follows:

Let ρ_n be a measure on $\Theta \setminus \Theta_n \times X_n \subseteq \Theta \times X_n$ s.t.

$$\rho_n\left((\theta, x) \in \Theta \setminus \Theta_n \times X_n \mid \forall y \in X_n, u(x, \theta, q_n(x)) \geq u(y, \theta, q_n(y))\right) = P(\Theta \setminus \Theta_n)$$

- i.e., ρ_n -a.s. all agents in Θ not in Θ_n maximize their utility in X_n at prices $q_n(\cdot)$. By assumption,

$$E_{\gamma_n + \rho_n}[c \mid x] \geq E_{\gamma_n}[c \mid x] = r_n(x), \forall x \in \overline{W}_n$$

with equality if $r_n(x) < c_0$, i.e., the types in Θ which not in Θ_n only increase costs. (Note that $\gamma_n + \rho_n$ may not be normalized; throughout this proof, when σ is a non-normalized measure, $E_\sigma[f] = \frac{1}{\sigma(\Theta)} \int_\Theta f d\sigma$.) Define $\pi_n(x) \geq 0$ s.t.

$$\frac{\gamma_n(x) + \rho_n(x)}{\gamma_n(x) + \rho_n(x) + \pi_n(x)} E_{\gamma_n + \rho_n}[c \mid x] = E_{\gamma_n}[c \mid x] = r_n(x), \forall x \in \overline{W}_n$$

(By assumption $\gamma_n(x) > 0$ for all $x \in \overline{W}_n$ hence this is well defined.) Hence, since cost of behavioral types is 0,

$$E_{\rho_n + \gamma_n + \pi_n}[c \mid x] = E_{\gamma_n}[c \mid x] = r_n(x), \forall x \in \overline{W}_n$$

Furthermore, $\pi_n(x) = 0$ if $x \in \overline{W}_n$ and $r_n(x) < c_0$, since no types in $\Theta \setminus \Theta_n$ purchases in the domain $r_n < c_0$, i.e., $\rho_n(\{(\theta, x) \mid r_n(x) < c_0\}) = 0$. We contend that $\pi_n(\overline{W}_n) \rightarrow 0$, as required of behavioral types. Indeed, by definition,

$$[\rho_n(x) + \gamma_n(x)] \cdot E_{\rho_n + \gamma_n}[c \mid x] = \gamma_n(x) \cdot E_{\gamma_n}[c \mid x] + [\pi_n(x) + \rho_n(x)] \cdot r_n(x), \forall x \in \overline{W}_n$$

Therefore,

$$\int c \cdot d(\gamma_n + \rho_n) = \int c \cdot d\gamma_n + \sum_{x \in \overline{W}_n} [\pi_n(x) + \rho_n(x)] \cdot r_n(x) = \int c \cdot d\gamma_n + \int r_n \cdot d\rho_n + \int r_n d\pi_n$$

i.e.,

$$c_0 \cdot \pi_n(\overline{W}_n) \leq \int r_n d\pi_n = \int (c - r_n) \cdot d\rho_n \rightarrow 0$$

where the second term vanishes due to our integrability requirement, and since ρ_n is supported on $\Theta \setminus \Theta_n$. Hence, denoting $\delta_n = \gamma_n + \rho_n + \pi_n$ (where π_n naturally induces a measure on the diagonal of $\overline{W}_n \times \overline{W}_n \subseteq (\Theta \cup \overline{W}_n) \times \overline{W}_n$) shows that the sequences of subspace $(\overline{W}_n)_{n \in \mathbb{N}}$, behavioral types $(\pi_n)_{n \in \mathbb{N}}$, and weak equilibrium $(r_n, \delta_n)_{n \in \mathbb{N}}$ witness (p, α) being an equilibrium. \square

F.2 Existence 2

We now prove Proposition 8. Before doing so, we present a useful generalization of the Arzela-Ascoli theorem. This generalization is found, e.g., Thm 17, Ch 7, of Kelley, “General Topology”. The key generalization in this version vis-a-vis more classical statements is the requirement that X be only locally compact (rather than compact), and the requirement that the functions f to be point-wise bounded (rather than uniformly bounded).

Theorem 5. *Let X be a locally compact metric space. Given a sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions on X , equicontinuous and point-wise bounded, there is a continuous function $f : X \rightarrow \mathbb{R}$ and a subsequence of $(f_n)_{n=1}^{\infty}$ converging to f uniformly on compact sets.*

We now prove Proposition 8:

Proof. The existence of the limit function $p(\cdot)$ with the required first property of $p(\cdot)$ in Proposition 1 follows from Theorem 5. Furthermore, $(\alpha_n)_{n=1}^{\infty}$ (or any of its subsequences) is tight (since $\alpha_n(\Theta_k \times \overline{X}) = P(\Theta_k)$ for all $n, k \in \mathbb{N}$) and hence it w.l.o.g. (passing to a subsequence) converges weakly to some measure α on $\Theta \times \overline{X}$.

We now need to show that $\alpha(\Theta \times (\overline{X} \setminus X)) = 0$. Suppose not, set $B := \alpha(\Theta \times \overline{X} \setminus X) > 0$. Fix some $y_0 \in X$, and fix some $D > \sup_n p_n(y_0)$. (By assumption, such $D < \infty$ exists.) We note that, by assumption, for each $\theta \in \Theta$, and each alternative $y \in X$, there is $q \geq 0$ s.t. $u(y_0, \theta, D) > u(y, \theta, q)$; by possibly decreasing D slightly, the continuity of utility in $\overline{X} \times \Theta \times \mathbb{R}_+$ shows that this statement is true for all $y \in \overline{X}$; and finally a standard continuity argument shows that q may be chosen independent of $y \in X$ (only dependent on θ); i.e., $\cap_{M>0} \{\theta \mid \exists y \in \overline{X} \text{ s.t. } u(y_0, \theta, D) \leq u(y, \theta, M)\} = \emptyset$. Fix M s.t. $P(\{\theta \mid \exists y \in \overline{X} \text{ s.t. } u(y_0, \theta, D) \leq u(y, \theta, M)\}) < \frac{1}{2}B$. By assumption, there is a neighborhood V of $\overline{X} \setminus X$ such that for all n large enough and all $y \in V$, $p_n(y) > M$. Therefore, $\alpha_n(\Theta \times V) \leq \frac{1}{2}B$ for all n large enough. By Portmanteau theorem, however, since V is open

$$\frac{1}{2}B \geq \liminf \alpha_n(\Theta \times V) \geq \alpha(\Theta \times V) \geq \alpha(\Theta \times \overline{X} \setminus X) = B > 0,$$

a contradiction. □

E.3 Condition (*) holds in simple insurance

We now show that condition (*), required by Propositions 1 and 8, holds in the context of simple insurance markets.

Lemma 7. *In the framework of insurance markets (i.e., under Assumption 1, Assumption 4, or Assumption 7, and where cost has the form $c(\theta, x) = \mu_\theta x$ and $w_\theta \geq \mu_\theta$), the condition (*) is satisfied when the other conditions of Proposition 1 or Proposition 8 hold.*

The intuition is as follows: price p (and hence p^n) are at least some $c_0 > 0$ in some neighborhood of full insurance. Moreover, types with high riskiness (and hence high willingness to pay) will purchase coverage in this neighborhood.

Proof. Fix some $0 < x_0 < 1$ for which $p(x_0) > 0$; by the break-even condition and the continuity of p , such x_0 exists. Hence there is $c_0 > 0$, a neighborhood U of $p(x_0)$, and N s.t. if $n > N$, $p_n \geq c_0$ in U ; since each p_n is monotonic, this means that there is $x_1 < x_0$ such that $p_n \geq c_0$ in $\{x > x_1\}$. Letting $\mu_n = \inf_{\Theta^n \setminus \Theta^{n-1}} \mu$, we have by assumption $\mu_n \rightarrow \infty$ monotonically; fix N s.t. $\mu_N \geq \frac{p(x_0)}{x_0 - x_1}$. Suppose by way of contradiction type $\theta \in \Theta \setminus \Theta^N$ purchases under p^n coverage $x < x_1$. Since $w_\theta \geq \mu_N \geq \frac{p(x_0)}{x_0 - x_1} > \frac{p(x_0) - p(x)}{x_0 - x}$, he must strictly prefer $(x_0, p(x_0))$ to $(x, p(x))$, a contradiction. □

G Equilibrium characterization in generalized Insurance

We now prove Theorem 2, or more precisely its generalization Proposition 6, which requires only the weaker Assumptions 7, 6, and 8. Henceforth, fix an equilibrium (p, α) of such an economy. Recall the notation

$$w_\theta(x, p) = -\frac{\partial u_\theta}{\partial x}(x, p) / \frac{\partial u_\theta}{\partial p}(x, p) \geq \mu_\theta$$

with equality iff $x = 1$. For this section we will also make repeated use of the pairs of the functions ψ^+, ψ^- defined on the support of α_X , the projection of the distribution α , to $[0, 1]$ (recall $X = [0, 1)$ or $X = [0, 1]$), by (17) and (18) of Section B. Throughout we rely on the continuity of prices in equilibrium (Lemma 18).

G.1 Coverage Weakly Increasing in Risk

First we prove that types with higher riskiness purchase strictly higher levels of insurance (except possibly at 0 insurance), and types of same riskiness purchase the same level. We begin with a weak version.

Lemma 8. *It holds α -a.s. that for each pair $(\theta_2, x_2), (\theta_1, x_1)$,*

$$x_2 > x_1 \Rightarrow \mu(\theta_2) \geq \mu(\theta_1)$$

This is also true if $X' \subseteq X$ is finite and α' is a weak equilibrium of the economy $[\Theta \cup X', P \cup \eta, X']$, where X' refers to behavioral types as well.⁴⁶

Informally, riskiness is (weakly) increasing in coverage. (This is later improved to strictly increasing.)

Proof. Suppose not. Then there are open subsets U, V of $\Omega \times [0, 1]$ with $\alpha_X(U) > 0$, $\alpha_X(V) > 0$, and such that for each $(\theta_1, x_1) \in U, (\theta_2, x_2) \in V, x_1 > x_2$ but $\mu_1 = \mu(\theta_1) < \mu_2 = \mu(\theta_2)$. Fix such a pair. Then $p(x_1)$ must be above the indifference curve of type θ_2 through $(x_2, p(x_2))$, and $p(x_2)$ must be above the indifference curve of type θ_1 through $(x_1, p(x_1))$. Since the latter indifference curve is strictly flatter, this is impossible. Somewhat more formally, if ϕ_j denotes the indifference curve of μ_j through (x_j, p_j) for $j = 1, 2$, then $\phi_1(x_1) = p(x_1) \leq \phi_2(x_1)$ and $\phi_2(x_2) = p(x_2) \leq \phi_1(x_2)$. However, $\phi'_j(x) = w_{\mu_j}(x, \phi_j(x))$ for $j = 1, 2$ a.e. and ϕ_1, ϕ_2 are absolutely continuous, and furthermore $w_{\mu_1} < w_{\mu_2}$ are continuous, hence $\phi_1(x') < \phi_2(x')$ in a neighborhood of x' of any point $x' \in [x_1, x_2]$ at which $\phi_1(x') = \phi_2(x')$, contradiction.

The same logic holds for weak equilibrium with behavioral types on a finite set of alternatives. \square

The following corollary proves the following intuitive property. If an agent with riskiness at least μ_0 purchases a contract with coverage less than x , then the price of x must be higher than the cost of type μ_0 purchasing that contract.

Corollary 5. *For $x < y < z$, we have*

$$\psi^+(x) \leq \psi^-(y) \leq \psi^+(y) \leq \psi^-(z)$$

This is also true if $X' \subseteq X$ is finite and α' is a weak equilibrium of the economy $[\Theta \cup X', P \cup \eta, X']$.

⁴⁶The conclusion then holds $\alpha'(\cdot \mid \Theta)$ -a.s., i.e., holds for types in Θ , not behavioral types for whom μ is not defined.

G.2 Breaking Even

Lemma 9. $\psi^- = \psi^+$ in $\text{supp}(\alpha_X) \cap (0, 1]$.

Intuitively, this means there is no pooling of types - different levels of riskiness purchase different levels of coverage.

Proof. It follows from the corollary that for each $0 \leq y < x < z \leq 1$, $p(y) \leq x \cdot \psi^-(x) \leq x \cdot \psi^+(x) \leq p(z)$, as $\text{id} \cdot \psi^- \leq p \leq \text{id} \cdot \psi^+ \alpha_X$ a.s. by (1). The continuity of prices gives the lemma for $x \in (0, 1)$. For $x = 1$, observe that if $\psi^-(1) < \psi^+(1)$, then since $\psi^- = \psi^+$ in $X \cap (0, 1)$, 1 must be an atom of 1; then by the same continuity of prices (Lemma 18), $p(1) \leq \psi^-(1)$, and yet p must be a strict average of $\psi^-(1), \psi^+(1)$, a contradiction. \square

Hence, denote $\psi = \psi^- = \psi^+$. By (1), we have

Lemma 10. $\alpha(\{\theta, x \mid p(x) \neq \mu(\theta) \cdot x\}) = 0$, and (equivalently),

$$p(z) = z \cdot \psi(z), \forall z \in \text{supp}(\alpha_X) \quad (26)$$

G.3 Coverage Strictly Increases with Risk

Lemma 11. ψ is strictly increasing in $\text{supp}(\alpha_X) \cap (0, 1]$.

Proof. Suppose not; let $a < b$ be such that $\psi(a) = \psi(b) = \mu_0$. Corollary 5 then implies that $p(x) = \mu_0 \cdot x$ for all $x \in [a, b]$. But since $w_{\mu_0} > \mu_0$ at all but full insurance, types with riskiness μ_0 would all prefer $(b, p(b) = \mu_0 \cdot b)$ over $(a, p(a) = \mu_0 \cdot a)$. \square

G.4 Coverage Continuous and Increasing in Risk

Corollary 6. There is a mapping $\sigma : \text{supp}(P_\mu) \rightarrow [0, 1]$, strictly increasing and continuous on $\text{supp}(P_\mu) \setminus \sigma^{-1}(\{0\})$, s.t. $\alpha\{(\theta, x) \mid x = \sigma(\mu(\theta))\} = 1$.

Proof. Let $W \subseteq \Theta$ such that $\alpha(\{\theta \in W\} \Delta \{x > 0\}) = 1$, i.e., those types which choose positive coverage. Let $\sigma = \psi^{-1}$. σ is well-defined P_μ -a.e. on $\text{supp}((P_\mu(\cdot \mid W)))$, and by the previous results is strictly monotonic. Extend σ to $\text{supp}(P_\mu(\cdot \mid \Theta \setminus W))$ by 0; by the previous results, this is well-defined (α -a.s., any types θ s.t. $\mu(\theta) \in \text{supp}((P_\mu(\cdot \mid W)) \cap \text{supp}(P_\mu(\cdot \mid \Theta \setminus W)))$ choose 0 coverage, i.e., are not in W). \square

G.5 Use of Assumption 8

Before continuing with the steps of proof, we bring here a simple result which embodies our use Assumption 8. First, a lemma:

Lemma 12. $\text{supp}(P_\mu) = \text{range}(\mu)$.

The follows from Assumption 8 together with the continuity of $\mu : \Theta \rightarrow \mathbb{R}_+$.

Proposition 11. *The set*

$$\{(\mu, p, x, q, y) \in \text{supp}(P_\mu) \times \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \times [0, 1] \mid \mu \text{ strictly prefers } (p, x) \text{ over } (q, y)\}$$

is open.

Proof. Suppose $u_{\mu_0}(p_0, x_0) > u_{\mu_0}(q_0, y_0)$. Suppose by way of contradiction there are sequences $\mu_n \rightarrow \mu_0$, $(p_n, x_n) \rightarrow (p_0, x_0)$, $(q_n, y_n) \rightarrow (q_0, y_0)$ s.t. μ_n weakly prefers (q_n, y_n) over (p_n, x_n) . Let $\theta_n \in \Omega$ s.t. $\mu(\theta_n) = \mu_n$; since $\{\theta \mid \mu(\theta) \leq \mu_0 + 1\}$ is compact, w.l.o.g. and by passing to a subsequence, we may assume $\theta_n \rightarrow \theta_0$ for some $\theta_0 \in \Theta$. Since μ is continuous, $\mu(\theta_0) = \mu_0$. Since

$$u(\theta_n, x_n, p_n) \leq u(\theta_n, y_n, q_n).$$

Taking the limit given by the continuity of utility,

$$u(\theta_0, x_0, p_0) \leq u(\theta_0, y_0, q_0),$$

a contradiction. □

G.6 Full insurance is in the Support

Lemma 13. *The supremum of the support of α_X is full insurance.*

Proof. Suppose $x^* < 1$ is the supremum of the support of α_X . By Corollary 5 (applied to a sequence of weak equilibria on finite grids which witness the equilibrium), we have for each $x > x^*$ and each $\mu \in \text{supp}(P_\mu)$, $p(x) \geq \mu \cdot x$. Hence, P_μ is compactly supported; denote $\bar{\mu} = \max \text{supp}(P_\mu)$. Hence, $p(x) \leq \bar{\mu} \cdot x$ for $x \geq x^*$ (price per insurance unit cannot be more than maximum costliness) and since p is continuous and prices are actuarially fair, $p(x^*) = \bar{\mu} \cdot x^*$; so $p(x) = \bar{\mu} \cdot x$ in $[x^*, 1]$.

Fix some $z > x^*$. Now, if $\bar{\mu}$ were an atom of P_μ , then for all $\theta \in \mu^{-1}(\bar{\mu})$, we could say that θ strictly prefers $(z, p(z) = \bar{\mu} \cdot z)$ for any to what he actually purchases, $(x^*, p(x^*) = \bar{\mu} \cdot x^*)$, as w_θ is greater than the slope of the price at all but full insurance, contradiction.. However, $\bar{\mu}$ need not be an atom of P_μ ; but by Lemma 12, $\bar{\mu} \in \text{range}(\mu)$. By Proposition 11, there is a neighborhood V of $(x^*, \bar{\mu} \cdot x^*)$ and neighborhood U of $\bar{\mu}$ s.t. if $(\sigma(\mu), p(\sigma(\mu))) \in V$ and $\mu \in U$, type μ prefers $(z, p(z))$ to what his purchased contract. Since price is continuous (Lemma 18), we may shrink U s.t. $(\sigma(\mu), p(\sigma(\mu))) \in V$ for all $\mu \in U$. For any

Proposition 1, Part (5)

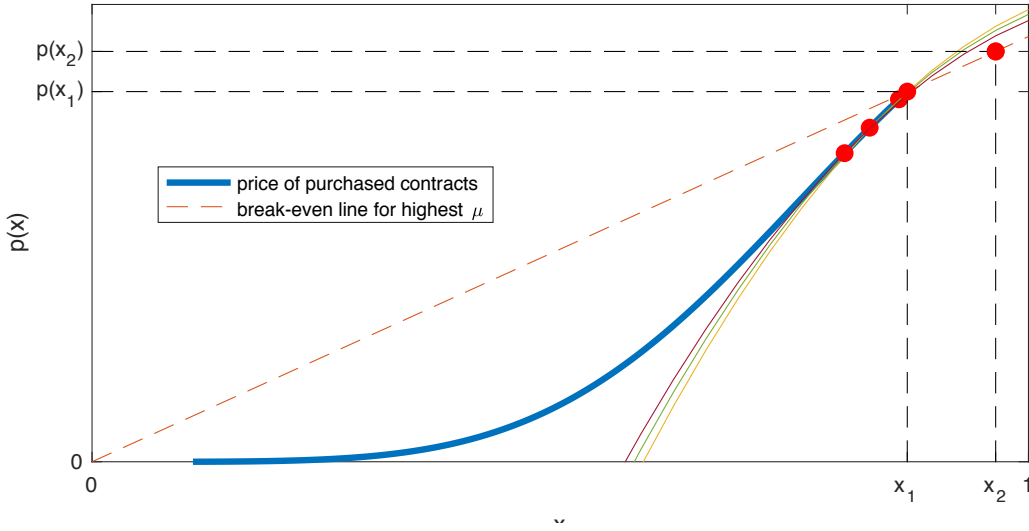


Figure 7: We consider, by way of contradiction, a hypothetical equilibrium where the supremum of contracts purchased is $x_1 < 1$. This implies, we show, that μ must be bounded by some $\bar{\mu}$, so the price of contracts $x > x_1$ is $p(x) = \bar{\mu}x$. Consider any $x_2 \in (x_1, 1)$. We then show that those agents who purchase x close to x_1 would prefer x_2 (at the price $p(x_2) = \bar{\mu}x_2$), a contradiction.

such $U, P_\mu(U) > 0$, hence a positive mass of types would wish to deviate, a contradiction. \square

The proof is illustrated in Figure 7 below.

The last argument in fact shows something stronger, which we will use later so we state it here:

Lemma 14. *There cannot exist and $\mu \in \text{supp}(P_\mu)$ and $z > \sigma(\mu)$ s.t. $p(z) \leq \mu \cdot z$ and $\alpha_X(\sigma(\mu), z) = 0$*

G.7 Lipschitz-Type Property

The fact that L -Lipschitzity of price in $[0, x_0]$ if $w_\theta \leq L$ in $\{x \leq x_0, p \leq p(x_0)\}$ for a.e. types purchasing coverage up to x_0 follows along the lines of Part 3 of Proposition 1 of AG, so we omit a complete proof. Essentially, the restriction of the economy to those types that choose coverage up to x_0 satisfies the framework and Lipschitz-ness conditions of that paper. As for the conclusion that $p(\cdot)$ is Lipschitz (for some constant) in $[0, x_0]$ without assuming a bound on w_θ , observe that since the coverage function is increasing in μ , and the willingness to pay is increasing in risk by Assumption 6. Take any θ with $\sigma(\mu(\theta)) > x_0$ and set $L = \max\{w_\theta(x, p) \mid x \leq x_0, p \leq p(x_0)\}$. It follows that $w_\theta \leq L$ in $\{x \leq x_0, p \leq p(x_0)\}$ for a.e. all types purchasing coverage up to x_0 .

G.8 Price as Approximate Upper Envelope

Here we prove the final property of equilibrium: Let $x \in (0, 1)$ with $p(x) > 0$. Let $\eta(\mu, x)$ denote the price at which type θ with riskiness μ is indifferent between $(\sigma(\mu), p(\sigma(\mu)))$ and $(x, \eta(\mu, x))$. Then, for every $y > x$ for which $\alpha_X((x, y]) > 0$,

$$p(x) = \zeta(x, y) := \text{ess} - \sup\{\eta(\theta, x) \mid x < \sigma(\mu(\theta)) \leq y\}.$$

Inequality (\geq) holds, as otherwise there would be a positive mass of types with $\sigma(\mu(\theta)) > x$ and $\eta(\theta, x) > p(x)$, and hence they would prefer $(x, p(x))$ over their purchased contract, which would result in a contradiction.

Conversely, suppose for some $x^* < y^*$ and $\delta > 0$, $\alpha_X((x^*, y^*]) > 0$ but $p(x^*) > \zeta(x^*, y^*)$. Clearly there means we have $\alpha_X((x^*, x^* + T]) = 0$ for some $T > 0$, since otherwise continuity of prices would give a contradiction. Let $\mu^* = \text{ess} - \inf\{\mu \in \text{supp}(P_\mu) \mid x^* < \sigma(\mu) \leq y^*\}$ be the infimum of all types that purchase insurance above x^* but at most y^* , and let $\mu_* = \text{ess} - \sup\{\mu \in \text{supp}(P_\mu) \mid \sigma(\mu) \leq x^*\}$; then $\mu_* < \mu^*$, otherwise the continuity of σ and of prices would finish the job. Recall by Lemma 12, $\mu^* \in \text{range}(\mu) = \text{supp}(P_\mu)$. Let $z^* = \sigma(\mu^*)$. We contend μ^* strictly prefers $(z^*, p(z^*))$ to $(x^*, p(x^*))$: Fix some $\zeta(x^*, y^*) < c < p(x^*)$; by assumption, for all P_μ -a.e. $\mu > \mu^*$, μ (strictly) prefers $(\sigma(\mu), p(\mu))$ to (x^*, c) , and hence by Lemma 11, μ^* (weakly) prefers $(z^*, p(z^*))$ to (x^*, c) , which he in turn strictly prefers to $(x^*, p(x^*))$.

Let $(\alpha_n, p_n)_{n=1}^\infty$ be a sequence of weak equilibria on finite grids $(X_n)_{n=1}^\infty$ converging to (α, p) in the appropriate sense.

Lemma 15. *For each $\varepsilon, \delta > 0$, for n large enough, under α_n a positive measure of types with riskiness in $[\mu^*, \mu^* + \delta)$ purchase coverage in $(x^* - \varepsilon, x^* + \varepsilon)$.*

Assuming the lemma (which we prove below) we will derive a contradiction. Let $x_n \rightarrow x^*$ and $z_n \rightarrow z^*$ with $x_n, z_n \in X_n$ for each n . Hence, by Proposition 11, there is $\varepsilon > 0$ s.t. if $|\mu - \mu^*| < \varepsilon$ for $\mu \in \text{supp}(P_\mu)$, if (p', x') is in an ε -neighborhood of $(x^*, p(x^*))$ and (q', z') is in a ε -neighborhood of $(z^*, p(z^*))$ μ' strictly prefers (q', z') to (p', x') . But by the above lemma, there are types $\mu_n \rightarrow \mu^*$ s.t. for n large enough, μ_n weakly prefers $(x_n, p_n(x_n))$ to $(z_n, p_n(z_n))$, a contradiction since for large enough n , $(x_n, p_n(x_n))$ and $(z_n, p_n(z_n))$ are in ε -neighborhoods of $(x^*, p(x^*))$ and $(z^*, p(z^*))$ respectively.

Now we return to the proof of the lemma: First we show that for each $\varepsilon > 0$ for large enough n , under α_n a positive measure of types with riskiness $\geq \mu^*$ purchase in $I_\varepsilon := (x^* - \varepsilon, x^* + \varepsilon)$. If there were such an $\varepsilon > 0$ for which this did not hold, then for all n large enough, $p^n(x) \leq \mu_* \cdot x$ for $x \in I_\varepsilon$; and hence $p(x) \leq \mu_* \cdot x$ for $x \in I_\varepsilon$. So in particular, there is at least one $x \in I := (\sigma(\mu_*), x^* + \varepsilon)$ s.t. $p(x) \leq \mu_* \cdot x$, and w.l.o.g., $\varepsilon \leq T$ so $\alpha_X(I) = 0$, contradicting Lemma 13. The reason we may choose positive mass of types

in $[\mu^*, \mu^* + \delta)$ if n is large enough is because fixing some $\mu' \in \text{supp}(P_\mu) = (\mu^*, \mu^* + \delta)$, we see μ' (and hence any $\mu \geq \mu'$ in $\text{supp}(P_\mu)$) strictly prefers $r' := (\sigma(\mu'), p(\sigma(\mu')))$ to (x^*, p^*) , and hence for large enough n would strictly prefer an option in X_n near r' instead of near (x^*, p^*) .

G.9 No Purchasing 0 Coverage

We sketch a proof showing that if in addition, Assumption 3 holds, then 0 is α -a.s. never purchased; the reader can complete the details. If P is discrete, with atoms $\mu_1 < \mu_2 < \dots$, then for some n , $\sigma(\mu_n) > 0$, and by backward induction using the 'upper-envelope' property of prices, $\mu_k > 0$ for all $k < n$ as well. Hence, assume P is continuous, but assume by way of contradiction that for some $\mu_* > \underline{\mu} = \inf \text{supp}(P_\mu)$, $\sigma(\mu_*) = 0$, and in particular 0 is an atom of α_X , the projection of α to alternatives.

First we contend $\text{supp}(\alpha_X) = [0, 1]$; by the previous properties, $\text{supp}(\alpha_X) = \{0\} \cup [\underline{x}, 1]$ for some $\underline{x} \geq 0$; if $\underline{x} > 0$, an imitation of the argument used to prove the upper-envelope property of price would give a contradiction, as the indifference curve of any type purchasing near \underline{x} would lie below $(0, 0)$. Next, we contend that Lemma 9 holds in this case as well, i.e., $\alpha_X(\cdot | J)$ for closed $J \subseteq (0, 1)$ is equivalent to the Lebesgue-measure: Indeed, the proof goes through nearly verbatim, replacing $w_x(z)$ with $w_x(z, p(z))$ where needed as willingness to pay now depends on prices as well, and relying on the fact that $\inf_{\mu \geq \mu_*, z \in J} w_\mu(z, p(z)) - \mu > 0$, i.e. that the insurance surplus for types with cost $\geq \mu_*$ is uniformly bounded away from 0 in J , which can be shown using the continuity of w and Assumption 8. Similarly to the proof Proposition 10, appropriately modified, it follows that for Lebesgue a.e. $x \in (0, \delta)$ for $1 > \delta > 0$,

$$p'(x) = w_{\tau(x)}(x, p(x)) = \tau(x) + w_{\tau(x)}(x, p(x)) - \tau(x) \geq \frac{p(x)}{x} + \inf_{x \in J} (w_{\tau(x)}(x, p(x)) - \tau(x)) = \frac{p(x)}{x} + W$$

for some $W > 0$. This equation encompasses that the insurance surplus is bounded away from 0 in the interval $(0, \delta)$ uniformly over all types with riskiness at least μ_* . Hence

$$\left(\frac{p(x)}{x}\right)' = \frac{p'(x)}{x} - \frac{p(x)}{x^2} \geq \frac{W}{x}$$

Hence,

$$\frac{p(\delta)}{\delta} \geq \frac{p(\delta)}{\delta} - \liminf_{y \rightarrow 0} \frac{p(y)}{y} = W \cdot \lim_{y \rightarrow 0} \ln\left(\frac{\delta}{y}\right) = \infty$$

a contradiction.

G.10 Weakening Assumption 8

We remark that Proposition 6 would hold if Assumption 8 would be replaced with the following weaker assumption, although the proof would be somewhat more technical and lengthy.

Assumption 9. For each $\mu' \in \text{supp}(P_\mu)$ with $\mu' > \underline{\mu}$, for each $0 < x_1 < x_2 < 1$, and each $0 < p_1 < p_2$, there is $\varepsilon > 0$ s.t. denoting $D = [x_1, x_2] \times [p_1, p_2]$ and $T = \{\theta \mid \mu' - \varepsilon < \mu(\theta) \leq \mu'\}$.

$$\inf_T \min_D (w_\theta(x, p) - \mu_\theta) > 0 \text{ and } \inf_T \min_D \frac{\partial u_\theta}{\partial p} < 0$$

We know for each $x < 1$ and $p \in \mathbb{R}$, $w_\theta(x, p) - \mu_\theta > 0$. Hence, by continuity, for each $\theta \in \Theta$, each $0 < x_1 < x_2 < 1$, and each $0 < p_1 < p_2$, $\min_D (w_\theta(x, p) - \mu_\theta) > 0$. However, we need this positivity to be uniform over all types whose riskiness μ is close to any given μ' . Similarly, we need the sensitivity to price to be bounded away from 0 in such a domain for all types with riskiness close enough to μ' .

H Existence in Generalized Insurance

We now prove Theorem 4: equilibrium existence in insurance markets exists under conditions more general than those stated in Section 3.

Fix a sequence of compact subsets $\Theta_1 \subseteq \Theta_2 \subseteq \dots \subseteq \Theta$ with $\Theta = \bigcup \Theta_n$. Let M^n be the essential supremum of μ w.r.t. $P(\cdot \mid \Theta^n)$. Then $M^1 \leq M^2 \leq M^3 \leq \dots$ with $M^n \rightarrow \infty$, $M^n \in \text{supp}(P_\mu)$ for each $n \in \mathbb{N}$. Let $P^n = P(\cdot \mid \Theta_n)$, and $\bar{X} = [0, 1]$. By AG, $\mathcal{E}^n = [\Theta^n, \bar{X}, P^n]$ has an equilibrium.

Fix one such equilibrium (α^n, p^n) for each economy. Proposition 6 implies that in each of these equilibria, there is a strictly increasing function $\sigma^n : \text{supp}(P_\mu^n) \rightarrow [0, 1]$ such that type with riskiness $\mu \in \text{supp}(P_\mu^n)$ purchases coverage $\sigma^n(\mu)$ α^n -a.s., with $\sigma^n(M^n) = 1$.

We will require the following two results.

Lemma 16. For each $\mu \in \text{supp}(P_\mu)$, $\limsup_{n \rightarrow \infty} \sigma^n(\mu) < 1$.

The intuition is simple: Fixing $\mu_1 < \mu_2$, then $\sigma^n(\mu_1) \leq \sigma^n(\mu_2)$ for all n , and if $\sigma^n(\mu_2)$ is close to 1, then $\sigma^n(\mu_1)$ should not be more (approximately) than the coverage z that makes type μ_2 indifferent between contract $(x = 1, p = \mu_2)$ and $(z, \mu_1 \cdot z)$.

Proof. For each θ_2, θ_1 with $\mu_2 = \mu(\theta_2) > \mu_1 = \mu(\theta_1)$ and each $u \in [0, 1]$, let $\phi(\mu_1, \mu_2, u)$ denote the unique $v < u$ s.t. type θ_2 is indifferent between contracts $(\mu_2 \cdot u, u)$ and $(\mu_1 \cdot v, v)$; such unique v exists as $w_{\theta_2}(x) > \mu_2 > \mu_1$ for all $x \in (0, 1)$, ϕ is continuous on $\mathbb{R}_{++} \times \mathbb{R}_{++} \times [0, 1]$, and $\phi(\mu_1, \mu_2, u) < u$ for $u \in (0, 1]$.

Suppose $\mu_1 \in \text{supp}(P_\mu)$ with $\lim_{n \rightarrow \infty} \sigma^{k_n}(\mu_1) = 1$ for some indices (k_n) . Fix some $\mu_2 > \mu_1$ with $\mu_2 \in \text{supp}(P_\mu)$, and hence w.l.o.g., $\mu_2 \in \text{supp}(P_\mu^{k_n})$ for all $n \in \mathbb{N}$. For each k_n , $\sigma^{k_n}(\mu_1) \leq \phi(\mu_1, \mu_2, \sigma^{k_n}(\mu_2))$, as otherwise type θ with $\mu(\theta) = \mu_1$ would instead choose coverage $\sigma^{k_n}(\mu_2)$. Then $\lim_{n \rightarrow \infty} \sigma^{k_n}(\mu_2) = 1$ as each σ^{k_n} is monotonically increasing. Hence,

$$1 = \lim_{n \rightarrow \infty} \sigma^{k_n}(\mu_1) \leq \lim_{n \rightarrow \infty} \phi(\mu_1, \mu_2, \sigma^{k_n}(\mu_2)) = \phi(\mu_1, \mu_2, \lim_{n \rightarrow \infty} \sigma^{k_n}(\mu_2) = 1) = u(\mu_1, \mu_2, 1) < 1$$

a contradiction. \square

Notice that Lemma (16) does not rely on on the condition given in Equation (14). Lemma 17 however, crucially, does:

Lemma 17. *For each $0 < m < 1$, there is $M > 0$ such that if $\mu > M$ and $n \in \mathbb{N}$ is such that $\mu \in \text{supp}(P_\mu^n)$, then $\sigma^n(\mu) > m$.*

The idea is to make observe types purchasing a sequence of riskinesses $\mu_1 < \mu_2 < \dots$, and bound the differences between the coverages chosen by two adjacent types in this sequence. (It is tempting to think we are making a reduction to the discrete case, but since the utility has a more general form that in Section 3.6, the arguments have to be slightly less direct.)

Proof. Fix types θ'', θ' and denote $\mu'' := \mu(\theta'') > \mu' := \mu(\theta')$, $x'' := \sigma(\mu'') > x' := \sigma(\mu') \geq 1 - \delta$, and $\nu'' := \underline{\nu}_\delta(\theta'')$, $\nu' := \underline{\nu}_\delta(\theta')$. Denote

$$\Delta\mu = \mu'' - \mu'$$

$$\Delta x = x'' - x'$$

$$\Delta p = p(x'') - p(x') = \mu'' \cdot x'' - \mu' \cdot x' = \mu' \cdot \Delta x + x'' \cdot \Delta\mu$$

It follows from Lemma 1 of Section A that

$$\Delta p \geq \Delta x \cdot \left[\mu' + \nu' \cdot \left(1 - \frac{x' + x''}{2}\right) \right] \quad (27)$$

Combining these,

$$x'' \cdot \Delta\mu \geq \Delta x \cdot \nu' \cdot \left(1 - \frac{x' + x''}{2}\right) \rightarrow \Delta x \leq \frac{\Delta\mu}{\nu'} \left(1 - \frac{x' + x''}{2}\right)^{-1}$$

Hence,

$$\Delta x \leq \frac{\Delta\mu}{\nu'} \cdot \frac{2}{2 - x' - x''} < \frac{\Delta\mu}{\nu'} \cdot \frac{1}{1 - x''}$$

Also, since $2 - x' - x'' \geq x'' - x' = \Delta x$,

$$\Delta x \leq \frac{\Delta\mu}{\nu'} \cdot \frac{2}{\Delta x} \rightarrow \Delta x \leq \sqrt{2 \cdot \frac{\Delta\mu}{\nu'}}$$

Now, by (14), there exists a sequence of types (θ_j) such that, denoting $\mu_j = \mu(\theta_j)$, we have $\mu_1 < \mu_2 < \dots$, and such that denoting $\nu_n = \nu_\delta(\theta_n)$,

$$\sum_{n=1}^{\infty} \frac{\mu_{n+1} - \mu_n}{\nu_n} < \infty$$

Fix $m, \eta > 0$ with $m < 1 - 4\eta < 1$, and choose K such that

$$\sum_{n=K}^{\infty} \frac{\mu_{n+1} - \mu_n}{\nu_n} < \eta^2$$

Each term is hence also less than η^2 . Suppose by way of contradiction, $\sigma(\theta_K) \leq m$. Then for each $n \geq K$, if $x_{n+1} > 1 - \eta$ we can say

$$x_{n+1} - x_n \leq \sqrt{2 \cdot \eta^2} < 2\eta$$

while if $x_{n+1} \leq 1 - \eta$, we can say

$$x_{n+1} - x_n \leq \frac{1}{\eta} \cdot \frac{\mu_{n+1} - \mu_n}{\nu_n}$$

From the first of these, we see that there is $N > K$ such that $1 - 3\eta < x_N < 1 - \eta$. Then,

$$x_N - x_K \leq \frac{1}{\eta} \sum_{n=K}^N \frac{\mu_{n+1} - \mu_n}{\nu_n} \leq \frac{1}{\eta} \cdot \eta^2 \rightarrow x_K \geq x_N - \eta > (1 - 3\eta) - \eta > m$$

a contradiction. Hence, we may take $M = \mu_K$. \square

As a result, the conditions of Proposition 8 hold: As remarked there, Assumption 1 - in particular, the quasi-linearity of utility in prices - implies that for every two alternatives $x, y \in X$, price $p \geq 0$, and type $\theta \in \Theta$, there is price q high enough s.t. $u(\theta, x, p) > u(\theta, y, q)$. As for the required properties of the equilibria $(p_n, \alpha_n)_{n=1}^{\infty}$:

1. First we show that if $[0, m] \subseteq [0, 1)$, then (p_n) is point-wise (in fact, uniformly) bounded and equicontinuous on $[0, m]$: Choose some $\theta_0 \in \Theta$ such that, denoting $\mu_0 = \mu(\theta_0)$, $\sigma^n(\mu_0) > m$ whenever $\mu_0 \in \text{supp}(P_\mu^n)$. Such θ_0 exists by Lemma 17. Then for all such n and all $x \leq m$, $p_n(x) \leq p_n(m) \leq \mu_0 \cdot m$, so we have the boundedness in $[0, m]$. Denote $w \equiv w_{\theta_0}$. Then for α -a.e. type θ that choose coverage in

$[0, m]$, $\sigma(\mu(\theta)) \leq \sigma(\mu_0) = \sigma(\mu(\theta_0))$ so by Assumption 3, $w_\theta \leq w_{\theta_0}$, and each type has Lipschitz utility (with the same Lipschitz constant as w_{θ_0} .)

2. Next we verify that for each $M \in \mathbb{R}$, there is a compact $K \subseteq [0, 1)$, s.t. for large enough n , $\inf_{x \notin K} p^n(x) \geq M$. Fix $M > 0$ and some $\mu \in \text{supp}(P_\mu)$ with $\mu > 2M$, let $N \in \mathbb{N}$ and $\frac{1}{2} < t < 1$ be such that $\sigma^n(\mu) < t$ for all $n > N$; such N, t are guaranteed by Lemma 16. By the monotonicity of each σ^n and p^n , for $n > N$ and $x \notin K := [0, t]$,

$$p^n(x) > p^n(t) \geq p^n(\sigma^n(\mu)) = \sigma^n(\mu) \cdot \mu \geq t \cdot 2M > M$$

3. Lemma 7 shows that the requirement (*) of Proposition 8 holds.

Hence, an equilibrium (p, α) of the economy $[X, \Theta, P]$ exists.

I Generalized Equilibrium Properties

This section shows that several properties of equilibrium derived by AG also hold in settings where costs are unbounded. (Note that Lemma 18 is used in the proof of Theorem 2, or more precisely its generalization Proposition 6.) Proposition 12 is used implicitly throughout, in particular the fact that in equilibrium, a.e. agent is selecting an optimal contract.

I.1 Continuity

The continuity (and in fact Lipschitz-ness) of prices was proven in AG (for the bounded environments they consider). Here, we prove the continuity of prices in generic unbounded settings.⁴⁷

Lemma 18. *If (p, α) is an AG-equilibrium, then p is continuous.*

Proof. Suppose $x_n \rightarrow x$ in X , and let $(p^n, \alpha^n)_{n=1}^\infty$ be the approximating sequence of weak equilibria with alternatives $(\bar{X}_n)_{n=1}^\infty$. By passing to a subsequence of (p^n, α^n) , we may assume that for each n , there is $y_n \in \bar{X}_n$ such that

$$|y_n - x_n| < \frac{1}{n} \text{ and } |p^n(y_n) - p(x_n)| < \frac{1}{n}$$

Hence, $y_n \rightarrow x$. Therefore, since $(p^n, \alpha^n)_{n=1}^\infty$ witnesses that (p, α) is an equilibrium, $p^n(y_n) \rightarrow p(x)$. By the second inequality, $p(x_n) \rightarrow p(x)$, as required. \square

⁴⁷The continuity of prices in the particular case of the utility functions of insurance markets discussed in this paper, as introduced in Section 5.1, follow from Lemma 3. 18 holds in a much more general setup.

I.2 Equilibrium is Weak Equilibrium

AG shows under their weaker assumptions that equilibria are, in particular, weak-equilibrium. This will also be true in our case although, to prove it, we need the following auxiliary result.

The following lemma, stated in greater generality than needed, may be of independent interest.

Lemma 19. *Let X be a locally compact separable metric space, (X_n) a sequence of finite subsets, $p : X \rightarrow \mathbb{R}$ continuous and for each $n \in \mathbb{N}$, $p_n : X_n \rightarrow \mathbb{R}$, s.t. if (x_n) is a sequence in X with $x_n \rightarrow x \in X$ s.t. $x_n \in X_n$ for each $n \in \mathbb{N}$, then $p_n(x_n) \rightarrow p(x)$.*

Then there are extensions of the p_n to continuous functions $\tilde{p}_n : X \rightarrow \mathbb{R}_+$ s.t. $\tilde{p}_n \rightarrow p$ uniformly on compact sets. In particular, if X is compact, then $\forall \varepsilon > 0$, there is $N \in \mathbb{N}$, s.t. $\forall n > N$ and $\forall x \in X_n$, $|p_n(x) - p(x)| < \varepsilon$.

The latter conclusion, for the case of compact X , follows from the first part. We note, however, that the latter conclusion actually already follows from the first step in the proof.

Proof. Let $(K_j)_{j=1}^\infty$ be an increasing sequence of compact sets with $X = \cup_j K_j$ and $K_j \subseteq K_{j+1}^\circ$; such exists as X is locally compact and separable metric. Fix $J \in \mathbb{N}$: We contend that $\forall \varepsilon > 0$, there is $N \in \mathbb{N}$, s.t. $\forall n > N$ and $\forall x \in X_n \cap K_J$, $|p_n(x) - p(x)| < \varepsilon$. Indeed, if not, there is $\varepsilon > 0$, a sequence $n_1 < n_2 < \dots$ of indices, a sequence (x_j) with $x_j \in X_{n_j} \cap K_J$, $|p_{n_j}(x_j) - p(x_j)| \geq \varepsilon$, and such that (x_j) converges; denote the limit $x \in K_J$. Hence, $p_{n_j}(x_j) \rightarrow p(x)$ by assumption. Since p is continuous by Lemma 18, $p(x_j) \rightarrow p(x)$. Together, these give a contradiction.

Hence, define $q_n : X_n \rightarrow \mathbb{R}$ by $q_n = p_n - p$. Denote $Y_n = X_n \cap K_n$, $\varepsilon_n = \max_{x \in Y_n} |q_n|$. By the last paragraph, $\varepsilon_n \rightarrow 0$. The Tietze extension theorem implies, for each $n \in \mathbb{N}$, the existence of a continuous extension \tilde{q}_n of q_n to X satisfying $\varepsilon_n = \max_{K_n} |\tilde{q}_n|$. (Formally, first extend the restriction of q_n to Y_n to a function \tilde{q}_n on K_n satisfying $\varepsilon_n = \max_{K_n} |\tilde{q}_n|$ via Tietze's theorem, and then extend it to a function on X agreeing with q_n on X_n in an arbitrary continuous way, again via Tietze's theorem.) Defining $\tilde{p}_n = \tilde{q}_n + p$ for each $n \in \mathbb{N}$ give the required extensions, since for any compact subset $K \subseteq X$, there is J s.t. for all $j > J$, $K \subseteq K_j$. \square

Now, the proof of Proposition 12 follows along lines similar to the corresponding Proposition in AG, with some care required since our setup allows for unbounded cost.

Proposition 12. *An equilibrium is also a weak equilibrium.*

Proof. Take a sequence (X_n, p_n, α_n) of weak equilibria on finite subsets $X_n \subseteq X$ which witnesses that (p, α) is an equilibrium. Let (\tilde{p}_n) correspond to (p_n) , (X_n) , and p as in

Lemma 18. For any continuous function $f : X \rightarrow \mathbb{R}$ with compact support, since the \tilde{p}_n are uniformly bounded on compact sets (p is continuous and $\tilde{p}_n \rightarrow p$ uniformly on compact sets), and since $\tilde{p}^n(x) = p^n(x) = E_{\alpha^n}[c \mid x]$ for all $x \in \text{supp}(\alpha^n)$,

$$\int_{\Theta \times X} f \cdot p \cdot d\alpha = \lim_{n \rightarrow \infty} \int_{\Theta \times X} f \cdot p \cdot d\alpha^n = \lim_{n \rightarrow \infty} \int_{\Theta \times X} f \cdot \tilde{p}^n \cdot d\alpha^n = \lim_{n \rightarrow \infty} \int_{\Theta \times X} f(x) \cdot c(x, \theta) d\alpha^n(x, \theta)$$

Now denoting by $K \subseteq X$ the compact support of f , we know that for each $\varepsilon > 0$, there is compactly support $g_\varepsilon : \Omega \rightarrow [0, 1]$ s.t. $\int_{\Theta} (1 - g_\varepsilon(\theta)) \max_{x \in K} c(x, \theta) dP(\theta) < \varepsilon$. Clearly,

$$\int_{\Theta \times X} f(x) \cdot c(x, \theta) d\alpha^n(x, \theta) = \int_{\Theta \times X} f(x) g_\varepsilon(\theta) \cdot c(x, \theta) d\alpha^n(x, \theta) + \int_{\Theta \times X} f(x) (1 - g_\varepsilon(\theta)) \cdot c(x, \theta) d\alpha^n(x, \theta)$$

$$\int_{\Theta \times X} f(x) \cdot c(x, \theta) d\alpha(x, \theta) = \int_{\Theta \times X} f(x) g_\varepsilon(\theta) \cdot c(x, \theta) d\alpha(x, \theta) + \int_{\Theta \times X} f(x) (1 - g_\varepsilon(\theta)) \cdot c(x, \theta) d\alpha(x, \theta)$$

Now,

$$\lim_{n \rightarrow \infty} \int_{\Theta \times X} f(x) g_\varepsilon(\theta) \cdot c(x, \theta) d\alpha^n(x, \theta) = \int_{\Theta \times X} f(x) g_\varepsilon(\theta) \cdot c(x, \theta) d\alpha(x, \theta)$$

and the errors terms are at most $\varepsilon \cdot \sup |f|$, and $\varepsilon > 0$ was arbitrary. Hence,

$$\int_{\Theta \times X} f \cdot p \cdot d\alpha = \int_{\Theta \times X} f(x) \cdot c(x, \theta) d\alpha(x, \theta)$$

and this was for any $f : X \rightarrow \mathbb{R}$ compactly supported. Hence, $p(x) = E_\alpha[c(x, \theta) \mid x]$ α -a.s.

Now, let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly monotonically increasing continuous funding with bounded range, e.g., $\phi(x) = \arctan(x)$ or $\phi(x) = \frac{x}{1+|x|}$. Since α_n is a weak equilibrium, it holds

$$u(\theta, \tilde{p}_n(x), x) = \sup_{x' \in X_n} u(\theta, \tilde{p}_n(x'), x'), \text{ for } \alpha_n - a.e. (\theta, x) \in \Theta \times X_n$$

Hence, it is also true that, denoting $v = \phi \circ u$

$$v(\theta, \tilde{p}_n(x), x) = \sup_{x' \in X_n} v(\theta, \tilde{p}_n(x'), x'), \text{ for } \alpha_n - a.e. (\theta, x) \in \Theta \times X_n$$

Let α' be a 'deviation to α' - i.e., a measure on $\Theta \times X$ whose projection to Θ is P , and letting (α'_n) be a sequence of measures on $(\Theta \cup X) \times X$, with α'_n supported on $(\Theta \cup X_n) \times X_n$ and $\alpha'_n \rightarrow \alpha$ weakly, we have since (p_n, α_n) is a weak equilibrium,

$$\int_{\Theta \times X} v(\theta, \tilde{p}_n(x), x) d\alpha_n \geq \int_{\Theta \times X} v(\theta, \tilde{p}_n(x), x) d\alpha'_n$$

$\alpha_n \rightarrow \alpha$, $\alpha'_n \rightarrow \alpha'$, so the families (α_n) and (α'_n) are tight, and v is bounded. Hence, for

each $\varepsilon > 0$, there is $\zeta_\varepsilon : \Theta \times X \rightarrow [0, 1]$ continuous and compactly supported, such that

$$\int_{\Omega \times X} (1 - \zeta_\varepsilon(x, \theta)) \cdot |v(\theta, \tilde{p}_n(x), x)| d\beta < \varepsilon, \text{ for } \beta = \alpha_n, \alpha, \alpha'_n, \alpha', n \in \mathbb{N}$$

Since $\tilde{p}_n \rightarrow p$ uniformly on the support of ζ_ε ,

$$\int_{\Theta \times X} \zeta_\varepsilon(x, \theta) v(\theta, \tilde{p}_n(x), x) d\alpha_n \rightarrow \int_{\Theta \times X} \zeta_\varepsilon(x, \theta) v(\theta, p(x), x) d\alpha$$

and

$$\int_{\Theta \times X} \zeta_\varepsilon(x, \theta) v(\theta, \tilde{p}_n(x), x) d\alpha'_n \rightarrow \int_{\Theta \times X} \zeta_\varepsilon(x, \theta) v(\theta, p(x), x) d\alpha'$$

Since this was for any compactly supported ζ_ε , it follows that

$$\int_{\Theta \times X} v(\theta, p(x), x) d\alpha \geq \int_{\Theta \times X} v(\theta, p(x), x) d\alpha'$$

Since this was for any measure α' on $\Theta \times X$ whose projection to Θ is P ,

$$v(\theta, p(x), x) = \sup_{x' \in X} v(\theta, p(x'), x'), \text{ for } \alpha - a.e. (\theta, x) \in \Theta \times X$$

and therefore

$$u(\theta, p(x), x) = \sup_{x' \in X} u(\theta, p(x'), x'), \text{ for } \alpha - a.e. (\theta, x) \in \Theta \times X$$

□

J Discrete types: Direct Construction

In this Section we provide a “direct” construction of the AG equilibrium for a simple insurance economy with discrete types, without using Proposition 1.

J.1 Equilibrium in bounded economies

We now show that, for a truncated economy \mathcal{E}^n , the allocation described in Proposition 4 is indeed an AG equilibrium.

Proof. Since utilities are quasi-linear and $p > 0$ on $(0, 1)$,⁴⁸ it is enough to approximate (p, α) , in the same manner described in Section 1, but on $X' = (x_0, 1)$ instead of $[0, 1]$ (as $p \equiv 0$ in $(0, x_0)$) and with η_n not necessarily strictly positive on the behavioral types \bar{X}^n ; afterwards the weight of the behavioral types could be increased slightly to be strictly

⁴⁸ $p(0) = 0$, but $\forall n, \bar{X}^n \subseteq (0, 1)$ in our construction to follow, so p is positive on \bar{X}^n .

positive in such a way that the price goes down by the same amount for each alternative in \bar{X}^n .

We will also index the sequence of economies by n . For each n , let \bar{X}^n be the set

$$\bar{X}^n = \{x_{ij} \mid i = 1, \dots, n, j = 1, \dots, n\} \cup \{x_1, \dots, x_n\}.$$

The contracts x_1, \dots, x_n are obtained as in Section 3.6, the contracts purchased by the first n types. That is, economy n has only the first n contracts x_1, \dots, x_n . Moreover, to each contract i are associated n behavioral types x_{i1}, \dots, x_{in} are distributed (e.g., evenly) strictly between x_{i-1} and x_i . (Recall that x_0 is the right-most point s.t. $p(x_0) = 0$, i.e., where type 1 is indifferent between $(0, x_0)$ and (p_1, x_1) .) The mass of agents at each x_{ij} (which denote η_n) is defined below.

As in AG, the behavioral agents in \bar{X}^n have riskiness $\mu = 0$, i.e., zero cost. We set prices $p_n \equiv p$ for contracts on \bar{X}^n . Moreover, we set the distribution of the weak equilibrium (α_n) such that

$$\begin{aligned} \alpha_n(\{\mu_i, x_i\}) &= P_i \left[1 - \frac{1}{n}\right], \forall i = 1, \dots, n \\ \alpha_n(\{\mu_{i+1}, x_{ij}\}) &= P_{i+1} \frac{1}{n^2}, \forall i, j = 1, \dots, n \end{aligned}$$

That is, of the original mass P_i of “regular” types μ_i , all but a $\frac{1}{n}$ -fraction choose x_i , while the rest evenly spread themselves between the contracts x_{i1}, \dots, x_{in} , such that the mass of type μ_i in each of these contracts is a share $\frac{1}{n^2}$ of the total mass P_i . Recall that $x_{i-1} < x_{i1} < \dots < x_{in} < x_i$ and moreover $p(x)$ is defined so that types μ_i are indifferent between all these contracts.

We also construct the distribution α_n such that, all types $k > n$ (each with mass P_k) purchase the highest coverage available (x_n):

$$\alpha_n(\{\mu_k, x_n\}) = P_k, \quad \forall k > n.$$

Since μ_n, ν_n increasing, this maximizes their utility when contracts x_k for $k > n$ are not available. This construction of α_n is illustrated by Figure 8.

We then define $\alpha_n(\{(x_{ij}, x_{ij})\}) = \eta_n(\{x_{ij}\})$ for all i, j to be the mass of behavioral types who purchase contract x_{ij} (which, recall, will also be purchased by some mass of types μ_i). We define $\eta_n(x_{ij})$ to satisfy

$$g_i(x_{ij}) = x_{ij} \mu_i \frac{P_i \cdot \frac{1}{n^2}}{P_i \cdot \frac{1}{n^2} + \eta_n(x_{ij})} \leq 1.$$

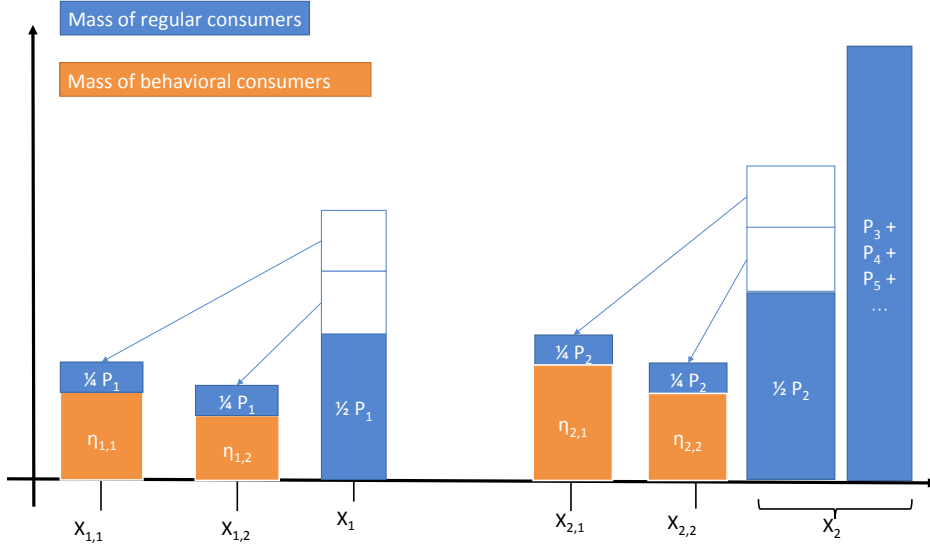


Figure 8: Illustration of α_n used in Section E. Specifically, the figure illustrates the second perturbation, α_3 .

This will imply that each contracts x_{ij} breaks even:

$$E_{\alpha_n}[\mu \cdot x \mid x_{ij}] = x_{ij} \mu_{i+1} \cdot \frac{\alpha_n(\mu_i, x_{ij})}{\alpha_n(\mu_i, x_{ij}) + \eta_n(x_{i,j})} + 0 = x_{ij} \mu_{i+1} \cdot \frac{P_i \cdot \frac{1}{n^2}}{P_i \cdot \frac{1}{n^2} + \eta_n(x_{i,j})} = g_i(x_{ij}) = p(x_{ij}) \quad (28)$$

Moreover, since $x_{i-1} \leq x_{ij} \leq x_i$, we also have

$$\frac{g_i(x_{i-1})}{x_i \mu_i} \leq \frac{P_i \cdot \frac{1}{n^2}}{P_i \cdot \frac{1}{n^2} + \eta_n(x_{ij})}$$

and therefore, as $n \rightarrow \infty$, we have $\sup_{x_{ij}} \eta_n(x_{ij}) \rightarrow 0$.

We also assume that, in economy n , there are no behavioral types purchasing contracts x_i for $i \leq n-1$:

$$\eta_n(x_i) = 0, \quad i = 1, \dots, n-1$$

Regarding the top contract x_n , the mass of behavioral types $\eta_n(x_n)$ is defined such that

$$\mu_n = \frac{p(x_n)}{x_n} = E_{\alpha_n}[\mu \mid x_n] = \frac{\mu_n P_n \frac{n-1}{n} + \sum_{j>n} \mu_j P_j}{P_n \frac{n-1}{n} + \sum_{j>n} P_j + \eta_n(x_n)}$$

i.e., $\eta_n(x_n)$ is chosen such that although the riskiest agents all choose the top contract, its price nonetheless satisfies $\mu_n = \frac{p(x_n)}{x_n}$. The fact that $\sum_n P_n \mu_n < \infty$ implies $\lim_{n \rightarrow \infty} \eta_n(x_n) = 0$.

In this way, for each i , the break even condition $E_{\alpha_n}[\mu x \mid x_i] = p_n(x_i) = p(x_i)$ holds for each $i = 1, \dots, n$ in (p_n, α_n) ; indeed, for each $i = 1, \dots, n-1$, only types μ_i purchase x_i , while for $i = n$ this results from our definition of $p_n(x_n) = p(x_n)$ and by (28). \square

J.2 Convergence to equilibrium of unbounded economy

We now prove that the equilibria described above for each truncated economy \mathcal{E}^n converge to the equilibrium of \mathcal{E} .

Proof. We claim that the sequence (p_n, α_n) demonstrates that (p, α) is an equilibrium. , $p_n \equiv p$ on \bar{X}^n and $\bar{X}^n \rightarrow X' = [x_0, 1)$ in the sense of Hausdorff.

Moreover, $\alpha_n \rightarrow \alpha$ weakly: Notice α_n is concentrated on the set of types $\{(\mu_k, \nu_k)\}_{k \in \mathbb{N}}$ and the behavioral types, with $\alpha_n(\mu_k, \nu_k) = \alpha(\mu_k, \nu_k) = P_k$. Define $\mathbb{I}_{k,m} = 1 \{k = m\}$ be an indicator function. Then,

$$\alpha_n(\mu_k, x_m) = \mathbb{I}_{k,m} \cdot P_m \left[1 - \frac{1}{n}\right] \rightarrow \mathbb{I}_{k,m} \cdot P_m = \alpha(\mu_k, x_m)$$

and for each $m \in \mathbb{N}$,

$$\alpha_n(\{x \in (x_{m-1}, x_m)\}) = P_m \frac{1}{n} \rightarrow 0 = \alpha(\{x \in (x_{m-1}, x_m)\}).$$

Hence for each $\delta < 1$, $\alpha_n(\cdot | \{x \leq \delta\}) \rightarrow \alpha(\cdot | \{x \leq \delta\})$ converges in total variation norm. This implies that $\alpha_n \rightarrow \alpha$ weakly.

Furthermore, α_n -a.s. the original agents $\{(\mu_n, \nu_n)\}_{n=1}^\infty$ are utility maximizing: agents of type $i \leq n$ are utility maximizing since they either choose the same option x_i in $\bar{X}^n \subseteq X = [0, 1)$, at the same price $p_n(x_i) = p(x_i)$, as they do when they can choose any alternative in X , or they choose an alternative $x_{i,1}, \dots, x_{i,n}$ which delivers the same utility as x_i at prices $p_n \equiv p$. Agents of type $k > n$ are utility maximizing since their willingness to pay for x is higher than that of type n , who (weakly) prefers the contract $x_n = \max[\bar{X}^n]$ to any other alternative in \bar{X}^n at prices $p_n \equiv p$. Therefore, each (p_n, α_n) is a weak equilibrium, so (p, α) is an equilibrium. \square