# Online Appendix to "Self-Fulfilling Debt Crises: A Quantitative Analysis"

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## A Timing, rollover risk, and crisis zone

In this section, we carefully define the crisis zone in the model. Recall that the timing within the period is as follows:

- Enter with state $\mathbf{S} = (B, \lambda, s)$;
- The government chooses its new portfolio of debt, $(B', \lambda')$;
- Lenders choose the price for the government bonds, $\{q^{(n)}(\mathbf{S}, B', \lambda')\}_n$, according to the no-arbitrage conditions (6)
- Finally, the government decides whether to default on its debt. The default decision is given by $\delta\left(\mathbf{S}, B', \lambda', \{q^{(n)}\}\right) \in \{0,1\}$, with $\delta = 1$ if

$$U\left(\tau Y(s_1) - B + \sum_{n=1}^{\infty} q^{(n)}\left[(1-\lambda')^{n-1}B' - (1-\lambda)^n B\right]\right) + \beta\mathbb{E}[V(B', \lambda', s')|\mathbf{S}] \geq \underline{V}(s_1),$$

  and $\delta = 0$ otherwise.

We allow the repayment decision $\delta$ to depend on arbitrary debt prices $\{q^{(n)}\}$ to have the notation for analyzing off-path situations. The problems in (5) and (7) are enough to determine the default decision along the equilibrium path given the pricing functions.

For notational convenience, it is useful to define the price of a portfolio of ZCB with a decaying factor $\lambda$ given that the government's portfolio is $(B', \lambda')$ as

$$Q(\mathbf{S}, B', \lambda'|\lambda) = \sum_{n=1}^{\infty}(1-\lambda)^{n-1}q^{(n)}(\mathbf{S}, B', \lambda').$$

We denote by $\mathcal{S}^{max}$ the *largest* region of the state space for which a default is possible. We can think of $\mathcal{S}^{max}$ as the collection of states in which the government defaults if lenders choose the *worst* possible price from the government's perspective conditional on satisfying the lenders' no-arbitrage condition. The next lemma characterizes the set $\mathcal{S}^{max}$. To this end, define the maximal value the government can attain if it faces fundamental prices but is restricted in the current period to have negative net issuances:

$$\Omega(\mathbf{S}) \equiv \max_{B', \lambda'} U\left(\tau Y(s_1) - B + \Delta^{\text{fund}}(\mathbf{S}, B', \lambda')\right) + \beta\mathbb{E}[V(B', \lambda', s')|\mathbf{S}] \quad \text{(A.1)}$$

subject to

$$\Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right) \leq 0.$$

**Lemma 1.** *Given* $V(B, \lambda, s)$ *and* $Q(\mathbf{S}, B', \lambda')$, $\mathbf{S} \in \mathcal{S}^{max}$ *if and only if*

$$\underline{V}(s) > \Omega(\mathbf{S}) \tag{A.2}$$

*Proof.* For the necessity part, note that if condition (A.2) does not hold, then the government will never default when the inherited state is $\mathbf{S}$ because it can attain a higher value than the default value by buying back part of the debt. Imposing the fundamental pricing function – the highest possible prices – in (A.1) is without loss of generality: because the government is buying back debt, a lower price will only increase the value of $\Omega$.

Consider now the sufficiency part. First note that $\mathbf{S} \in \mathcal{S}^{max}$ if for all $(B', \lambda')$ such that $\Delta^{\text{fund}}(\mathbf{S}, B', \lambda') \geq 0$ we have

$$U\left(\tau Y\left(s_1\right) - B\right) + \beta \mathbb{E}\left[V\left(B', \lambda', s'\right) | \mathbf{S}\right] < \underline{V}(s_1), \tag{A.3}$$

and for all $(B', \lambda')$ such that $\Delta^{\text{fund}}(\mathbf{S}, B', \lambda') < 0$ we have

$$U\left(\tau Y\left(s_1\right) - B + \Delta^{\text{fund}}\left(\mathbf{S}, B', \lambda'\right)\right) + \beta \mathbb{E}\left[V\left(B', \lambda', s'\right) | \mathbf{S}\right] < \underline{V}(s_1). \tag{A.4}$$

In condition (A.3) we use the fact that when net issuances are positive, $\Delta^{\text{fund}}(\mathbf{S}, B', \lambda') \geq 0$, the worst price for the government is zero. In condition (A.4) we use the fact that when net issuances are negative, $\Delta^{\text{fund}}(\mathbf{S}, B', \lambda') < 0$, the worst price for the government is the fundamental price. If conditions (A.3) and (A.4) are satisfied, it is then rational for lenders to expect a default and it is optimal for the government to default. We can further simplify condition (A.3) by noticing that it is sufficient to check such condition only for $(B', \lambda')$ such that $\Delta^{\text{fund}}(\mathbf{S}, B', \lambda') = 0$ because the continuation value $\mathbb{E}[V(B', \lambda', s') | \mathbf{S}]$ is decreasing in $B'$. Combining this simplified condition (A.3) with condition (A.4) implies that $\mathbf{S} \in \mathcal{S}^{max}$ if (A.2) holds, proving the claim. *Q.E.D.*

We can then define the crisis zone as $\mathcal{S}^{crisis} = \mathcal{S}^{max} \setminus \mathcal{S}^{fund}$.

# B Three-period model

We consider a three-period version of our model to illustrate in the most transparent way the key trade-offs that govern the optimal maturity composition of debt. At $t = 0$ the government can issue two types of securities: a zero coupon bond maturing in period 1,

$b_{01} \geq 0$, and a zero coupon bond maturing in period 2, $b_{02} \geq 0$. In period 1, the government decides whether to default. If there is no default, the government can issue a bond maturing in period 2, $b_{12}$. We allow for negative values of $b_{12}$ and interpret these as buybacks of outstanding long-term bonds.

It is convenient to present the model starting from the last period. In this appendix, a subscript on $s$ denotes time. At $t = 2$, inheriting a state $(b_{02}, b_{12}, s_2)$ the government chooses whether to default on the previously issued debt ($\delta_2 = 0$) or not ($\delta_2 = 1$) to maximize

$$V_2\left(b_{02} + b_{12}, s_2\right) = \max_{\delta_2} \delta_2 U\left(\tau Y_2 - b_{02} - b_{12}\right) + (1 - \delta_2)\underline{V}_2.$$

At $t = 1$, inheriting a state $(b_{01}, b_{02}, s_1)$, the government issues $b_{12}$ and it decides whether to default ($\delta_1 = 0$). The decision problem at $t = 1$ is

$$V_1\left(b_{01}, b_{02}, s_1\right) = \max_{\delta_1, G_1, b_{12}} \delta_1 \left\{U\left(G_1\right) + \beta \mathbb{E}_1[V_2\left(b_{02} + b_{12}, s_2\right)]\right\} + (1 - \delta_1)\underline{V}_1$$

subject to

$$G_1 + b_{01} \leq \tau Y_1\left(s_1\right) + q_{12}\left(b_{01}, b_{02}, s_1, b_{12}\right) b_{12}.$$

Finally at $t = 0$ the government issues both short- and long-term debt to solve

$$V_0\left(s_0\right) = \max_{G_0, b_{01}, b_{02}} U\left(G_0\right) + \beta \mathbb{E}_0[V_1\left(b_{01}, b_{02}, s_1\right)]$$

subject to

$$G_0 + D_0 \leq \tau Y_0 + q_{01}\left(s_0, b_{01}, b_{02}\right) b_{01} + q_{02}\left(s_0, b_{01}, b_{02}\right) b_{02},$$

with $D_0$ being the debt inherited from the past. To avoid issues associated with the dilution of legacy debt, we assume that the government does not inherit long-term debt. We further assume that $D_0$ is sufficiently small that the government does not default at $t = 0$. Price schedules $q_{01}$, $q_{02}$, and $q_{12}$ must be consistent with lenders' no-arbitrage condition,

$$\begin{aligned}
q_{01}\left(s_0, b_{01}, b_{02}\right) &= \mathbb{E}_0\left[m\delta_1\left(s_1, b_{01}, b_{02}\right)\right] \\
q_{02}\left(s_0, b_{01}, b_{02}\right) &= \mathbb{E}_0\left[m^2\delta_1\left(s_1, b_{01}, b_{02}\right)\delta_2\left(s_2, b_{02} + b_{01}\right)\right] \\
q_{12}\left(b_{01}, b_{02}, s_1, b_{12}\right) &= \delta_1(s_1, b_{01}, b_{02})\mathbb{E}_1\left[m\delta_2\left(s_2, b_{02} + b_{01}\right)\right],
\end{aligned}$$

where for simplicity we assume that lenders are risk neutral: $M(s_0, s_1) = M(s_1, s_2) = m$.

## B.1 Maturity choices and rollover risk

We next show that expectations of rollover crisis generate a preference for the government to issue long-term bonds at $t = 0$. In the extreme case in which all default risk at $t = 0$ reflects rollover risk, the government at $t = 0$ issues only long-term debt.

To illustrate the relation between maturity choices and rollover risk, we assume that a rollover crisis occurs with probability $\pi$ if the government is in the crisis zone at $t = 1$.

**Proposition 1.** *In the three-period economy, if there is only rollover risk and fundamental defaults never happen at $t = 1, 2$, then $b_{01} = 0$ and all debt is long term.*

*Proof.* By way of contradiction, suppose $\{b_{01}, b_{02}, b_{12}(s_1)\}$ is an equilibrium outcome with $b_{01} > 0$, and in period 1 it is always optimal to repay if the borrower is facing fundamental prices but a rollover crisis can arise in some states $s_1$ with associated output level $Y_1$ such that

$$U(\tau Y_1 - b_{01}) + \beta \mathbb{E}_1[V_2(b_{02}, s_2)] < \underline{V}_1 \tag{A.5}$$

hold.

Consider the following variation: increase $b_{02}$ by $\varepsilon/q_{02} > 0$ and decrease $b_{01}$ by $\varepsilon/q_{01} > 0$ so that $G_0$ is unchanged at the original price. We next show that under the assumption that there is no fundamental default risk, the variation can replicate the consumption pattern $(G_1, G_2)$ prescribed by the original allocation conditional on not having a rollover crisis. In fact, since there is no default risk between $t = 1$ and $t = 2$, conditional on not having a rollover crisis at $t = 1$, we have that $q_{12} = m$. Hence, optimality implies that at the original allocation, the following Euler equation is satisfied:

$$mU'(G_1) = \beta \mathbb{E}_1[U'(G_2)]. \tag{A.6}$$

Hence, achieving the same $G_1$ and $G_2$ is optimal and budget feasible if the government inherits $(b_{01} - \varepsilon/q_{01}, b_{02} + \varepsilon/q_{02})$ because the government can just decrease $b_{12}$ by $\varepsilon/q_{02}$ and

$$\begin{aligned} Y_1 - (b_{01} - \varepsilon/q_{01}) + m(b_{12} - \varepsilon/q_{02}) &= Y_1 - b_{01} + mb_{12} + \varepsilon\left(\frac{1}{q_{01}} - \frac{m}{q_{02}}\right) \\ &= Y_1 - b_{01} + mb_{12} = G_1, \end{aligned}$$

where in the second line we used the fact that under our assumptions, $q_{02} = mq_{01}$.

Finally, we turn to show that the proposed variation reduces the crisis zone, and so it increases the prices of debt in period zero and in turn increases consumption in period 0. To this end, note that under the original allocation, condition (A.5) holds for some states $s_1$ and there is no fundamental default risk, so for all $s_1$,

$$U(\tau Y_1 - b_{01} + mb_{12}) + \beta \mathbb{E}_1[V_2(b_{02} + b_{12}, s_2)] \geq \underline{V}_1.$$

Condition (A.5) and the equation above imply that the $b_{12}$ that solves (A.6) is greater than zero. This observation, (A.6), and concavity of $U$ imply that

$$q_{12}U'(\tau Y_1 - b_{01}) > \beta \mathbb{E}_1[V_2'(b_{02} + b_{12}, s_2)] \iff \frac{1}{q_{01}}U'(\tau Y_1 - b_{01}) > \frac{1}{q_{02}}\beta \mathbb{E}_1[V_2'(b_{02} + b_{12}, s_2)]$$

where in the second relation we used the fact that $q_{12} = \frac{q_{02}}{q_{01}} = \frac{m^2 \pi \Pr(\text{crisis zone})}{m \pi \Pr(\text{crisis zone})}$. So we have that

$$\begin{aligned} U(\tau Y_1 - b_{01} + \varepsilon/q_{01}) + \beta \mathbb{E}_1[V_2(b_{02} + \varepsilon/q_{02}, s_2)] \;\approx\; & \{U(\tau Y_1 - b_{01}) + \beta \mathbb{E}_1[V_2(b_{02}, s_2)]\} \\ + & \left\{\frac{1}{q_{01}}U'(\tau Y_1 - b_{01}) + \frac{1}{q_{02}}\beta \mathbb{E}_1[V_2'(b_{02}, s_2)]\right\}\varepsilon \end{aligned}$$

and so

$$U(\tau Y_1 - b_{01} + \varepsilon/q_{01}) + \beta \mathbb{E}_1[V_2(b_{02} + \varepsilon/q_{02}, s_2)] > U(\tau Y_1 - b_{01}) + \beta \mathbb{E}_1[V_2(b_{02}, s_2)]. \quad \text{(A.7)}$$

Since under our variation the borrower is in the crisis zone if

$$U(\tau Y_1 - b_{01} + \varepsilon/q_{01}) + \beta \mathbb{E}_1[V_2(b_{02} + \varepsilon/q_{02}, s_2)] \leq \underline{V}_1, \quad \text{(A.8)}$$

the inequality (A.7) implies that the probability of being in the crisis zone is smaller under our variation because (A.8) is satisfied for a lower output level than (A.5). Hence, bond prices at $t = 0$, $q_{01} = m\pi \Pr(\text{crisis zone})$, and $q_{02} = m^2 \pi \Pr(\text{crisis zone})$, increase and the government can increase consumption in the first period. So the variation increases utility, a contradiction. Therefore, we must have that $b_{01} = 0$. *Q.E.D.*

## B.2 Incentive channel

We now show how the incentive channel discussed in Section 3generates a preference for the government to issue short-term bonds. Consider now a situation in which there is no rollover risk, $\pi = 0$, and $Y_0$ and $Y_1$ are deterministic. $Y_2$ is the only source of uncertainty, and the uncertainty is revealed in $t = 2$. Because output is deterministic at $t = 1$, issuing long-term debt at time $t = 0$ does not entail hedging benefits. Hence, this environment isolates the incentive channel. The following proposition shows that the government at $t = 0$ issues only short-term debt.

**Proposition 2.** *In the three-period economy, if there is no rollover risk and there are no shocks in $t = 1$, then the optimal solution must have $b_{02} = 0$ if the probability of default in $t = 2$ is positive.*[1]

*Proof.*It is helpful to use a primal approach to solve for the equilibrium outcome. In this proof, we will show that issuing only one-period debt replicates the solution of a more relaxed problem in which the government in period 0 chooses debt issuances in period 1 subject to a no-default constraint.

To set up the primal problem, note that absent rollover risk and uncertainty in period 1, if the government plans to default, $\delta_1 = 0$, then $q_{01}$ and $q_{02}$ equal zero, so the government cannot raise resources in period zero and it is without loss of generality to set debt issuances equal to zero. Thus, we can impose $\delta_1 = 1$ and the following *no-default constraint*:

$$U(G_1) + \beta \mathbb{E}_1 \left[ \max \left\{ U(G_2(s_2)), \underline{V}_2(s_2) \right\} \right] \geq \underline{V}_1, \tag{A.9}$$

where the left side is the value along the equilibrium path for the government in period 1. Note that we imposed that in period 2, the government is defaulting everytime the value of default is above the value of repaying. This ensures that in period 2, the government's incentives to repay are satisfied.

Thus, an equilibrium outcome solves the following programming problem:

$$\max_{b_{01}, b_{02}, b_{12}, G_0, G_1, G_2(s_2)} U(G_0) + \beta \mathbb{E}_0 \left[ U(G_1) + \beta \max \left\{ U(G(s_2)), \underline{V}_2(s_2) \right\} \right] \tag{A.10}$$

subject to budget constraints

$$\begin{aligned} G_0 + D_0 &\leq q_{01} b_{01} + q_{02} b_{02} + \tau Y_0 \\ G_1 + b_{01} &\leq q_{12} b_{12} + \tau Y_1 \\ G_2 + b_{02} + b_{12} &\leq \tau Y_2(s_2) \end{aligned}$$

the no-default constraint (A.9), and the *issuance constraint*

$$U(G_1) + \beta \mathbb{E}_1 U(G_2(s_2)) \geq V_1(b_{01}, b_{02}), \tag{A.11}$$

where debt prices are given by the pricing equations

$$\begin{aligned} q_{01} &= m, \quad q_{02} = m q_{12}, \\ q_{12} &= m \Pr\left( U(G(s_2)) \geq \underline{V}_2(s_2) \right). \end{aligned}$$

[1]A sufficient condition for this is that $\beta/m$ is sufficiently low or $D_0$ sufficiently large.

It is immediate to verify that an equilibrium outcome solves the above problem and the converse is also true. The default and issuance constraints capture the sources of time inconsistency. The default constraint (A.9) captures the fact that the time zero government cannot choose allocations that attain a value lower than the value of default since future governments at $t = 1$ can always choose such an option if it is ex-post optimal. The issuance constraint (A.11) captures the fact that the time zero government cannot control debt issuances of the government in period 1. Such issuances must be optimal from the $t = 1$ government's perspective given its inherited state $(b_{01}, b_{02})$.

It is convenient to replace the issuance constraint with a first-order condition that characterizes the solution to the right side. To this end, notice that starting at $\left(b_{01}^*, b_{02}^*\right)$ in period $t = 1$, the optimal $b_{12}$ chosen by period 1 government is such that

$$0 = \left(q_{12}^* + \frac{\partial q_{12}}{\partial b_2} b_{12}\right) U'\left(\tau Y_1 - b_{01}^* + q_{12} b_{12}\right) - \int_{\mathcal{Y}_2\left(b_{02}^*, b_{12}\right)} U'\left(\tau Y_2\left(s_2\right) - b_{02}^* - b_{12}\right) \mu_2\left(s_2\right) ds_2 \tag{A.12}$$

where $q_{12} = m \Pr\left(U\left(\tau Y_2 - \left(b_{02}^* + b_{12}^*\right)\right) \geq \underline{V}_2\right)$ and

$$\mathcal{Y}_2\left(b_{02}, b_{12}\right) \equiv \left\{s_2 : U\left(\tau Y_2\left(s_2\right) - \left(b_{02} + b_{12}\right)\right) \geq \underline{V}_2\left(s_2\right)\right\}$$

is the set of output levels $Y_2\left(s_2\right)$ for which the government does not default in period 2. We can then replace (A.11) with (A.12) in the programming problem (A.10). We will refer to (A.12) as the issuance constraint in first-order condition form.

We now show that short-term debt is desirable because it relaxes the issuance constraint. To this end, consider a relaxed version of (A.10) in which we drop the issuance constraint in first-order condition form (A.12). Such a relaxed problem has a continuum of solutions because the split between long- and short-term debt issued in period zero is indeterminate. Let $\left\{b_{01}^*, b_{02}^*, b_{12}^*, G_0^*, G_1^*, G_2^*\left(s_2\right)\right\}$ be a generic solution to this relaxed programming problem. The optimality condition for $b_{12}$ for this relaxed problem is

$$\begin{aligned} 0 \;=\; & \frac{m}{1+\lambda} \frac{\partial q_{12}}{\partial b_2} b_{02}^* U'\left(G_0^*\right) + \\ & \left(q_{12}^* + \frac{\partial q_{12}}{\partial b_2} b_{12}^*\right) U'\left(\tau Y_1 - b_{01}^* + q_{12}^* b_{12}^*\right) - \int_{\mathcal{Y}_2\left(b_{02}^*, b_{12}^*\right)} U'\left(\tau Y_2\left(s_2\right) - b_{02}^* - b_{12}^*\right) \mu_2\left(s_2\right) ds_2 \end{aligned} \tag{A.13}$$

where $\lambda$ is the multiplier on the no-default constraint (A.9). The first term on the right side is the effect of an increase in $b_{12}$ on the price of long-term debt in period 0, the root of the time inconsistency problem we are considering here.

We next show that if $b_{02}^* = 0$, then the government at $t = 0$ can achieve the value of this relaxed problem in the more constrained problem (A.10). To see this, it is sufficient to check that the issuance constraint is met. It is immediate that if $b_{02}^* = 0$, then the first-order condition (A.13) implies the issuance constraint in first-order condition form. Hence, the solution to the relaxed problem can be implemented when $b_{02}^* = 0$.

The final step in the proof is to show that $b_{02}^* = 0$ is necessary when the solution to (A.10) is such that there are defaults in $t = 2$ in some states. Note that (A.13) and (A.12) can be jointly satisfied if and only if $\frac{\partial q_{12}}{\partial b_2} b_{02}^* = 0$, so at least one of the following conditions must be satisfied: i) $b_{02}^* = 0$, ii) no default at $t = 2$ so that $\partial q_{12}/\partial b_2 = 0$. Hence, if there are defaults in $t = 2$, then $b_{02}^* = 0$ is indeed necessary. *Q.E.D.*

## B.3 Insurance channel

We now turn to illustrate the insurance channel. To isolate this channel, we consider an economy in which there is no rollover risk, $\pi = 0$, and the current government can choose debt issued by future governments so that the incentive channel just described is not operative. We can think of this as studying the best arrangements where any deviations from the prescribed path of plays are punished with a reversion to $\underline{V}_t$.

To illustrate that long-term debt is a better instrument than short term debt to provide insurance absent outright default, we consider a minimalistic stochastic structure. In period $t = 1$, $s_1 \in \{s_L, s_H\}$ with $Y_1(s_L) < Y_1(s_H)$. The output at time 2 is again distributed in a continuous fashion, as in the previous example. Let $\mu_2(s_2)$ be the associated probability distribution. We assume that the realization of $s_1$ does not affect the distribution of $s_2$ and $\mu_2(s_2) Y_2(s_2)$ is increasing.[2] This last assumption is a sufficient condition for the debt Laffer curve in period 1 to be concave.

The equilibrium outcome solves a problem similar to the one considered in the previous subsection without the issuance constraint. Moreover, we will assume that $\underline{V}_1(s_1)$ is sufficiently small and it is not optimal to default in period 1, so the no-default constraint

$$U(G_1(s_1)) + \beta \int \max\{U(G_2(s_1, s_2)), \underline{V}_2(s_2)\} \mu_2(s_2) ds_2 \geq \underline{V}_1(s_1) \tag{A.14}$$

is slack for all $s_1$. Thus, an outcome $x = \{b_{01}, b_{02}, b_{12}, G_0, G_1(s_1), G_2(s_1, s_2)\}$ solves

$$\max_x U(G_0) + \beta \sum_{s_1} \mu(s_1) \left[ U(G_1(s_1)) + \beta \int \max\{U(G_2(s_1, s_2)), \underline{V}_2(s_2)\} \mu_2(s_2) ds_2 \right] \tag{A.15}$$

[2] Of course this is satisfied if output in period 2 is distributed uniformly.

subject to budget constraints

$$\begin{aligned} G_0 + D_0 &\leq q_{01}b_{01} + q_{02}b_{02} + \tau Y_0 \\ G_1(s_1) + b_{01} &\leq q_{12}(s)\, b_{12}(s) + \tau Y_1(s_1) \\ G_2(s_1, s_2) + b_{02} + b_{12}(s) &\leq \tau Y_2(s_2), \end{aligned}$$

where debt prices are given by

$$\begin{aligned} q_{01} &= m, \quad q_{02} = m \sum_{s_1 \in \{s_L, s_H\}} \mu(s_1)\, q_{12}(s_1), \\ q_{12}(s_1) &= \sum_{s_1} \mu(s_1) \Pr\left(U(G_2(s_1, s_2)) \geq \underline{V}_2(s_2) \,|s_1\right). \end{aligned}$$

**Proposition 3.** *In the three-period economy described above, if there is no rollover risk, no default in period 1, and default in some states in period 2, then the government at time 0 issues only long-term debt and $b_{01} = 0$.*

*Proof.* Assuming there is no default in period $t = 1$ (and so the default constraint at $t = 1$ is slack), the solution of problem (A.15) must satisfy the first-order necessary conditions with respect to $b_{01}$,$b_{02}$ ,and $b_{12}(s_1)$:

$$0 = U'(G_0)\, m - \beta \textstyle\sum_{s_1 \in \{s_L, s_H\}} \mu(s_1)\, U'(G_1(s_1)) + \eta_{01} \tag{A.16}$$

$$\begin{aligned} 0 \;=\; & U'(G_0) \left[ q_{02} + \frac{\partial q_{02}}{\partial b_{02}} b_{02} \right] + \\ & \sum_{s_1 \in \{s_L, s_H\}} \mu(s_1) \left[ \frac{\partial q_{12}(s_1)}{\partial b_{02}} b_{12}(s_1)\, \beta U'(G_1(s_1)) - \int_{\mathcal{Y}_2(b_{02}, b_{12}(s_1))} \beta^2 U'(G_2(s_1, s_2))\, \mu_2(s_2)\, ds_2 \right] \end{aligned} \tag{A.17}$$

$$\begin{aligned} 0 \;=\; & m \frac{\partial q_{12}(s_1)}{\partial b_2} b_{02} U'(G_0) + \\ & \left( q_{12}(s_1) + \frac{\partial q_{12}(s_1)}{\partial b_2} b_{12}(s_1) \right) \beta U'(G_1(s_1)) - \int_{\mathcal{Y}_2(b_{02}, b_{12}(s))} \beta^2 U'(G_2(s_1, s_2))\, \mu_2(s_2)\, ds_2 \end{aligned} \tag{A.18}$$

where $\eta_{01}$ is the multiplier on the non-negativity constraint for $b_{01}$ and we used the fact that $q_{02} = m \sum_{s_1} \mu(s_1)\, q_{12}(s_1)$. Combining (A.17) with (A.18) and using the fact that $q_{02} = m \sum_{s_1} \mu(s_1)\, q_{12}(s_1)$, we obtain

$$0 = U'(G_0)\, m - \beta \sum_{s_1 \in \{s_L, s_H\}} \mu(s_1) U'(G_1(s_1)) \frac{q_{12}(s_1)}{\sum_{s_1 \in \{s_L, s_H\}} \mu(s_1) q_{12}(s_1)}. \tag{A.19}$$

There can be two cases: either $G_1(s_1)$ is not constant across $s_1$ or $G_1(s_1)$ is constant across $s_1$. We consider the first case and later show that the second case cannot arise.

Suppose by way of contradiction that $G_1(s_1)$ is not constant and $b_{01} > 0$, and so the multiplier $\eta_{01}$ in (A.16) equals zero. Combining (A.16) with (A.19) and using $\eta_{01} = 0$, we obtain

$$\begin{aligned} 0 &= \sum_{s_1 \in \{s_L, s_H\}} \mu(s_1) U'(G_1(s_1)) \left[1 - \frac{q_{12}(s_1)}{\sum_{s_1 \in \{s_L, s_H\}} \mu(s_1) q_{12}(s_1)}\right] \\ &= \text{Cov}\left(U'(G_1(s_1)), \frac{q_{12}(s_1)}{\sum_{s_1 \in \{s_L, s_H\}} \mu(s_1) q_{12}(s_1)}\right). \end{aligned} \tag{A.20}$$

Since $G_1(s_1)$ is not constant by assumption, for the covariance above to be zero, we need the condition that

$$q_{12}(s_1) = m \Pr\left(U\left(\tau Y_2 - b_{02} - b_{12}(s_1)\right) \geq \underline{V}_2\right)$$

does not depend on $s_1$, which is equivalent to having the condition that $b_{12}(s_1)$ does not depend on $s_1$. Then, all the terms in (A.18) other than $G_1(s_{\prime})$ do not depend on $s_1$. Hence, for (A.18) to hold at $s_L$ and $s_H$, it must also be that $G_1(s_1)$ does not depend on $s_1$. This is a contradiction.

We now turn to show that we cannot attain perfect insurance in that $G_1(s_1)$ cannot be constant across $s_1$. Suppose by way of contradiction that $G_1(s_1) = G_1$ for all $s_1$. So from the government budget constraint in period 1, it must be that

$$0 < \tau\left[Y_1(s_H) - Y_1(s_L)\right] = q_{12}(s_L) b_{12}(s_L) - q_{12}(s_H) b_{12}(s_H).$$

Hence, since it is always optimal to choose debt levels on the increasing side of the "debt Laffer curve," it must be that $b_{12}(s_L) > b_{12}(s_H)$. Consider now the first-order conditions for debt issuance in period 1, (A.18), in state $s_H$ and $s_L$. They can be combined to obtain

$$\begin{aligned} &\beta U'(G_1)\left[\left(q_{12}(s_L) + \frac{\partial q_{12}(s_L)}{\partial b_2} b_{12}(s_L)\right) - \left(q_{12}(s_H) + \frac{\partial q_{12}(s_H)}{\partial b_2} b_{12}(s_H)\right)\right] \\ &= \left[\int_{\mathcal{Y}_2(b_{02}, b_{12}(s_L))} \beta^2 U'(G_2(s_L, s_2)) \mu_2(s_2) \, ds_2 - \int_{\mathcal{Y}_2(b_{02}, b_{12}(s_H))} \beta^2 U'(G_2(s_H, s_2)) \mu_2(s_2) \, ds_2\right], \end{aligned} \tag{A.21}$$

where we used (A.16) to substitute for $G_0$. Since $b_{12}(s_L) > b_{12}(s_H)$ and the Laffer curve is concave given our assumption about $\mu_2(s_2)$, the left side of (A.21) negative. But the left side of (A.21) is positive because $b_{12}(s_L) > b_{12}(s_H)$. Thus, we reached a contradiction. Therefore, we must be in the first case in which $G_1(s_H) \neq G_1(s_L)$ and $b_{01} = 0$ and all debt issued at $t = 0$ is long term. *Q.E.D.*

## C Data appendix

**Real gross domestic product (GDP)**. *OECD Quarterly National Accounts*, GDP expenditure approach, volume estimates (reference year 2010), 2000:Q1-2013:Q4.

**Debt-to-output ratio**. Debt is the face value of outstanding debt securities of the central government obtained from *OECD Quarterly Public Sector Debt*, expressed in millions of euros at current prices. We obtain this series for the period 2000:Q1-2013:Q4, seasonally adjust it, and scale it by GDP at current prices.[3]

**Debt maturity**. We use detailed information on outstanding bonds issued by the Italian central government to construct an indicator of debt maturity for the 2008:Q1-2013:Q4 period. Every quarter, the Italian Treasury publishes a list of all outstanding bonds issued by the central government.[4] We can classify these bonds into four main categories: i) *Buoni ordinari del Tesoro* (BOT); ii) *Certificati del Tesoro Zero Coupon* (CTZ); iii) *Buoni del Tesoro poliannuali* (BTP); iv) *Certificati di credito del Tesoro* (CCT).

The first two categories are zero coupon bonds with a maturity of up to two years. BTP are fixed coupon bonds, with a scheduled payment occurring every six months. CCT are variable coupon bonds, with a scheduled payment occurring every six months. The coupon per unit of principal is computed as a deterministic function of the prevailing yield on BOT. Specifically, letting $r_{\text{BOT}}$ to be the annualized yield on the last auction of a BOT. The coupon on the CCT is $r_{\text{BOT}} \times 0.5 + \text{spread}$, where the spread is specified in the contract (typically 15 basis points).

At a given quarter $t$, we use this information to construct a sequence of payments (principal and coupons) that the government has promised to make for any future date. We denote by $C_t^{(1)}$ the payments due within a year, $C_t^{(2)}$ those due between 1 and 2 years, etc. This calculation does not require an approximation for BOT, CTZ and BTP, because we have information on the principal due at maturity and the series of coupons that each instrument pays over its life. For CTZ, instead, we need to infer the prevailing yield on BOT at future dates in order to compute future coupon payments. We approximate those yields using the time $t$ yield on BOTs with a residual maturity of 1 year.

[3]To seasonally adjust the series, we estimate a linear regression

$$b_t = \gamma t + \sum_{j=1}^{4} \delta_{j,t} + e_t,$$

where $b_{j,t}$ is outstanding debt (in logs) at time $t$ quarter $j$, and $\delta_{j,t}$ are quarterly dummies. The seasonally adjusted series is then $\tilde{B}_t = \exp\{b_t\} - \exp\{\sum_{j=1}^{4} \delta_{j,t}\}$.

[4]The list can be downloaded at http://www.dt.tesoro.it/en/debito_pubblico/dati_statistici/scadenze_titoli_suddivise_anno/index.html.

After computing the sequence of payments we calculate the weighted average life of principal and coupon payments as

$$\sum_{n=1}^{N} n \frac{C_t^{(n)}}{C_t},$$

where $C_t = \sum_{n=1}^{N} C_t^{(n)}$. This indicator maps exactly to $1/\lambda_t$ in our model.

We can also use these data to construct the average maturity of new issuances. Specifically, we can define net issuances between period $t$ and $t+1$ for a given maturity $n$ as $\Delta_t^{(n)} = C_{t+1}^{(n)} - C_t^{(n+1)}$, and the average maturity of new issuances is then

$$\sum_{n=1}^{N} n \frac{\Delta_t^{(n)}}{\Delta_t},$$

where $\Delta_t = \sum_{n=1}^{N} \Delta_t^{(n)}$.

**Term structure of Italian interest rates**. Data on the term structure of Italian government bonds is obtained from *Datastream*. Datastream provides an estimate of the Italian yield curve by fitting a polynomial on the yields on several government securities that differ by residual maturity.[5] We use the parameters of this curve to generate nominal bond yields for all maturities between $n = 1$ and $n = 80$ quarters for the 2000:M1-2013:M12 period. We convert yields into bond prices, and construct $Q_t^{\text{ita}}(\lambda)$ using equation (**??**).

**Term structure of German interest rates**. Data on the term structure of ZCB for German federal government securities is obtained from the *Bundesbank online database*. We collect monthly data on the parameters of the Nelson and Siegel (1987) and Svensson (1994) model for the period 1973:M1-2013:M12, and we generate nominal bond yields for all maturities between $n = 1$ and $n = 80$ quarters. These data are used to estimate the stochastic discount factor, and to construct $Q_t^{\text{ger}}(\lambda)$ as explained in Section **??**.

[5] In the Italian case, Datastream uses BTP with a maturity of up to 30 years. The fitting curve is a polynomial of 3rd degree, estimated by OLS on daily data. The series mnemonic are GVIL03(CM05) for a bond with residual maturity of 5 years, GVIL03(CM10) for a bond with residual maturity of 10 years, etc.

# D The lenders' stochastic discount factor

The real stochastic discount factor $M_{t,t+1} = \exp\{m_{t,t+1}\}$ follows the process

$$\begin{aligned} m_{t,t+1} &= -(\phi_0 + \phi_1 \chi_t) - \frac{1}{2}\kappa_t^2 \sigma_\chi^2 + \kappa_t \varepsilon_{\chi,t+1}, \\ \chi_{t+1} &= \rho_\chi \chi_t + \varepsilon_{\chi,t+1} \qquad \varepsilon_{\chi,t+1} \sim \mathcal{N}(0,1), \\ \kappa_t &= \kappa_0 + \kappa_1 \chi_t. \end{aligned} \tag{A.22}$$

The parameters we need to estimate are $\theta_{\text{sdf}} = [\phi_0, \phi_1, \kappa_0, \kappa_1, \rho_\chi]$. We estimate $\theta_{\text{sdf}}$ by fitting (A.22) to the German nominal yield curve. Toward this end, we first enrich $M_{t,t+1}$ with a process for inflation and use the lenders' Euler equation to express nominal bond prices as a function of deep parameters and state variables. We can then estimate the joint process for inflation and for the stochastic discount factor using the method of simulated moments.

The process for inflation is a standard first-order autoregressive,

$$\Delta p_{t+1} = \mu_p(1-\rho_p) + \rho_p \Delta p_t + \varepsilon_{p,t+1} \qquad \varepsilon_{p,t+1} \sim \mathcal{N}(0, \sigma_p^2), \tag{A.23}$$

where $\Delta p_{t+1}$ is the first difference in log CPI. Further, we assume that $\Delta p_{t+1}$ and $m_{t,t+1}$ are potentially correlated, as the innovations $[\varepsilon_{\chi,t}, \varepsilon_{p,t}]$ are jointly normal, with covariance given by $\rho_{\chi,p}\sigma_p$.

We now proceed by characterizing the behavior of nominal bond prices under the assumed process for $(m_{t,t+1}, \Delta p_{t+1})$ and by explaining the estimation procedure.

## D.1 Bond prices, yields, and expected excess returns

Let $q_t^{€,(n)}$ be the price of a risk-free nominal ZCB maturing in $n$ quarters. These prices satisfy the recursion

$$q_t^{€,(n)} = \mathbb{E}_t \left[ M_{t,t+1} \exp\{-\Delta p_{t+1}\} q_{t+1}^{€,(n-1)} \right], \tag{A.24}$$

with initial condition $q_t^{€,(0)} = 1$.

We can use equations (A.22), (A.23), and (A.24) to express the prices of nominal zero coupon bonds as a function of the model parameters and the state variables $[\chi_t, \Delta p_t]$. Specifically, and following the steps in Ang and Piazzesi (2003), we can show that $\{q_t^{€,(n)}\}$ satisfies

$$\log\left(q_t^{€,(n)}\right) = A_n + B_n \chi_t + C_n \Delta p_t, \tag{A.25}$$

where $A_n, B_n$, and $C_n$ can be recursively computed:

$$\begin{aligned}
A_{n+1} &= -\phi_0 + A_n + B_n\left[\mu_\chi(1-\rho_\chi) + \left(\kappa_0 + \frac{B_n}{2}\right)\sigma_\chi^2\right] \\
&\quad + [C_n - 1]\left[(1-\rho_p)\mu_p + [C_n - 1]\frac{\sigma_p^2}{2} + (\kappa_0 + B_n)\rho_{\chi,p}\sigma_p\right], \\
B_{n+1} &= -\phi_1 + B_n(\rho_\chi + \kappa_1\sigma_\chi^2) + \kappa_1[C_n - 1]\rho_{\chi,p}\sigma_p, \qquad \text{(A.26)} \\
C_{n+1} &= (C_n - 1)\rho_p,
\end{aligned}$$

with initial conditions $A_0 = B_0 = C_0 = 0$.

Given equation (A.25) and the coefficients in (A.26), we can compute key moments of the yield curve.

For example, log-yields on a nominal bond maturing in one quarter are $r_t^{€,(1)} = -\log\left(q_t^{€,(1)}\right)$. Given our representation for prices, we have

$$r_t^{€,(1)} = \left\{\phi_0 + \left[\mu_p(1-\rho_p) - \frac{\sigma_p^2}{2} + \kappa_0\rho_{\chi,p}\sigma_p\right]\right\} + \left[\phi_1 + \kappa_1\rho_{\chi,p}\sigma_p\right]\chi_t + \rho_p\Delta p_t. \qquad \text{(A.27)}$$

We can then express the mean and the variance of $r_t^{€,(1)}$ as a function of the model parameters:

$$\mathbb{E}[r_t^{€,(1)}] = (\phi_0 + \mu_p) - \frac{\sigma_p}{2} + \kappa_0\rho_{\chi,p}\sigma_p, \qquad \text{(A.28)}$$

$$\text{var}[r_t^{€,(1)}] = \frac{(\phi_1 + \kappa_1\rho_{\chi,p}\sigma_p)^2}{1-\rho_\chi^2} + \frac{\rho_p^2\sigma_p^2}{1-\rho_p^2} + \frac{(\phi_1 + \kappa_1\rho_{\chi,p}\sigma_p)\rho_p\rho_{\chi,p}\sigma_p}{1-\rho_\chi\rho_p}, \qquad \text{(A.29)}$$

We can also express holding period excess log-returns on a ZCB maturing in $n$ periods. By definition, this is equal to

$$rx_{t+1}^{€,(n)} = \log\left(\frac{q_{t+1}^{€,(n-1)}}{q_t^{€,(n)}}\right) + \log\left(q_t^{€,(1)}\right).$$

This object compares the realized returns from purchasing a bond maturing in $n$ periods at time $t$ and selling it at $t+1$ relative to the return one obtains from purchasing a bond with

Table A-1: Summary statistics: inflation, yields and holding period returns

| | Mean | Standard deviation | Sharpe Ratio |
|---|---|---|---|
| $\Delta p_t$ | 2.59 | 2.35 | |
| $r_t^{€,(1)} - \Delta p_t$ | 2.15 | 2.38 | |
| $r_t^{€,(20)} - \Delta p_t$ | 2.92 | 2.26 | |
| $rx_{t+1}^{€,(4)}$ | 0.22 | 1.92 | 0.12 |
| $rx_{t+1}^{€,(8)}$ | 0.92 | 3.96 | 0.23 |
| $rx_{t+1}^{€,(12)}$ | 1.49 | 5.71 | 0.26 |
| $rx_{t+1}^{€,(16)}$ | 1.95 | 7.23 | 0.27 |
| $rx_{t+1}^{€,(20)}$ | 2.31 | 8.57 | 0.27 |

*Notes: The sample period is 1973:Q1-2013:Q4. The inflation rate is first difference in log CPI. Variables are reported as annualized percentages.*

a one-period maturity at time $t$. Substituting for log prices, we can rewrite this object as

$$\begin{aligned} rx_{t+1}^{€,(n)} &= \underbrace{[A_{n-1} - A_n + A_1 + C_{n-1}\mu_p(1-\rho_p)]}_{\tilde{A}_n} + \underbrace{(B_{n-1}\rho_\chi - B_n + B_1)}_{\tilde{B}_n}\chi_t + \\ &+ \underbrace{(C_{n-1}\rho_p - C_n + C_1)}_{\tilde{C}_n}\Delta p_t + B_{n-1}\varepsilon_{\chi,t+1} + C_{n-1}\varepsilon_{p,t+1}. \end{aligned} \tag{A.30}$$

Taking conditional expectations on both sides, we obtain

$$\mathbb{E}_t[rx_{t+1}^{€,(n)}] = \tilde{A}_n + \tilde{B}_n\chi_t, \tag{A.31}$$

as $\tilde{C}_n = 0$ given our assumptions on the process for $\Delta p_{t+1}$.

This equation shows that expected excess returns on long-term bonds are linear functions of $\chi_t$. Our empirical strategy consists of estimating the parameters of the stochastic discount factor so that our model is consistent with the behavior of an empirical counterpart for $\mathbb{E}_t[rx_{t+1}^{€,(20)}]$ and with the behavior of yields on short-term bonds. We now explain the details of our empirical strategy.

## D.2 Estimation of $\theta_{\mathbf{sdf}}$

We estimate the parameters $\theta_{\text{sdf}}$ and the parameters of the inflation process, $[\mu_p, \rho_p, \sigma_p, \rho_{\chi,p}]$ using data on the term structure of nominal ZCB and inflation for Germany. The sample period is 1973:Q1-2013:Q4. Table A-1 reports summary statistics on inflation, nominal yields, and realized excess log returns as a function of $n$.

The yield curve slopes up on average: yields on five-year bonds are, on average, 80 basis points higher than yields on bonds maturing next quarter. We can also see that long-term bonds earn a positive excess return on average. For example, holding a five-year bond and selling it off next quarter earns, on average, an annualized premium of 2.40% relative to investing in a bond that matures next quarter. Excess returns on long-term bonds increase monotonically with $n$, and so does their Sharpe ratio.

We first estimate an AR(1) process for inflation and set $[\mu_p, \rho_p, \sigma_p]$ to the estimated values.[6] The remaining parameters, $[\theta_{\text{sdf}}, \rho_{\chi,p}]$, are jointly estimated to fit the following:

- The mean and variance of yields on bonds with a one-quarter maturity, $r_t^{€,(1)} - \Delta p_t$;
- The correlation between $r_t^{€,(1)} - \Delta p_t$ and $\Delta p_t$;
- The statistical properties of an estimate of $\mathbb{E}_t[rx_{t+1}^{€,(20)}]$.

We estimate $\mathbb{E}_t[rx_{t+1}^{€,(20)}]$ using the procedure developed in Cochrane and Piazzesi (2005) and routinely used in the literature. This procedure consists first in estimating an OLS regression of $\overline{rx}_{t+1}^{€} = \sum_n rx_{t+1}^{€,(n)}$ on a vector of log-forward rates,

$$\overline{rx}_{t+1}^{€} = \gamma_0 + \gamma' \mathbf{f}_t + \overline{\eta}_t, \tag{A.32}$$

and then estimating by OLS the following regression:

$$rx_{t+1}^{€,(n)} = \alpha_n + b_n(\hat{\gamma}_0 + \hat{\gamma}' \mathbf{f}_t) + \eta_t^n. \tag{A.33}$$

The proxies for $\mathbb{E}[rx_{t+1}^{€,(n)}]$ are then the fitted values of equation (A.33).

In implementing this procedure, we average excess log returns across $n = 4, 8, 12, 16, 20$, and the vector $\mathbf{f}_t$ includes the risk-free rate and the log forward rates for these five maturities. The top panel of Table A-2 reports the results for the estimation of equation (A.32), while the bottom panel reports the estimates of equation (A.33) for $n = 4, 8, 12, 16, 20$.

Relative to the analysis of Cochrane and Piazzesi (2005) on U.S. data, excess returns are less predictable, as measured from the $R^2$ of the above regressions: in U.S. data, the $R^2$ varies between 0.31 and 0.37 while in our case it varies between 0.10 and 0.19. This result mirrors Dahlquist and Hasseltoft (2013), who also estimated the Cochrane and Piazzesi (2005) regressions using German bonds. In that paper, the authors argue that the performance of the Cochrane and Piazzesi (2005) factor in forecasting excess returns is significantly higher than that of other factors used in the literature. Similar to Cochrane and Piazzesi (2005), we find that the sensitivity of excess returns to their factor (the estimated $b_n$'s) increases with the maturity of the bonds.

[6] When fitting equation (A.23) to German data we find that $\hat{\mu}_p = 0.006$, $\hat{\rho}_p = 0.470$ and $\hat{\sigma}_p = 0.005$.

Table A-2: Cochrane and Piazzesi (2005) regressions

| | $\gamma_0$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ | $\gamma_4$ | $\gamma_5$ | $\gamma_6$ | $R^2$ |
|---|---|---|---|---|---|---|---|---|
| Estimates of equation (A.32) | -0.001 | -1.65 | 5.02 | -21.95 | 48.01 | -46.17 | 16.95 | 0.12 |
| | (-0.29) | (-3.18) | (2.94) | (-2.02) | (1.57) | (-1.22) | (1.01) | |

| | | $a_n$ | $b_n$ | $R^2$ |
|---|---|---|---|---|
| Estimates of equation (A.33) | 4 | -0.001 | 0.45 | 0.19 |
| | | (-2.40) | (6.26) | |
| | 8 | -0.000 | 0.77 | 0.13 |
| | | (-0.38) | (4.93) | |
| | 12 | 0.000 | 1.03 | 0.11 |
| | | (0.15) | (4.54) | |
| | 16 | 0.000 | 1.27 | 0.11 |
| | | (0.30) | (4.42) | |
| | 20 | 0.001 | 1.49 | 0.10 |
| | | (0.34) | (4.36) | |

We summarize this information by including $[\hat{a}_{20}, \hat{b}_{20}, \hat{\sigma}_{\eta^{20}}]$ in the targets of the method of simulated moments, along with the parameters of an AR(1) model estimated on the first-stage factor, $x_t = \hat{\gamma}_0 + \hat{\gamma}'\mathbf{f}_t$.

The parameters $[\theta_{\text{sdf}}, \rho_{\chi,p}]$ are then estimated by simulated method of moments. That is, we choose $[\theta_{\text{sdf}}, \rho_{\chi,p}]$ to minimize a weighted distance between these moments and the corresponding statistics computed in model-simulated data. The weighting matrix is diagonal, with the inverse of each sample moment (in absolute value) on the main diagonal. The model-implied statistics are computed on a long simulation ($T = 20000$).[7] Table A-3 reports the estimated parameters along with measures of the in-sample fit.

# E Numerical solution

It is convenient to simplify the objects of the recursive equilibrium. First, let us drop $\xi$ from the state vector and record the face value of debt, $\hat{B} = B/\lambda$, instead of the debt coming due next period, $B$, and have $\mathbf{S} = [\hat{B}, \lambda, y, \chi, \pi]$ and $\mathbf{s} = [y, \chi, \pi]$. As we shall see momentarily, $\mathbf{S}$

[7]In simulations, we add small measurement errors to the forward rates in order to avoid multicollinearity when estimating the Cochrane and Piazzesi (2005) first-stage regression.

Table A-3: Estimated parameters and model fit

| Parameters | $\phi_0$ | $\phi_1$ | $\kappa_0$ | $\kappa_1$ | $\rho_\chi$ | $\rho_{\chi,p}$ |
|---|---|---|---|---|---|---|
| | 0.005 | 0.002 | 0.161 | 0.374 | 0.513 | 0.676 |

| Moment | Data | Model |
|---|---|---|
| Mean($r_t^{€,(1)} - \Delta p_t$) | 2.148 | 2.153 |
| Stdev($r_t^{€,(1)} - \Delta p_t$) | 2.381 | 1.059 |
| Corr($r_t^{€,(1)} - \Delta p_t$,$\Delta p_t$) | -0.2729 | -0.2783 |
| $\hat{a}_x$ | 0.002 | 0.002 |
| $\hat{\rho}_x$ | 0.490 | 0.521 |
| $\hat{\sigma}_x$ | 0.004 | 0.010 |
| $\hat{a}_{20}$ | 0.001 | 0.000 |
| $\hat{b}_{20}$ | 1.485 | 1.339 |
| $\hat{\sigma}_{\eta^{20}}$ | 0.020 | 0.032 |

*Notes: The moments $(\hat{a}_x, \hat{\rho}_x, \hat{\sigma}_x)$ are the estimates of an AR(1) process for the fitted values of equation (A.32). Yields are reported as annualized percentages.*

will be enough to solve for the recursive equilibrium. Second, we can represent the decision problem of the government as choosing the face value $\hat{B}$ and the decay parameter $\lambda$ rather than choosing the entire portfolio of ZCB. We denote the pricing schedule for a portfolio with decay parameter $\lambda$, given that the realization of the exogenous state is $\mathbf{s}$ and given the choices $(\hat{B}', \lambda')$ for the government to be

$$Q\left(\mathbf{S}, \hat{B}', \lambda' | \lambda\right) = \sum_{n=1}^{\infty} \lambda(1-\lambda)^{n-1} q^{(n)}\left(\mathbf{S}, \hat{B}', \lambda'\right).$$

Note that we need to price an arbitrary $\lambda$ portfolio, given government choices $(\hat{B}', \lambda')$, in order to know the market value of the portfolio repurchased by the government.[8] We can verify that the pricing schedule solves the recursion

$$Q\left(\mathbf{s}, \hat{B}', \lambda' | \lambda\right) = \mathbb{E}\left\{M\left(s_1, s_1'\right)\delta\left(\mathbf{S}', \xi'\right)\left[\lambda + (1-\lambda) Q\left(\mathbf{s}', \hat{B}'', \lambda'' | \lambda\right)\right] | \mathbf{S}\right\}, \tag{A.34}$$

with $\hat{B}'' = \hat{B}'\left(s', \hat{B}', \lambda'\right)$ and $\lambda'' = \lambda'\left(s', \hat{B}', \lambda'\right)$.

Under this formulation, and assuming that the government is not defaulting today given $\mathbf{S}$ and $\xi$, we can write the net issuances of bonds as

$$\Delta\left(\mathbf{S}, \hat{B}', \lambda'\right) = Q\left(\mathbf{s}, \hat{B}', \lambda' | \lambda'\right)\hat{B}' - Q\left(\mathbf{s}, \hat{B}', \lambda' | \lambda\right)(1-\lambda)\hat{B}. \tag{A.35}$$

[8] See Sánchez, Sapriza, and Yurdagul (2018) for a discussion of this issue.

With this notation, we can write the decision problem for the government using three simple sub-problems. We define the value of repaying conditional on lenders rolling over the debt, $V^R_{\text{roll}}(\mathbf{S})$, as follows:

$$V^R_{\text{roll}}(\mathbf{S}) = \max_{\hat{B}',\lambda'} \left\{ U(\tau Y - \lambda \hat{B} + \Delta\left(\mathbf{S}, \hat{B}', \lambda'\right)) + \beta \mathbb{E}[V(\hat{B}', \lambda', \mathbf{s}')|\mathbf{S}] \right\}, \tag{A.36}$$

where $\Delta\left(\mathbf{S}, \hat{B}', \lambda'\right)$ is defined in equation (A.35) and $Y = \exp\{y\}$. The value of repaying conditional on lenders not rolling over the debt, $V^R_{\text{no roll}}(\mathbf{S})$, is

$$V^R_{\text{no roll}}(\mathbf{S}) = \left\{ U(\tau Y - \lambda \hat{B}) + \beta \mathbb{E}[V(\hat{B}(1-\lambda), \lambda, \mathbf{s}')|\mathbf{S}] \right\}, \tag{A.37}$$

while the value of defaulting, $V^D(y, \chi)$, is

$$V^D(y,\chi) = \left\{ U(\tau Y[1 - d(Y)]) + \beta\{\psi \mathbb{E}[V(0, \overline{\lambda}, y', \chi', \pi')|\mathbf{S}] + (1-\psi)\mathbb{E}[V^D(y', \chi')|\mathbf{S}]\} \right\}. \tag{A.38}$$

Note that $V^D(.)$ does not depend on $\pi$ because this process is iid. The value function of the government can then be written as

$$V(\mathbf{S}, \xi) = \begin{cases} V^R_{\text{roll}}(\mathbf{S}) & \text{if } V^R_{\text{no roll}}(\mathbf{S}) \geq V^D(y,\chi) \\ V^R_{\text{roll}}(\mathbf{S}) & \text{if } V^R_{\text{no roll}}(\mathbf{S}) < V^D(y,\chi) \text{ and } \xi = 0 \\ V^D(y,\chi) & \text{if } V^R_{\text{no roll}}(\mathbf{S}) < V^D(y,\chi) \text{ and } \xi = 1. \end{cases}$$

This value function, its associated policy functions, and the pricing function in equation (A.34) are enough to determine the equilibrium outcome path.[9]

The numerical solution of the model consists of approximating the pricing schedule $Q$ and the value functions $\{V^R_{\text{roll}}(\mathbf{S}), V^R_{\text{no roll}}(\mathbf{S}), V^D(y,\chi)\}$. We approximate the value functions using a mixture of projection and discrete state space methods. We let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_\lambda}\}$ and $\mathcal{B} = \{B_1, \ldots, B_{N_B}\}$ be the set of decaying factors and debt levels over which we approximate the value function. The value functions are approximated using piece-wise smooth functions. That is, $V^R_{\text{roll}}(.)$ is approximated as follows:

$$V^R_{\text{roll}}(\lambda_j, B_k, \mathbf{s}) = {\gamma^R_{\text{roll},(\lambda_j,B_k)}}' \mathbf{T}(\mathbf{s}),$$

where $\mathbf{s} = [y, \chi, \pi] \in \mathcal{S}$ is a realization of the exogenous state variables from a set of points $\mathcal{S}$, $\gamma^R_{\text{roll},(\lambda_j,B_k)}$ is a vector of coefficients, and $\mathbf{T}(.)$ is a vector collecting Chebyshev's polynomials. The value of repaying conditional on the lenders not rolling over the debt and the value

[9] On the outcome path, the price of a bond portfolio is either zero – in the case of a fundamental default or a rollover crisis – or equal to the price defined in (A.34) if there is repayment in the current period.

of defaulting are defined in a similar fashion, and we denote by $\gamma^R_{\text{no roll},(\lambda_j,B_k)}$ and $\gamma^D$ the coefficients that parametrize those values. The pricing schedule $Q$ in equation (A.34) is approximated on a grid. Specifically, let $\tilde{\mathcal{B}} = \{\tilde{B}_1, \ldots, \tilde{B}_{N_{\tilde{\mathcal{B}}}}\}$ be the set of debt levels that the government can choose. The pricing schedule is then approximated on $\tilde{\mathcal{B}} \times \Lambda \times \Lambda \times \mathcal{S}^q$. We allow the grid for the exogenous state variables and the one for debt choices to be different from the one used in the approximation of the value function. This degree of flexibility allows us to have finer grids for the pricing schedule than for value functions, which is helpful because the former are typically highly nonlinear in sovereign debt models.

We index the numerical solution by $[\Gamma, Q]$, with $\Gamma = \{[\gamma^R_{\text{roll},(\lambda_j,B_k)}, \gamma^R_{\text{no roll},(\lambda_j,B_k)}]_{j,k}, \gamma^D\}$ collecting the coefficients that parametrize the value functions. The numerical solution is obtained via value function iteration. The algorithm is as follows:

**Step 0: Defining the state space and the polynomials.** Specify the set of values in $\mathcal{B}, \Lambda, \tilde{\mathcal{B}}$. Set upper and lower bounds for the exogenous state variables $(y, \chi, \pi)$ and construct individual grids for each exogenous state. Construct a tensor grid $\mathcal{S}$ for the exogenous state variables and the associated Chebyshev's polynomials $\mathbf{T}(.)$. These are used for the approximation of the value functions. Construct a tensor grid $\mathcal{S}^q$ for the approximation of the pricing schedule.

**Step 1: Update value functions.** Start with a guess for the value and pricing functions, $(\Gamma^c, Q^c)$. For each point in $\mathcal{B} \times \Lambda \times \mathcal{S}$, update the value functions using the definitions in equations (A.36)-(A.38). Denote by $\Gamma^u$ the updated guess and by $[r^R_{\text{roll}}, r^R_{\text{no roll}}, r^D]$ the distance between the initial guess and its update.

**Step 2: Update pricing function.** For each exogenous state $\mathbf{s}$ in $\mathcal{S}^q$ and for each $(B', \lambda', \lambda) \in \tilde{\mathcal{B}} \times \Lambda \times \Lambda$, evaluate the right-hand side of equation (A.34) using $(\Gamma^c, Q^c)$ and the policy functions associated with $\Gamma^c$. Denote this value by $\hat{Q}^u(s, B', \lambda'|\lambda)$, and let $r^Q$ be the distance between $Q^c$ and $\hat{Q}^u$. Update the pricing schedule as

$$Q^u(.) = a\hat{Q}^u(.) + (1-a)Q^c(.) \qquad a \in (0,1).$$

**Step 3: Iteration.** If $\max\{r^R_{\text{roll}}, r^R_{\text{no roll}}, r^D\} \leq c_1$ and $r^Q \leq c_2$, stop the algorithm. If not, set $(\Gamma^u, q^u)$ as the new guess and repeat Steps 1-2. $\square$

Regarding the specifics of the algorithm, we set the upper and lower bounds for $y$ and $\chi$ to be equal to +/- 3 standard deviations of the stochastic processes, while the grid for $\pi$ is between 0 and 3.5 standard deviations. We select 5 equally spaced points between these bounds for the approximation of the value function. The set $\mathcal{S}$, therefore, contains 125 distinct points. For the approximation of the pricing function, instead, we consider 51 points on the $y$ dimension, and 5 points on the $\chi$ and $\pi$ dimension. The set $\mathcal{S}^q$ contains, therefore, 1275 distinct points. The upper and lower bounds for $B$ are $[0, 16]$. When approximating the value function, we construct $\mathcal{B}$ using 81 equally spaced points in this interval. The grid for $\lambda$ contains 11 equally spaced values within the interval $[1/(4 \times 8), 1/(4 \times 5)]$. This interval implies a range of $+/-$ 8 standard deviations around an observed maturity of 6.5 years, the Italian pre-crisis level. The grid for debt choices over which the pricing schedule is defined, $\tilde{\mathcal{B}}$, consists of 650 points in the $[0, 16]$ interval. The grid has 50 equally spaced points on the $[0, 6)$ segment, 500 points on the $[6, 12)$ segment, and 100 points on the $[12, 16]$ segment.

When updating the guess for the value and pricing functions, we compute expectations over future outcomes using Gauss-Hermite quadrature, with $n = 5$ points on each random variable. We use the polynomial approximation to compute the value functions associated to states that are not on our grid, while we use linear interpolation for bond prices and policy functions. Finally, we compute the distance for the value functions using the sup norm in logs. For the pricing function, we compute the sup norm for equilibrium prices and square it. The value functions converge at 0.00006 level while equilibrium prices at 0.000001.

A simulation of the model consists in obtaining the default decision, $\delta_t$, the characteristics of the new debt portfolio, $(\hat{B}_{t+1}, \lambda_{t+1})$, and the equilibrium price of a portfolio of type $\lambda$, $Q_t(\lambda)$ for a given realization $\{y_t, \chi_t, \pi_t, \xi_t\}_{t=1}^T$ and initial conditions $(\hat{B}_0, \lambda_0)$.[10] Let $\{y_t, \chi_t, \pi_t, \xi_t\}$ be the time $t$ realization of the stochastic process and let $(\hat{B}_t, \lambda_t)$ be the initial condition for the face value and decaying parameter of the debt portfolio inherited by the government. Assuming that the government is not currently in default, we can obtain the value for $\{\delta_t, \hat{B}_{t+1}, \lambda_{t+1}, Q_t(\lambda)\}$ as follows

1. Given the state variables $\mathbf{S}_t = [\hat{B}_t, \lambda_t, y_t, \chi_t, \pi_t]$, we compute $V^R_{\text{roll}}(\mathbf{S}_t)$, $V^R_{\text{no roll}}(\mathbf{S}_t)$ and $V^D(y_t, \chi_t)$ using $\Gamma$. We use the Chebyshev's polynomials and linear interpolation across the $B_t$ and $\lambda_t$ dimensions to evaluate these value functions at a point $\mathbf{S}_t$ that is not in our grid.
2. If $V^{\text{roll}}(\mathbf{S}_t) < V^D(y_t, \chi_t)$, or if $V^{\text{no roll}}(\mathbf{S}_t) < V^D(y_t, \chi_t) \leq V^{\text{roll}}(\mathbf{S}_t)$ and $\xi_t = 1$, then the government enters a default state. In such a case, we set $\delta_t = 0$, $\hat{B}_{t+1} = 0$, $\lambda_{t+1} = \bar{\lambda}$, and $Q_t(\lambda) = \text{NaN}$.
3. If the government is not in a default state, we set $\delta_t = 1$. The choices for $(\hat{B}_{t+1}, \lambda_{t+1})$ and equilibrium prices $Q_t(\lambda)$ are obtained by linearly interpolating the policy functions at $\mathbf{S}_t$ and the pricing schedule at $(\hat{B}_{t+1}, \lambda_{t+1}, y_t, \chi_t, \pi_t)$.

[10] To obtain a simulation for $\xi_t$, we draw a random variable $u_t \sim \text{Uniform}(0, 1)$. We set $\xi_t = 0$ if $u_t \geq \pi_{t-1}$, and $\xi_t = 1$ otherwise.

If the government is currently in default, we draw a random variable $v_t \sim \text{Uniform}(0,1)$. If $v_t > \psi$, then the government remains in default and we set $\delta_t = 0$, $\hat{B}_{t+1} = 0$ and $\lambda_{t+1} = \bar{\lambda}$ and $Q_t(\lambda) = \text{NaN}$. Else, the government exits a default state, and the simulation is obtained by following step 1-3 above.

# F Details of the counterfactual experiment

We now detail the counterfactual experiment of Section 5. First, we explain how we use the particle filter to extract information on the sequence of $\{\pi_t\}$. Second, we discuss how we generate the decomposition of Figure 5.

## F.1 Particle filtering

From Section 5 we have that the state-space representation of the model is

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{g}(\mathbf{S}_t) + \eta_t \\ \mathbf{S}_t &= \mathbf{f}(\mathbf{S}_{t-1}, \varepsilon_t). \end{aligned}$$

The first equation is the measurement equation, with $\eta_t$ being a vector of iid Gaussian errors with a variance-covariance matrix equal to $\Sigma$. The second equation is the transition equation, describing the law of motion for the model's state variables. The vector $\varepsilon_t$ collects the innovations to the structural shocks $y_t$, $\chi_t$, and $\pi_t$. The functions $g(.)$ and $f(.)$ are generated using the numerical procedure previously described.

Let $\mathbf{Y}^t = [\mathbf{Y}_1, \ldots, \mathbf{Y}_t]$, and denote by $p(\mathbf{S}_t|\mathbf{Y}^t)$ the conditional distribution of the state vector given observations up to period $t$. Although the conditional density of $\mathbf{Y}_t$ given $\mathbf{S}_t$ is known and Gaussian, there is no analytical expression for the density $p(\mathbf{S}_t|\mathbf{Y}^t)$. We use the particle filter to approximate this density for each $t$. The approximation is done via a set of pairs $\{\mathbf{S}_t^i, \tilde{w}_t^i\}_{i=1}^N$, in the sense that

$$\frac{1}{N}\sum_{i=1}^{N} f(\mathbf{S}_t^i)\tilde{w}_t^i \overset{\text{a.s.}}{\to} \mathbb{E}[f(\mathbf{S}_t)|\mathbf{Y}^t],$$

and it is used to obtain the (mean) trajectory of the state vector over the sample. We refer to $\mathbf{S}_t^i$ as a particle and to $\tilde{w}_t^i$ as its weight. The algorithm used to approximate $\{p(\mathbf{S}_t|\mathbf{Y}^t)\}_t$ builds on Kitagawa (1996), and it goes as follows:

**Step 0: Initialization.** Set $t = 1$. Initialize $\{\mathbf{S}_0^i, \tilde{w}_0^i\}_{i=1}^N$ and set $\tilde{w}_0^i = 1\ \forall i$.

**Step 1: Prediction.** For each $i = 1, \ldots, N$, obtain a realization for the state vector $\mathbf{S}^i_{t|t-1}$ given $\mathbf{S}^i_{t-1}$ using the simulation procedure described in online Appendix E.

**Step 2: Filtering.** Assign to each particle $\mathbf{S}^i_{t|t-1}$ the weight

$$w^i_t = p(\mathbf{Y}_t|\mathbf{S}^i_{t|t-1})\tilde{w}^i_{t-1}.$$

**Step 3: Resampling.** Rescale the weights $\{w^i_t\}$ so that they add up to unity, and denote these rescaled values by $\{\tilde{w}^i_t\}$. Sample $N$ values for the state vector with replacement from $\{\mathbf{S}^i_{t|t-1}, \tilde{w}^i_t\}^N_{i=1}$, and denote these draws by $\{\mathbf{S}^i_t\}_i$. Set $\tilde{w}^i_t = 1\ \forall i$. If $t < T$, set $t = t + 1$ and go to Step 1. If not, stop. $\square$

In our exercise, the measurement equation includes $y_t$ and $\chi_t$, and the variance of the measurement error associated to these variables is set to zero. Thus, at every prediction step of the filter, we set the innovations $\varepsilon^i_{y,t}$ and $\varepsilon^i_{\chi,t}$ so that $y^i_t$ and $\chi^i_t$ are equivalent to their empirical counterpart. The non-trivial part of the algorithm consists in filtering $\pi_t$. Regarding the tuning of the filter, we set $N = 100,000$.

## F.2 Counterfactual experiment

We now discuss how we use the approximation to $\{p(\mathbf{S}_t|\mathbf{Y}^t)\}^{2012:Q2}_{t=2008:Q1}$ along with the structural model to generate the decomposition presented in Figure 5.

Let $\text{spr}^{\text{data}}_t$ be the interest rate spread at time $t$, and let $\hat{\text{spr}}^{\text{model}}_t$ be

$$\hat{\text{spr}}^{\text{model}}_t = \sum_{i=1}^{N} g_{\text{spr}}(\mathbf{S}^i_t)\tilde{w}^i_t,$$

where $g_{\text{spr}}(.)$ is the policy function for equilibrium interest rate spreads. The measurement error component in Figure 5 is defined as $\text{spr}^{\text{data}}_t - \hat{\text{spr}}^{\text{model}}_t$.

The fundamental component of interest rate spreads is generated as follows. For each $S^i_t$, and given the choices $B^i_{t+1}$, $\lambda^i_{t+1}$, evaluate the interest rate spread on a $\lambda$-type portfolio by setting $\pi_t = 0$,

$$\text{spr}^{i,\text{fund}}_t(\lambda) = \frac{\lambda[1 - Q(B^i_{t+1}, \lambda^i_{t+1}, y^i_t, \chi^i_t, 0|\lambda)]}{Q(B^i_{t+1}, \lambda^i_{t+1}, y^i_t, \chi^i_t, 0|\lambda)} - r^{\text{risk free}}_t(\lambda).$$

Given $\{\text{spr}^{i,\text{fund}}_t\}_{i\in N, t\in T}$, we next construct, for each $t$,

$$\hat{\text{spr}}^{\text{fund}}_t = \frac{1}{N}\sum_{i=1}^{N} \text{spr}^{i,\text{fund}}_t \tilde{w}^i_t \approx \mathbb{E}[\text{spr}^{\text{fund}}_t|\mathbf{Y}^t].$$

This is the (average) interest rate spread implied by the model under the assumption that $\pi_t$ was zero over the 2008:Q1-2012:Q2 period, and it is the fundamental component of interest rate spreads in Figure 5. The non-fundamental component of the spreads is then defined as $\hat{\text{spr}}_t^{\text{model}} - \hat{\text{spr}}_t^{\text{fund}}$ for each $t$.

# G Sensitivity

In this section we perform a sensitivity analysis with the aim of assessing the robustness of our results.

**Welfare gains from lengthening debt maturity.** In the first experiment, we use the structural model to measure the welfare gains from lengthening debt maturity under the hypothesis that rollover risk was a key determinant of interest rate spreads during the Italian debt crisis. This exercise checks the robustness of our results with respect to model's misspecification. Small welfare gains would indicate that the results in the previous section could be easily overturned by determinants of debt maturity that have been neglected in our analysis. Large welfare gains would imply that our results are more robust to model's misspecification.

To this end, we set the state variables $(\lambda_t, y_t, \chi_t)$ to their empirical counterpart in 2011:Q4, and we fix $\lambda_{t+1}$ to the observed debt maturity at the end of the quarter. We then choose a combination of $(B_t, B_{t+1}, \pi_t)$ that delivers a government deficit of 3.5% of GDP, the level of interest rate spreads observed in 2011:Q4, and the highest probability of a rollover crisis next period. By doing so, the model replicates the key features of the Italian economy in 2011:Q4 *and* generates a sizable role for rollover risk. Given these states and choices, we can compute the certainty equivalent government spending $G_t^*$,

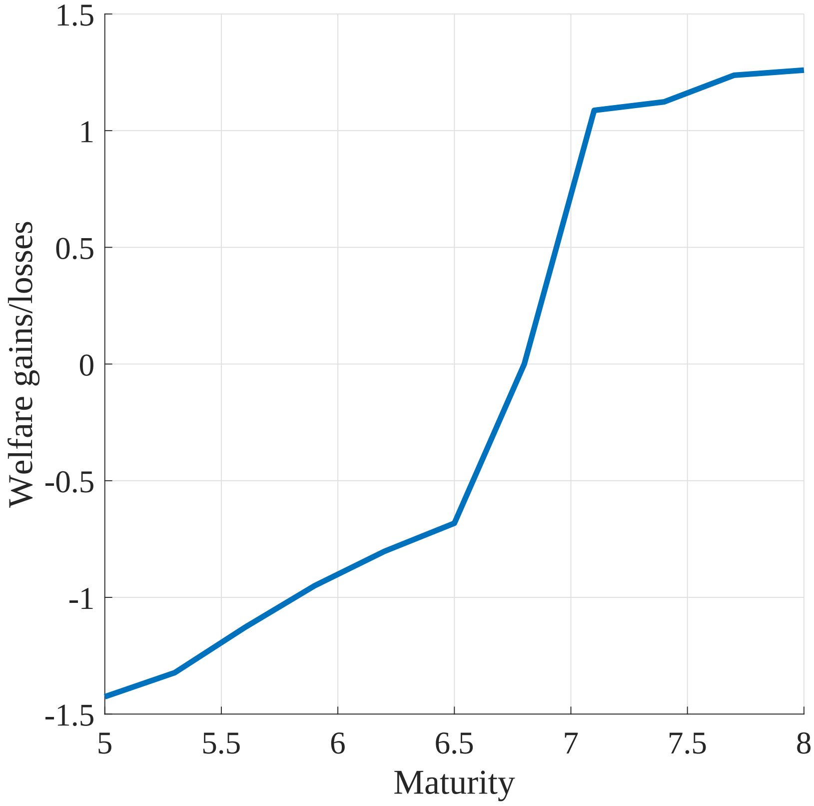
$$\frac{1}{1-\beta}\frac{(G_t^{\star} - \underline{G})^{1-\sigma}}{1-\sigma} = V_t,$$

where $V_t$ is the value for the government given the state variables selected above and the choices $(B_{t+1}, \lambda_{t+1})$. Because $(B_{t+1}, \lambda_{t+1})$ are not chosen optimally in this experiment, there are welfare gains/losses from changing them. Our focus is to measure the welfare gains that the government would obtain by changing $\lambda_{t+1}$.

Figure A-1 reports the certainty equivalent government spending for different values of $\lambda_{t+1}$ in percentage deviations from the benchmark $G_t^*$.[11] By construction, this measure

[11] When changing $\lambda_{t+1}$ we adjust $B_{t+1}$ so that government spending is the same as in the benchmark.

Figure A-1: Gains from lenghtening debt maturity in presence of high rollover risk

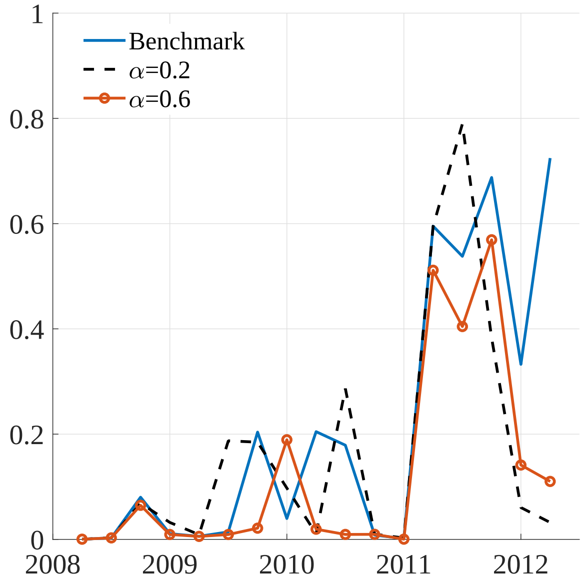



equals zero for the maturity level observed in 2011:Q4. We can see that welfare is increasing in debt maturity. By lengthening the maturity of the stock of debt from 6.7 years to 8, for example, the government would increase its certainty equivalent consumption by 1.25%. This is sizable number, especially when compared to the welfare costs of business cycle fluctuations documented in the literature.

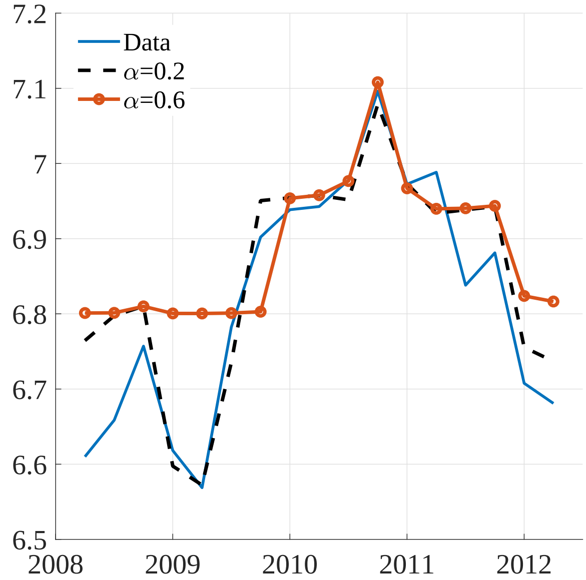
This analysis shows that if rollover risk was sizable at the peak of the crisis, then the government would have strong incentives to lengthen maturity relative to what we have observed in the data. The reason why the model produces such large welfare gains can be explained as follows. First, in the above experiment $\pi$ equals 0.22. Thus, a government falling in the crisis zone faces a substantial risk of a rollover crisis. Second, the model needs large output costs of default (roughly 8% of GDP in our parametrization) in order to reproduce the large debt-to-output ratio of the Italian economy. This makes rollover crises particularly costly from the perspective of the government. Third, lengthening debt maturity from 6.7 years to 8 years reduces the probability of falling in the crisis zone next quarter, thus having a first order effect on welfare.

Thus, we conclude that our our main results are robust to model misspecification, in the sense that small perturbations of our environment will not lead to a different conclusion

Figure A-2: Rollover risk and debt maturity: sensitivity to the parameter $\alpha$




(a) Rollover risk component




(b) Debt maturity

regarding the importance of rollover risk in our sample.

**Sensitivity on utility costs of changing maturity.** We now assess the robustness of our main result to the parameters of the utility costs of changing maturity. For this purpose, we solve the model for two alternative parametrizations. In the first parametrization we set $\alpha$ to 0.2, which implies a reduction of 50% in the costs of adjusting maturity relative to our benchmark of $\alpha = 0.4$. In the second parametrization, we solve the model by setting $\alpha$ to 0.6. In both experiments, we keep all remaining parameters to their value in Table 1. We then repeat the exercise of Section 5 under these two alternative parametrizations of the model.

Panel (a) of Figure A-2 reports the rollover risk component of interest rate spreads in these two parametrizations and compares them to our benchmark results. We can verify that the measured rollover risk component does not vary substantially when varying $\alpha$, which implies that our decomposition is robust to alternative values for the cost of changing maturity.

# H OMT in the model

We model OMT as follows. At the beginning of each period, after all uncertainty is realized, the government can ask for assistance. In such case, the central bank (CB) commits to buying government bonds in secondary markets at a price $q_{CB}^{(n)}(\mathbf{S})$ that may depend on the state of the economy, $\mathbf{S}$. The CB assistance is conditional on the face value of government debt at the end of the period, $B'$, being below a limit $\bar{B}_{CB}(\mathbf{S},\lambda') < \infty$, also set by the CB. The limit can depend on the state of the economy and on the maturity of the stock of the debt portfolio, and it captures the conditionality of the assistance in the secondary markets. OMT is therefore fully characterized by a policy rule $\left(q^{(n)}CB(\mathbf{S}), \bar{B}_{CB}(\mathbf{S},\lambda')\right)$. We assume that the CB finances bond purchases with a lump-sum tax levied on the lenders, and that such transfers are small enough that they do not affect the stochastic discount factor $M_{t,t+1}$.

The problem for the government described in (6) changes as follows. Letting $a \in \{0,1\}$ be the decision to request CB assistance, with $a = 1$ for the case in which assistance is requested, the government problem is

$$V(\mathbf{S}) = \max_{\delta \in \{0,1\}, B', \lambda', G, a \in \{0,1\}} \delta \left\{ U(G) + \beta \mathbb{E}[V(\mathbf{S}') | \mathbf{S}] \right\} + (1-\delta) \underline{V}(s_1) \tag{A.39}$$

subject to

$$\begin{aligned}
G + B &\leq \tau Y(s_1) + \Delta(\mathbf{S}, a, B', \lambda'), \\
\Delta(\mathbf{S}, a, B', \lambda') &= \sum_{n=1}^{\infty} q^{(n)}(\mathbf{S}, a, B', \lambda') \left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right] \\
B' &\leq \bar{B}_{CB}(\mathbf{S},\lambda') \quad \text{if } a = 1.
\end{aligned}$$

The lenders no-arbitrage condition requires that for $n \geq 1$,

$$q^{(n)}(\mathbf{S}, a, B', \lambda') = \max\{a q_{CB}^{(n)}(\mathbf{S}) \mathbb{I}_{\{B' \leq \bar{B}_{CB}(\mathbf{S},\lambda')\}}; \delta(\mathbf{S}) \mathbb{E}\left\{ M(s_1, s_1') \delta(\mathbf{S}') q^{(n-1)'} | \mathbf{S} \right\}, \tag{A.40}$$

where $q^{(n-1)'} = q^{(n-1)}(s', B'', \lambda'')$ with $B'' = B'(s', B', \lambda')$, $\lambda'' = \lambda'(s', B', \lambda')$, $a' = a(s', B', \lambda')$, and the convention $q_0' = 1$. The max operator on the right side of equation (A.40) reflects the option that lenders now have to sell the bond to CB at the price $q^{(n)}CB$ in case the government asks for assistance ($a = 1$). Because of that, pricing schedules now depend on current *and* future decisions of the government to activate OMTs. Given a policy rule $\left(q_{CB}^{(n)}, \bar{B}_{CB}\right)$, the recursive competitive equilibrium with OMT is defined as in Section 2.

We now turn to showing that an appropriately designed policy rule can uniquely implement the equilibrium outcome that would arise in the absence of rollover risk, that is, if $\pi_t = 0$ for all possible histories. We refer to such an outcome as the *fundamental equilibrium outcome* and denote the objects of a recursive competitive equilibrium associated with it with a superscript asterisk. The fundamental equilibrium outcome is our normative benchmark.[12]

[12]We abstract from policy interventions that aim to ameliorate inefficiencies arising from incomplete markets

**Proposition 4.** *The OMT rule can be chosen such that the fundamental equilibrium outcome is uniquely implemented and assistance is never activated along the path. In such case, the equilibrium with OMT is a weak Pareto improvement relative to the equilibrium without it, strict if the equilibrium outcome without OMT does not coincide with the fundamental equilibrium.*

*Proof.* Given $V^*$ and $q^{(n),*}$, let $\mathcal{S}^{\text{crisis}}(V^*)$ be the crisis zone associated with the fundamental equilibrium value function. Construct the policy rule $\left(q_{CB}^{(n)}, \bar{B}_{CB}\right)$ so that for all $\mathbf{S} \in \mathcal{S}^{\text{crisis}}(V^*)$ there exists at least one $(B', \lambda')$ with $B' \leq \bar{B}_{CB}(\mathbf{S}, \lambda')$ such that if the government asks for assistance, then it prefers to repay rather than default:

$$U\left(\tau Y - B + \sum_n q_{CB}^{(n)}(\mathbf{S})\left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right]\right) + \beta \mathbb{E}\left[V^*\left(B', \lambda', s'\right)|\mathbf{S}\right] \geq \underline{V}(s_1), \tag{A.41}$$

and the fundamental equilibrium is always preferable than asking for assistance, in that for all $(B', \lambda')$ such that $B' \leq \bar{B}_{CB}(\mathbf{S}, \lambda')$,

$$U\left(\tau Y - B + \sum_n q_{CB}^{(n)}(\mathbf{S})\left[(1-\lambda')^{n-1} B' - (1-\lambda)^n B\right]\right) + \beta \mathbb{E}\left[V^*\left(B', \lambda', s'\right)|\mathbf{S}\right] \leq V^*(\mathbf{S}). \tag{A.42}$$

Clearly it is possible to find policy rules that satisfy (A.41) and (A.42). An obvious example is to set $q_{CB}^{(n)}(\mathbf{S}) = q^{(n)^*}\left(s, B'(\mathbf{S}), \lambda^{*\prime}(\mathbf{S})\right)$ and $\bar{B}_{CB}(\mathbf{S}, \lambda) = B^{*\prime}(\mathbf{S})$ if $\lambda = \lambda^{*\prime}(\mathbf{S})$ and zero otherwise.

Under (A.41) and (A.42), no self-fulfilling run is possible, the optimal $B'$ and $\lambda'$ are the same that arise in the fundamental equilibrium, and the government has no incentives to activate OMT along the equilibrium path. Hence, given a policy rule that satisfies (A.41) and (A.42), there exists a recursive equilibrium with OMT that implements the fundamental equilibrium outcome for any sunspot process $\{s_{2t}\}$.[13] *Q.E.D.*

# I Rollover risk and public debt management in 1980s Italy

This section documents in details an example of a government extending the maturity of its debt while facing rollover problems. Using a narrative approach, we analyze the experience of the Italian Treasury department in the early 1980s.

and consider OMT rules targeted at eliminating "bad" equilibria. Such features will also survive in models with complete markets or in environments where some notion of constrained efficiency can be achieved as in Dovis (2019).

[13] We cannot establish that given a policy rule that satisfies (A.41) and (A.42), the fundamental equilibrium is the unique recursive equilibrium with OMT. This is because there may be multiple fixed point of the operator that defines a recursive equilibrium. Hence, the fact that $(V^*, q^*)$ is a fixed point of such an operator for an arbitrary sunspot process does not necessarily imply that there is not another fixed point $(V, q)$.

Two main factors at the beginning of the 1980s contributed to placing the Italian government at risk of a rollover crisis. First, the Italian government needed to refinance almost its entire debt, which was roughly 60% of GDP at the time, within the span of a year. Following the chronic inflation of the 1970s, in fact, investors became discouraged from holding long duration bonds that were unprotected from inflation risk, and the average residual maturity of Italian debt went from a peak value of 9.2 years in 1972 to 1.1 years in 1980 (Pagano, 1988). Second, and in an effort to increase the independence of the central bank, a major institutional reform freed the Bank of Italy from the obligation of buying unsold public debt in auctions.[14]

The short residual life of government debt coupled with the loss of central bank financing meant that the Italian government had to use primary markets to refinance its maturing debt. However, these markets were not well developed at the time, and private demand for government bonds was weak and volatile (Campanaro and Vittas, 2004). The left panel of Figure A-3 reports statistics regarding the placement of Italian Treasury securities during the 1981-1982 period. The solid line plots the private bid-to-cover ratio for Italian Treasury securities. This ratio averaged only 0.65 over this period, with a standard deviation of 0.25. The dashed line reports the ratio between the quantity sold and the Treasury's target. Until July 1981, this ratio was equal to 1 because of the statutory requirement for the central bank to buy unsold bonds. Following the reform of the central bank, though, the Treasury became exposed to variation in the private demand of bonds.

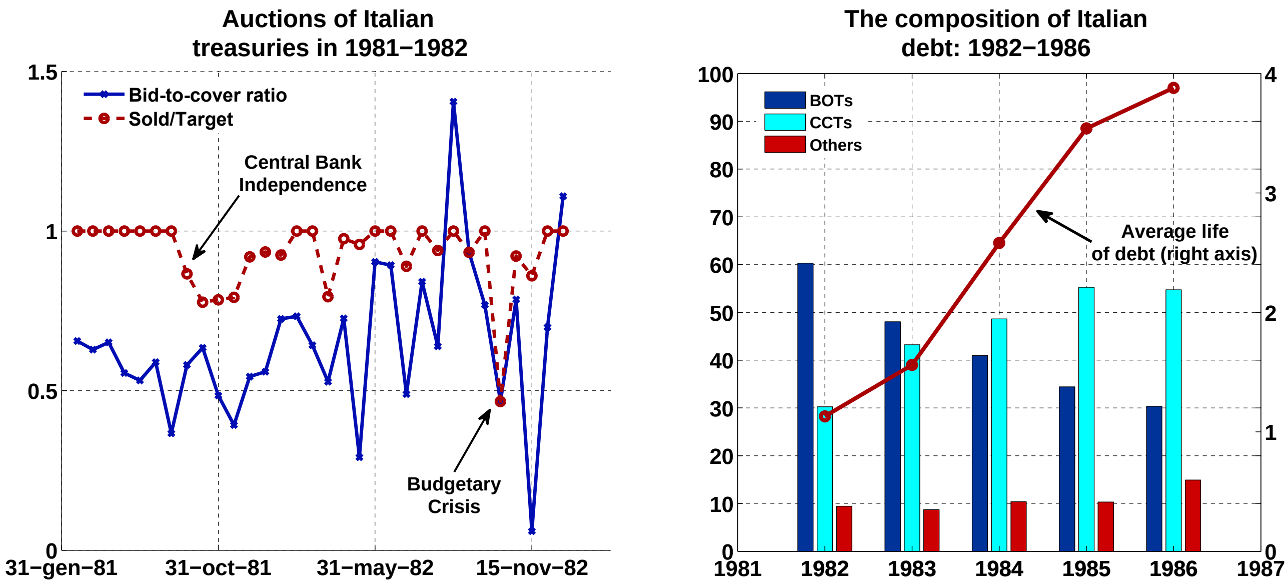
The possibility that rollover problems may eventually lead to a debt crisis became evident in the last quarter of 1982. On the auction of October 15, private demand covered only 46% of the Treasury's needs, and the central bank decided not to purchase unsold bonds. The Treasury was thus forced to use the overdraft account it had with the Bank of Italy to cover its financing needs, reaching the statutory limit. This led to a budgetary crisis, which further depressed private demand of bonds out of fears of a debt restructuring.[15] While the Parliament later voted in a law that allowed a temporal overshoot of the overdraft account (Scarpelli, 2001), these events exposed to policymakers the risks implicit in refinancing large amounts of debt in short periods of time.

The response of the Italian government to these events is consistent with the logic of our identification strategy. As documented in Alesina, Prati, and Tabellini (1989) and in Scarpelli

[14]Starting from 1975, the Bank of Italy was required to act as a residual buyer of all the public debt that was unsold in the auctions. This resulted in a massive increase in the share of public debt held by the Bank of Italy, reaching a maximum of 40% in 1976. See Tabellini (1988) for a discussion of the historical context underlying the "divorce" between the Bank of Italy and the Italian Treasury.

[15]These fears were not without motivation. Rino Formica, ministry of Finance at the time, publicly called for an agreement with bondholders that would allow the Treasury to reimburse only part of its debt. Beniamino Andreatta, ministry of the Treasury, strongly opposed this view. This controversy, known in the public debate as "lite delle comari," eventually led to the fall of the Italian government on December 1, 1982.

Figure A-3: Rollover risk and public debt management: Italy in the early 1980s




*Notes: The statistics in the left panel are constructed using data from Bank of Italy, Supplements to the Statistical Bulletin-Financial Markets. The statistics in the right panel are constructed using data from the Italian Treasury. The bar indicates the percentage of a particular class of bonds over total outstanding debt. The line is the average life of outstanding debt (reported in years on the right axis.*

(2001), the Treasury actively pursued a policy to extend the life of its public debt. Financial innovation was the main tool used for this purpose, with the introduction of new types of bonds whose interest payments were indexed to the prevailing nominal rate. These *Certificati di Credito del Tesoro* (CCT) were palatable to investors because they offered protection from inflation risk, and at the same time they had longer maturity than the *Buoni Ordinari del Tesoro* (BOT), the prevailing form of bond financing at the time.[16] The right panel of Figure A-3 reports the composition of the outstanding Italian debt (bars) along with its residual average life during the 1982-1986 period. We can see that the Treasury quickly replaced BOTs with CCTs as the main source of public financing. The efforts of the Treasury were successful in increasing the maturity of outstanding debt, with its residual average life more than tripling within the span of four years.

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[16]Indexed securities such as CCT are not subject to refinancing and rollover problems but are essentially equal to short-term debt for the incentive to generate ex-post inflation because any effort to generate ex-post inflation will not reduce the real value of debt. See Missale and Blanchard (1994).

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