Online Appendix

The Pass-Through of Sovereign Risk

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A Numerical Solution

A.1 Equilibrium conditions

The states of the model are $\mathbf{S} = [\tilde{K}, \tilde{B}, \tilde{P}, \Delta z, g, s, d]$. The controls $\{\tilde{C}(\mathbf{S}), R(\mathbf{S}), \alpha(\mathbf{S}), Q_B(\mathbf{S})\}$ solve the residual equations

$$E_{\mathbf{S}}\left[\beta \frac{\tilde{C}(\mathbf{S})}{\tilde{C}(\mathbf{S}')} e^{-\Delta z'} R(\mathbf{S})\right] - 1 = 0, \tag{A.1}$$

$$\mathbb{E}_{\mathbf{S}}\left\{\beta \frac{\tilde{C}(\mathbf{S})}{\tilde{C}(\mathbf{S}')} e^{-\Delta z'}[(1-\psi)+\psi\alpha(\mathbf{S}')]\left[\frac{(1-\delta)Q_K(\mathbf{S}') + \alpha \frac{\tilde{Y}(\mathbf{S}')}{\tilde{K}'(\mathbf{S})} e^{\Delta z'}}{Q_K(\mathbf{S})}\right]\right\} - \lambda\mu(\mathbf{S}) = 0, \tag{A.2}$$

$$\mathbb{E}_{\mathbf{S}}\left\{\beta \frac{\tilde{C}(\mathbf{S})}{\tilde{C}(\mathbf{S}')} e^{-\Delta z'}[(1-\psi)+\psi\alpha(\mathbf{S}')]\left[1-d'D\right]\left[\frac{\pi + (1-\pi)\left[\iota + Q_b(\mathbf{S}')\right]}{Q_B(\mathbf{S})}\right]\right\} - \lambda\mu(\mathbf{S}) = 0, \tag{A.3}$$

$$\alpha(\mathbf{S}) - \frac{(1-\psi)+\psi R(\mathbf{S})\mathbb{E}_{\mathbf{S}}\left[\beta \frac{\tilde{C}(\mathbf{S})}{\tilde{C}(\mathbf{S}')} e^{-\Delta z'}\alpha(\mathbf{S}')\right]}{1-\mu(\mathbf{S})} = 0, \tag{A.4}$$

where equation (A.1) is the Euler equation for households' savings and the Lagrange multiplier $\mu(\mathbf{S})$ is given by

$$\mu(\mathbf{S}) = \max\left\{1 - \left[\frac{\mathbb{E}_{\mathbf{S}}\left\{\beta \frac{\tilde{C}(\mathbf{S})}{\tilde{C}(\mathbf{S}')} e^{-\Delta z'}[(1-\psi)+\psi\alpha(\mathbf{S}')]R(\mathbf{S})\right\}\tilde{N}(\mathbf{S})}{\lambda[Q_K(\mathbf{S})\tilde{K}'(\mathbf{S}) + Q_B(\mathbf{S})\tilde{B}'(\mathbf{S})]}\right], 0\right\}. \tag{A.5}$$

The endogenous state variables $[\tilde{K}, \tilde{B}, \tilde{P}]$ evolve as follows:

$$\tilde{K}'(\mathbf{S}) = \left\{(1-\delta)\tilde{K} + \Phi\left[e^{\Delta z}\left(\frac{\tilde{Y}(\mathbf{S})(1-g) - \tilde{C}(\mathbf{S})}{\tilde{K}}\right)\right]\tilde{K}\right\} e^{-\Delta z}, \tag{A.6}$$

$$\tilde{B}'(\mathbf{S}) = \frac{[1-dD]\{\pi + (1-\pi)[\iota + Q_B(\mathbf{S})]\}\tilde{B}e^{-\Delta z} + \tilde{Y}(\mathbf{S})g - \left(t^* + \gamma_\tau \tilde{B}e^{-\Delta z}\right)}{Q_B(\mathbf{S})}, \tag{A.7}$$

$$\tilde{P}'(\mathbf{S}) = R(\mathbf{S})[Q_K(\mathbf{S})\tilde{K}'(\mathbf{S}) + Q_B(\mathbf{S})\tilde{B}'(\mathbf{S}) - \tilde{N}(\mathbf{S})]. \tag{A.8}$$

The state variable $\tilde{P}$ measures the detrended *cum* interest deposits that bankers pay to households at the beginning of the period, and is sufficient to keep track of the evolution of aggregate bankers' net worth. Indeed, the aggregate net worth of bankers can be expressed as

$$\begin{aligned} \tilde{N}(\mathbf{S}) \quad = \quad & \psi\left\{\left[(1-\delta)Q_K(\mathbf{S}) + \alpha\frac{\tilde{Y}(\mathbf{S})}{\tilde{K}}e^{\Delta z}\right]\tilde{K} + [1-dD]\left[\pi + (1-\pi)\left[\iota + Q_B(\mathbf{S})\right]\right]\tilde{B} - \tilde{P}\right\}e^{-\Delta z} \\ & + \omega[Q_K(\mathbf{S})\tilde{K} + Q_B(\mathbf{S})\tilde{B}]e^{-\Delta z}. \end{aligned} \tag{A.9}$$

Using the intratemporal Euler equation of households, we can write detrended output as

$$\tilde{Y}(\mathbf{S}) = \left[\chi^{-1}\frac{(\tilde{K}e^{-\Delta z})^\alpha}{\tilde{C}(\mathbf{S})}\right]^{\frac{1-\alpha}{\alpha+\nu-1}}(\tilde{K}e^{-\Delta z})^\alpha. \tag{A.10}$$

The exogenous state variables $[\Delta z, \log(g), s]$ evolve as follows:

$$\Delta z' = (1-\rho_z)\gamma + \rho_z\Delta z + \sigma_z\varepsilon_z', \tag{A.11}$$

$$\log(g') = (1-\rho_g)g^* + \rho_g\log(g) + \sigma_g\varepsilon_g', \tag{A.12}$$

$$s' = (1-\rho_s)s^* + \rho_s s + \sigma_s\varepsilon_s', \tag{A.13}$$

while d follows

$$d' = \begin{cases} 1 & \text{with probability} \quad \frac{\exp\{s\}}{1+\exp\{s\}} \\ 0 & \text{with probability} \quad 1 - \frac{\exp\{s\}}{1+\exp\{s\}}. \end{cases} \tag{A.14}$$

A.2 Algorithm for numerical solution

I approximate the control variables of the model using piecewise smooth functions, parametrized by $\gamma = \{\gamma^x_{d=0}, \gamma^x_{d=1}\}_{x=\{\tilde{C},\alpha,Q_b,R\}}$. The law of motion for a control variable x is described by

$$x(d, \tilde{\mathbf{S}}) = (1-d)\gamma^x_{d=0}{}'\mathbf{T}(\tilde{\mathbf{S}}) + d\gamma^x_{d=1}{}'\mathbf{T}(\tilde{\mathbf{S}}), \tag{A.15}$$

where $\tilde{\mathbf{S}} = [\tilde{K}, \tilde{P}, \tilde{B}, \Delta z, g, s]$ and $\mathbf{T}(.)$ is a vector collecting Chebyshev's polynomials. Define $\mathcal{R}(\gamma^c, \{d^i, \tilde{\mathbf{S}}^i\})$ to be a 4×1 vector collecting the left-hand side of the residual equations (A.1)-(A.4) for the candidate solution γ^c evaluated at $\{d^i, \tilde{\mathbf{S}}^i\}$. The numerical solution of the model consists of choosing γ^c so that $\mathcal{R}(\gamma^c, \{d^i, \tilde{\mathbf{S}}^i\}) = 0$ for a set of collocation points $\{d^i, \tilde{\mathbf{S}}^i\} \in \{0, 1\} \times \tilde{\mathcal{S}}$.

The choice of collocation points and the associated Chebyshev's polynomials follows the method of Smolyak (Krueger, Kubler, and Malin, 2010), specifically the implementation proposed by Judd, Maliar, Maliar, and Valero (2014). Conditional expectations when evaluating $\mathcal{R}(\gamma^c, \{d, \tilde{\mathbf{S}}\})$ are calculated following the approach of Judd, Maliar, and Maliar (2011). To give an example of this latter, suppose we wish to compute $\mathbb{E}_{d^i, \tilde{\mathbf{S}}^i}[y(d', \tilde{\mathbf{S}}')]$, where y is an integrand of interest.[1] Given a candidate solution γ^c, we can compute y at every collocation point using the model's equilibrium conditions. Next, we can construct an implied policy function for y, $\{\gamma^y_{d=0}, \gamma^y_{d=1}\}$, via a Chebyshev's regression. Using the law of total probability, the conditional expectation of interest can be expressed as

$$\begin{aligned}\mathbb{E}_{d^i, \tilde{\mathbf{S}}^i}[y(d', \tilde{\mathbf{S}}')] \quad = \quad & (1 - \text{Prob}\{d' = 1 | \tilde{\mathbf{S}}^i\})\mathbb{E}_{\tilde{\mathbf{S}}^i}[\gamma^{y\,\prime}_{d=0}\mathbf{T}(\tilde{\mathbf{S}}')] + \\ & \text{Prob}\{d' = 1 | \tilde{\mathbf{S}}^i\}\mathbb{E}_{\tilde{\mathbf{S}}^i}[\gamma^{y\,\prime}_{d=1}\mathbf{T}(\tilde{\mathbf{S}}')],\end{aligned} \tag{A.16}$$

where $\text{Prob}\{d' = 1 | \tilde{\mathbf{S}}^i\} = \frac{e^{s^i}}{1+e^{s^i}}$. Judd, Maliar, and Maliar (2011) propose a simple procedure to evaluate integrals of the form $\mathbb{E}_{\tilde{\mathbf{S}}^i}[\gamma^{y\,\prime}_{d=1}\mathbf{T}(\tilde{\mathbf{S}}')]$. Proposition 1 of their paper establishes that, under certain conditions, the expectation of a polynomial can be calculated via a linear transformation $\mathcal{I}$ of the coefficient vector γ^y_d, where $\mathcal{I}$ is a function of the deep parameters of the model. The authors provide general formulas for the transformation $\mathcal{I}$.

The algorithm for the numerical solution of the model is as follows:

Step 0.A: Defining the grid and the polynomials. Set upper and lower bounds on the state variables $\tilde{\mathbf{S}} = [\tilde{K}, \tilde{P}, \tilde{B}, \Delta z, g, s]$. Given these bounds, construct a μ-level Smolyak grid and the associated Chebyshev's polynomials $\mathbf{T}(.)$ following Judd et al. (2014).[2]

Step 0.B: Precomputing integrals. Compute $\mathcal{I}$ using the formulas in Judd, Maliar, and Maliar (2011).

[1]For example, y could be $e^{-\Delta z'}/\tilde{C}(\mathbf{S}')$ in equation (A.1).

[2]The grid is constructed on a rotation of $\tilde{K}$ and $\tilde{P}$ in order to account for the tight correlation between these two variables. The rotation matrix is generated by applying the singular value decomposition to data simulated from a third-order perturbation of the model without sovereign risk.

Step 1: Equilibrium conditions at the candidate solution. Start with a guess for the model's policy functions γ^c. For each $(d^i, \tilde{\mathbf{S}}^i)$, use γ^c and equation (A.15) to compute the value of control variables $\{\tilde{C}(d^i, \tilde{\mathbf{S}}^i), \alpha(d^i, \tilde{\mathbf{S}}^i), Q_B(d^i, \tilde{\mathbf{S}}^i), R(d^i, \tilde{\mathbf{S}}^i)\}$. Given the control variables, solve for the endogenous state variables next period using the model's equilibrium conditions. Given the value of control and state variables, compute the value of every integrand in equations (A.1)-(A.4) at $(d^i, \tilde{\mathbf{S}}^i)$. Collect these integrands in the matrix $\mathbf{y}$.

Step 2: Evaluate conditional expectations. For each $d = \{0, 1\}$, run a Chebyshev regression for the integrand in $\mathbf{y}$, and denote by γ_d^y the implied policy function for an element $y \in \mathbf{y}$. Conditional expectations are calculated using equation (A.16) and the matrix $\mathcal{I}$.

Step 3: Evaluate residual equations. Given conditional expectations, compute the Lagrange multiplier using equation (A.5). Evaluate the residuals $\mathcal{R}(\gamma^c, \{d^i, \tilde{\mathbf{S}}^i\})$ at every collocation point $(d^i, \tilde{\mathbf{S}}^i)$. The dimension of the vector of residuals is four times the cardinality of the state space. Denote by r the Euclidean norm for this vector.

Step 4: Iteration. If $r \leq 10^{-20}$, stop the algorithm. If not, update the guess and repeat Step 1-4. □

The specifics for the algorithm are as follows. The bounds on $[\Delta z, g]$ are +/- 3 standard deviations from their mean. The bounds on s are larger and set to $[s^* - 4.75, s^* + 4.75]$. The bounds on the endogenous state variables are set to +/- 3 standard deviations from their balanced growth values. The standard deviation is calculated by simulating a third-order perturbation of the model without sovereign risk. For the Smolyak grid, I choose $\mu = 3$, and I use Gaussian numerical quadrature (15 points) to compute the matrix $\mathcal{I}$. Finally, I find the zeros of the residual equation using a variant of the Newton algorithm.

A.3 Accuracy of numerical solution

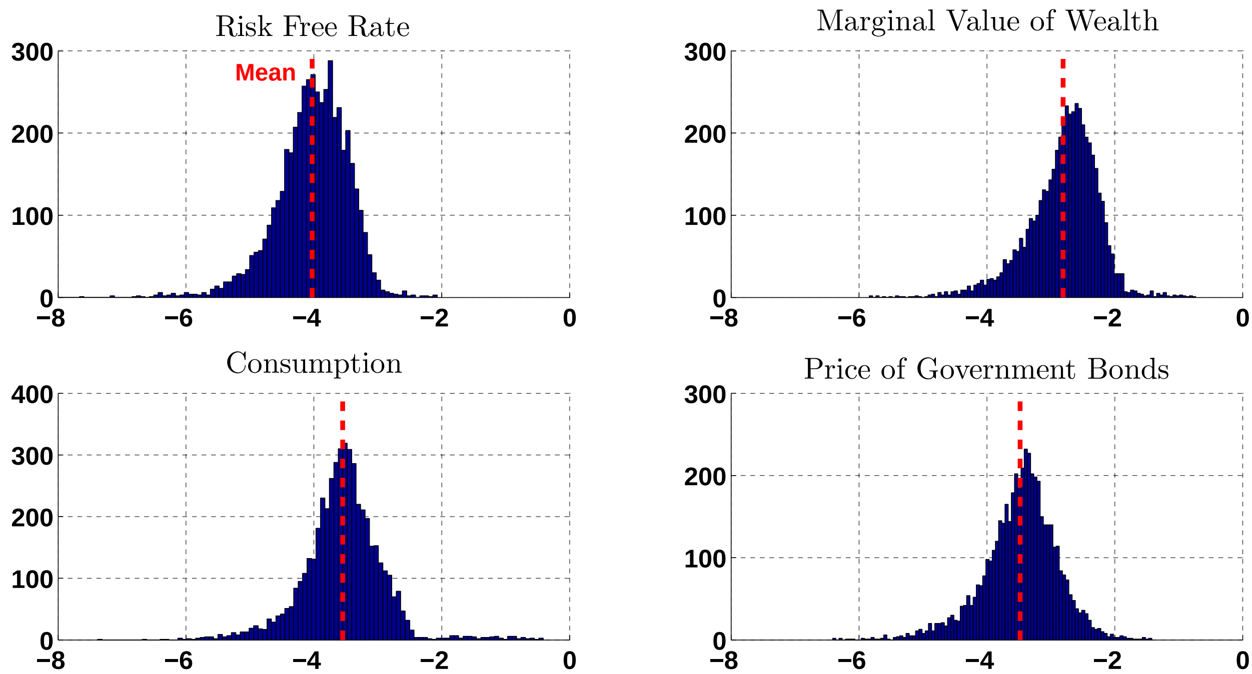
I check the accuracy of the numerical solution by computing the errors of the residual equations (Judd, 1992). Specifically, I proceed as follows. Let γ^* denote the solution to the model. First, I simulate the model forward for 5,000 periods. This gives a simulation for the state variable of the model $\{d_t, \tilde{\mathbf{S}}_t\}_{t=1}^{5000}$. Second, for each pair $(d_t, \tilde{\mathbf{S}}_t)$, I calculate the errors of the residual equations $\mathcal{R}(\gamma^*, \{d_t, \tilde{\mathbf{S}}_t\})$. As an example, let us consider equation

(A.2). Then, the residual error at $(d_t, \tilde{\mathbf{S}}_t)$ for this equation is defined as

$$\lambda\mu(d_t, \tilde{\mathbf{S}}_t) - \mathbb{E}_{\mathbf{d_t}, \tilde{\mathbf{S}}_\mathbf{t}} \left\{ \beta \frac{\tilde{C}(d_t, \tilde{\mathbf{S}}_t)}{\tilde{C}(\mathbf{S}')} e^{-\Delta z'} [(1-\psi) + \psi\alpha(\mathbf{S}')] \left[\frac{(1-\delta)Q_K(\mathbf{S}') + \alpha \frac{\tilde{Y}(\mathbf{S}')}{\tilde{K}'(d_t, \tilde{\mathbf{S}}_t)} e^{\Delta z'}}{Q_K(d_t, \tilde{\mathbf{S}}_t)} \right] \right\},$$

where the model's policy functions are used to generate the value for endogenous variables at $(d_t, \tilde{\mathbf{S}}_t)$.[3] Following standard practice, I report the decimal log of the absolute value of these residual errors. Figure A-1 reports the density (histogram) of those errors.

Figure A-1: **Residual equation errors**



Notes: The histograms report the residual equation errors in decimal log basis. The dotted line marks the mean residual equation error.

On average, residual equation errors are on the order of -4 for the risk-free rate, -3.5 for consumption and the price of government securities, and -3 for the marginal value of wealth. These numbers are comparable to values reported in the literature for models of similar complexity, and they are still very reasonable.

[3]By construction, the residual errors are zero at the collocation points. These residual errors provide a measure of how large are the discrepancies between the decision rule derived from the numerical algorithm and those implied by the model's equilibrium conditions in other points of the state space.

B Empirical Analysis

B.1 Estimating the model without sovereign risk

The model without sovereign risk has five state variables $\mathbf{S}_t = [\hat{K}_t, \hat{P}_t, \hat{B}_t, \Delta z_t, g_t]$. Let $\mathbf{Y}_t$ be a 2×1 vector of observables collecting output growth and the time series for the Lagrange multiplier on the leverage constraint. The state-space representation is

$$\mathbf{Y}_t = f_{\tilde{\theta}}(\mathbf{S}_t) + \eta_t \qquad \eta_t \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}) \tag{A.17}$$

$$\mathbf{S}_t = g_{\tilde{\theta}}(\mathbf{S}_{t-1}, \varepsilon_t) \qquad \varepsilon_t \sim \mathcal{N}(\mathbf{0}, \mathbf{I}). \tag{A.18}$$

The first equation is the measurement equation, with η_t being a vector of Gaussian measurement errors. The second equation is the transition equation, which represents the law of motion for the model's state variables. The vector ε_t represents the innovation to the structural shocks Δz_t and g_t. The functions $f_{\tilde{\theta}}(.)$ and $g_{\tilde{\theta}}(.)$ are generated using the numerical procedure described in Appendix A applied to the model without sovereign risk. I characterize the posterior distribution of $\tilde{\theta}$ using Bayesian methods. I denote by $p(\tilde{\theta})$ the prior on $\tilde{\theta}$. In what follows, I provide details on the evaluation of the likelihood function and on the posterior sampler adopted.

B.1.1 Likelihood evaluation

Let $\mathbf{Y}^t = [\mathbf{Y}_1, \ldots, \mathbf{Y}_t]$, and denote by $p(\mathbf{S}_t|\mathbf{Y}^{t-1};\theta)$ the conditional distribution of the state vector given observations up to period $t-1$. The likelihood function for the state-space model of interest can be expressed as

$$\mathcal{L}(\mathbf{Y}^T|\theta) = \prod_{t=1}^{T} p(\mathbf{Y}_t|\mathbf{Y}^{t-1};\theta) = \prod_{t=1}^{T}\left[\int p(\mathbf{Y}_t|\mathbf{S}_t;\theta)p(\mathbf{S}_t|\mathbf{Y}^{t-1};\theta)d\mathbf{S}_t\right]. \tag{A.19}$$

Although the conditional density of $\mathbf{Y}_t$ given $\mathbf{S}_t$ is known and Gaussian, there is no analytical expression for the density $p(\mathbf{S}_t|\mathbf{Y}^{t-1},\theta)$. I use the auxiliary particle filter of Pitt and Shephard (1999) to approximate this density. This approximation is then used to estimate the likelihood function.

Step 0: Initialization. Set $t = 1$. Initialize $\{\mathbf{S}_0^i, \tilde{\pi}_0^i\}_{i=1}^N$ from the model's ergodic distribution and set $\tilde{\pi}_0^i = \frac{1}{N}\ \forall i$.

Step 1: Prediction. For each $i = 1, \ldots, N$, draw $\mathbf{S}^i_{t|t-1}$ values from the proposal density $g(\mathbf{S}_t|\mathbf{Y}^t, \mathbf{S}^i_{t-1})$.

Step 2: Filtering. Assign to each $\mathbf{S}^i_{t|t-1}$ the particle weight

$$\pi^i_t = \frac{p(\mathbf{Y}_t|\mathbf{S}^i_{t|t-1};\theta)p(\mathbf{S}_t|\mathbf{S}^i_{t|t-1};\theta)}{g(\mathbf{S}_t|\mathbf{Y}^t, \mathbf{S}^i_{t-1})}.$$

Step 3: Sampling. Rescale the particles $\{\pi^i_t\}$ so that they add up to unity, and denote these rescaled values by $\{\tilde{\pi}^i_t\}$. Sample N values for the state vector with replacement from $\{\mathbf{S}^i_{t|t-1}, \tilde{\pi}^i_t\}^N_{i=1}$. Call each draw $\{\mathbf{S}^i_t\}$, and set $\tilde{\pi}^i_t = \frac{1}{N}$. If $t < T$, set $t = t+1$ and go to Step 1. If not, stop. □

The likelihood function of the model is then approximated as

$$\mathcal{L}(\mathbf{Y}^T|\theta) \approx \left(\prod_{t=1}^{T} \left[\frac{1}{N} \sum_{i=1}^{N} p(\mathbf{Y}_t|\mathbf{S}^i_{t|t-1};\theta) \right] \right).$$

Regarding the tuning of the filter, I set $N = 100000$. The matrix $\mathbf{\Sigma}$ is diagonal, and the diagonal elements equal 25% of the variance of the observable variables. The choice for the proposal density $g(\mathbf{S}_t|\mathbf{Y}^t, \mathbf{S}^i_{t-1})$ is more involved. I sample the structural innovations ε_t from $\mathcal{N}(m_t, \mathbf{I})$. Then, I use the model's transition equation (A.18) to obtain $\mathbf{S}^i_{t|t-1}$. The center for the proposal distribution for ε_t is generated as follows. Let $\overline{\mathbf{S}}_{t-1}$ be the mean for $\{\mathbf{S}^i_{t-1}\}$ over i. Then, m_t is the vector of innovations ε solving the program

$$\text{argmin}_\varepsilon \left\{ [\mathbf{Y}_t - f_{\tilde{\theta}}(g_{\tilde{\theta}}(\overline{\mathbf{S}}_{t-1}, \varepsilon))]'[\mathbf{Y}_t - f_{\tilde{\theta}}(g_{\tilde{\theta}}(\overline{\mathbf{S}}_{t-1}, \varepsilon))] + \varepsilon' \mathbf{\Sigma}^{-1} \varepsilon \right\}.$$

The first part of the objective function pushes ε toward values that can rationalize the observation $\mathbf{Y}_t$. The second part imposes a penalty for shocks that are far away from 0, their unconditional mean.

B.1.2 Posterior sampler

I characterize the posterior density of $\tilde{\theta}$ using a Random Walk Metropolis Hastings with proposal density given by

$$q(\tilde{\theta}^p|\tilde{\theta}^{m-1}) \sim \mathcal{N}(\tilde{\theta}^{m-1}, c\mathbf{H}).$$

The sequence of draws $\{\tilde{\theta}^m\}$ is generated as follows:

1. Initialize the chain at $\tilde{\theta}^1$.

2. For $m = 2, \ldots, M$, draw $\tilde{\theta}^p$ from $q(\tilde{\theta}^p|\tilde{\theta}^{m-1})$. The jump from $\tilde{\theta}^{m-1}$ to $\tilde{\theta}^p$ is accepted ($\tilde{\theta}^m = \tilde{\theta}^p$) with probability $\min\{1, r(\tilde{\theta}^{m-1}, \tilde{\theta}^p|\mathbf{Y}^T)\}$, and rejected otherwise ($\tilde{\theta}^m = \tilde{\theta}^{m-1}$). The probability of accepting the draw is

$$r(\tilde{\theta}^{m-1}, \tilde{\theta}^p|\mathbf{Y}^T) = \frac{\mathcal{L}(\mathbf{Y}^T|\tilde{\theta}^p)p(\tilde{\theta}^p)}{\mathcal{L}(\mathbf{Y}^T|\tilde{\theta}^{m-1})p(\tilde{\theta}^{m-1})}.$$

First, I run the chain for $M = 10,000$ with $\mathbf{H}$ being the identity matrix and $c = 0.001$. The chain is initialized at the prior mean. I drop the first 5,000 draws, and I use the remaining draws to initialize a second chain and to construct a new candidate density. This second chain is initialized at the mean of the 5,000 draws. Moreover, the variance-covariance matrix $\mathbf{H}$ is set to the empirical variance-covariance matrix of these 5,000 draws. The parameter c is fine-tuned to obtain an acceptance rate of roughly 60%. I run the second chain for $M = 60,000$. Posterior statistics are based on the latter 10,000 draws.

B.2 Approximating $\mathbb{E}_t[\hat{\Lambda}_{t,t+1}|d_{t+1} = 1]$

I approximate $\mathbb{E}_t[\hat{\Lambda}_{t,t+1}|d_{t+1} = 1]$ as follows:

$$\mathbb{E}_t[\hat{\Lambda}_{t,t+1}|d_{t+1} = 1] \approx \mathbb{E}_t[\hat{\Lambda}_{t,t+1}] + \kappa \text{Var}_t[\hat{\Lambda}_{t,t+1}]^{\frac{1}{2}}, \tag{A.20}$$

where $\kappa > 0$ is a hyperparameter. The idea of equation (A.20) is that the marginal value of wealth for bankers is above average in a sovereign default because they are more likely to face funding constraints: κ parameterizes the number of standard deviations by which $\mathbb{E}_t[\hat{\Lambda}_{t,t+1}|d_{t+1} = 1]$ is greater than $\mathbb{E}_t[\hat{\Lambda}_{t,t+1}]$.

I use the model restrictions to generate the terms $\{\mathbb{E}_t[\hat{\Lambda}_{t,t+1}], \text{Var}_t[\hat{\Lambda}_{t,t+1}]^{\frac{1}{2}}\}$. Specifically, we can write $\hat{\Lambda}_{t,t+1}$ as a function of observables and of model parameters estimated in the first step

$$\hat{\Lambda}_{t,t+1} = \beta e^{-\Delta \log(c_{t+1})}[(1 - \psi) + \psi \lambda \text{lev}_{t+1}], \tag{A.21}$$

where Δc_{t+1} is the growth rate of real personal consumption expenditure and lev_{t+1} is the leverage of financial intermediaries. Conditional on the posterior mean of $[\beta, \psi, \lambda]$, I generate a time series for $\hat{\Lambda}_{t,t+1}$. The time series $\{\mathbb{E}_t[\hat{\Lambda}_{t,t+1}], \text{Var}_t[\hat{\Lambda}_{t,t+1}]^{\frac{1}{2}}\}$ are generated by fitting an AR(1) on $\log(\hat{\Lambda}_{t,t+1})$. Next, I use $\{\mathbb{E}_t[\hat{\Lambda}_{t,t+1}], \text{Var}_t[\hat{\Lambda}_{t,t+1}]^{\frac{1}{2}}\}$ and equation (A.20) to approximate $\mathbb{E}_t[\hat{\Lambda}_{t,t+1}|d_{t+1} = 1]$. The hyperparameter κ is selected with the help of the structural model. I consider a set of values $\kappa^i \in \{1, 3, 5\}$ and I select the value that minimizes, in model simulated data, average root mean square errors for the approximation of $\mathbb{E}_t[\hat{\Lambda}_{t,t+1}|d_{t+1} = 1]$. This gives a value of $\kappa = 3$.

C Refinancing Operations

It is instructive to first consider the stationary problem. The government allows bankers to borrow up to $\overline{m}$ at the fixed interest rate R_m, and this intervention is financed through lump-sum taxation. Moreover, these loans are not subject to limited enforcement problems. The decision problem of the banker becomes

$$
\begin{aligned}
v^b(n;\mathbf{S}) &= \max_{a_B,a_K,b,m} \mathbb{E}_{\mathbf{S}} \left\{ \Lambda(\mathbf{S}',\mathbf{S}) \left[(1-\psi)n' + \psi v^b(n';\mathbf{S}') \right] \right\}, \\
n' &= \sum_{j=\{B,K\}} [R_j(\mathbf{S}',\mathbf{S}) - R(\mathbf{S})]Q_j(\mathbf{S})a_j + [R_m - R(\mathbf{S})]m - R(\mathbf{S})b, \\
\lambda \left[\sum_{j=\{B,K\}} Q_j(\mathbf{S})a_j - m \right] &\leq v^b(n;\mathbf{S}), \\
m &\in [0,\overline{m}], \\
\mathbf{S}' &= \mathbf{\Gamma}(\mathbf{S}).
\end{aligned}
$$

Assuming that $m \geq 0$ does not bind, the first-order condition with respect to m is

$$
\mathbb{E}_s \left\{ \Lambda(\mathbf{S}',\mathbf{S}) \left[(1-\psi) + \psi \frac{\partial v^b(n';\mathbf{S})}{\partial n'} \right] \right\} [R(\mathbf{S}) - R_m] + \lambda\mu(\mathbf{S}) = \chi(\mathbf{S})
$$

It can be shown, following the logic of Proposition 1, that $v^b(n;\mathbf{S}) = \alpha(\mathbf{S})n + x(\mathbf{S})$, with $x(\mathbf{S}) \geq 0$. The leverage constraint becomes

$$
\frac{\sum_j Q_j(\mathbf{S})a_j}{n} \leq \frac{\alpha(\mathbf{S}) + x(\mathbf{S})}{\lambda} + m.
$$

Note that refinancing operations have two main effects on bankers. First, to the extent that $R_m < R(\mathbf{S})$, they represent an implicit transfer to banks and they contribute to an increase in their net worth. Second, the policy relaxes the incentive constraint of bankers. This happens because of two distinct reasons: i) the loan from the government is not subject to the limited enforcement problem, and ii) the value function of bankers increases as a result of the subsidized loan.

The longer term refinancing operations (LTROs) are a nonstationary version of the refinancing operations described above. The government allows bankers to borrow up to $\overline{m}$ in period $t = 1$, and it receives the principal and interest in a later period T. I assume that the policy was unexpected by agents. At time $t = 1$, agents are perfectly informed about the time path of the loans, and they believe that the policy will not be implemented in the future. Note that the decision rules under LTROs are time dependent: the dynamics at $t = 1$ will be different from those at $t = T - 1$, since in the latter case, bankers are getting closer to the repayment stage and will display a different behavior. In order to solve for the path of model's decision rules, I follow a backward induction procedure. From period $t = T + 1$ onward, the decision rules are those in the absence of policy. Thus, at $t = T$, agents use those decision rules to form expectations. By solving the equilibrium in this scenario, we can obtain decision rules for $\tilde{C}_T(\mathbf{S}), R_T(\mathbf{S}), \alpha_T(\mathbf{S}), Q_{B,T}(\mathbf{S})$. At $t = T - 1$ we proceed in the same way, this time using $\tilde{C}_T(\mathbf{S}), R_T(\mathbf{S}), \alpha_T(\mathbf{S}), Q_{B,T}(\mathbf{S})$ to form expectations. More specifically, the policy functions in the transition $\{\tilde{C}_t(\mathbf{S}), R_t(\mathbf{S}), \alpha_t(\mathbf{S}), Q_{B,t}(\mathbf{S})\}_{t=1}^{T}$, are derived as follows:

1. **Period** T: Solve the model using $\{\tilde{C}(\mathbf{S}), R(\mathbf{S}), \alpha(\mathbf{S}), Q_B(\mathbf{S})\}$ to form expectations. The Lagrange multiplier is modified as follows:

$$\mu_T(\mathbf{S}) = \max\left\{1 - \frac{\mathbb{E}_{\mathbf{S}}\{\hat{\Lambda}_{T+1}(\mathbf{S}', \mathbf{S})\} R_T(\mathbf{S})(N_T(\mathbf{S}) - m)}{\lambda\left[Q_{B,T}(\mathbf{S})B'_T(\mathbf{S}) + Q_{K,T}(\mathbf{S})K'_T(\mathbf{S})\right]}, 0\right\}$$

   Denote the solution by $\{\tilde{C}_T(\mathbf{S}), R_T(\mathbf{S}), \alpha_T(\mathbf{S}), Q_{B,T}(\mathbf{S})\}$.

2. **Period** $t = T - 1, \ldots, 1$: Solve the model using $\{\tilde{C}_{t+1}(\mathbf{S}), R_{t+1}(\mathbf{S}), \alpha_{t+1}(\mathbf{S}), Q_{B,t+1}(\mathbf{S})\}$ to form expectations. The Lagrange multiplier is modified as follows:

$$\mu_t(\mathbf{S}) = \max\left\{\frac{\lambda\left[TA_t(\mathbf{S}) - m\mathbf{1}_{t=1}\right] - \mathbb{E}_{\mathbf{S}}\{\hat{\Lambda}_{t+1}(\mathbf{S}', \mathbf{S})\} R_t(\mathbf{S})(N_t(\mathbf{S}) + m\mathbf{1}_{t=1}) - \psi\mathbb{E}_{\mathbf{S}}[\Lambda_{t+1}(\mathbf{S}')\alpha_{t+1}(\mathbf{S}')x_{t+1}(\mathbf{S}')]}{\lambda\left[Q_{B,t}(\mathbf{S})B'_t(\mathbf{S}) + Q_{K,t}(\mathbf{S})K'_t(\mathbf{S}) - m\mathbf{1}_{t=1}\right]}, 0\right\},$$

   where x_t follows the recursion $x_t(\mathbf{S}) = \frac{\lambda m\mu_t(\mathbf{S}) + \psi\mathbb{E}_S[\Lambda_{t+1}(\mathbf{S}')\alpha_{t+1}(\mathbf{S}')x_{t+1}(\mathbf{S}')]}{1 - \mu_t(\mathbf{S})}$ and $TA_t(\mathbf{S})$ is the market value of total assets. The initial condition of this recursion is $x_T(\mathbf{S}) = -\alpha_T(\mathbf{S})m$. Store the solution. □

In order to simulate the effect of the policy for a given initial condition $\overline{\mathbf{S}}$, I use the following algorithm

Simulating LTROs: Let $\{\gamma_t^{\text{ltro}})\}_{t=1}^{T}$ be the model solution under LTROs, and let γ^* be the solution in absence of the policy.

1. Draw N simulations for the structural shocks, $\{\varepsilon_t\}_{t=1}^{T}(n)$.
2. For each $\{\varepsilon_t\}_{t=1}^{T}(n)$, compute the path for outcome variable x using the model solution $\{\gamma_t^{\text{ltro}})\}_{t=1}^{T}$ and initializing the simulations at $\overline{\mathbf{S}}$. Denote this sequence by $\{\mathbf{x}^{\text{ltro}}\}_{t=1}^{T}(n)$. Repeat the same procedure for the model without LTROs, γ^*. Denote this sequence by $\{\mathbf{x}^{\text{no ltro}}\}_{t=1}^{T}(n)$.

3. For each n, compute $\{\mathbf{x}_t^{\text{ltro}}(n) - \mathbf{x}_t^{\text{no ltro}}(n)\}_{t=1}^{T}$, and average over n. □

D Sensitivity to "Appendix C: Evidence from the Cross Section of Italian Stock Returns"

This section reports the results for three alternative specifications of the asset pricing model estimated in Appendix C of the paper. Specifically, I consider three variations: i) the cross-sectional regression is estimated using only the first set of portfolios, ii) the cross-sectional regression is estimated using only the second set of portfolios, and iii) two pass estimates are computed using the full sample 1999:Q1-2011:Q3. The results are reported, respectively, in Table A-1, Table A-2 and Table A-3.

We can verify that the results in the benchmark specification are robust to the set of portfolios we use in the cross-sectional regression. When including the crisis period in the sample, though, the fit deteriorates and the intercept is estimated to be negative. This reflects the extremely negative realized returns observed over the 2008-2011 period.

Table A-1: **Cross-sectional regression, 15 portfolios by industry and size**

	Benchmark	CAPM	Three-factor FF	No Leverage
Intercept	0.51	22.23	25.86	-0.49
	(0.33)	(1.50)	(2.38)	(0.41)
$\beta^{\hat{\Lambda}}$	16.24			
	(3.91)			
β^{MKT}		-20.70	-24.40	
		(5.82)	(6.04)	
β^{SMB}			3.02	
			(3.66)	
β^{HML}			3.61	
			(3.92)	
$\beta^{\hat{\Lambda},\psi=0}$				0.08
				(0.26)
R^2	0.53	0.39	0.40	0.00
MAPE	4.81	5.16	5.23	6.27
$T^2(\chi^2)$	9.32	10.21	9.15	13.96
p-value	0.81	0.75	0.69	0.45

Notes: The dependent variable is the sample mean of annualized excess returns for the 15 portfolios constructed by sorting stocks by industry and size. See the note to Table A-3 in the paper for details on the statistics reported.

Table A-2: **Cross-sectional regression, 10 portfolios by betas**

	Benchmark	CAPM	Three-factor FF	No Leverage
Intercept	0.29	5.21	2.66	4.21
	(0.33)	(0.59)	(0.47)	(0.25)
$\beta^{\hat{\Lambda}}$	12.66			
	(3.93)			
β^{MKT}		-2.56	-1.34	
		(5.68)	(5.67)	
β^{SMB}			5.88	
			(3.61)	
β^{HML}			10.64	
			(3.85)	
$\beta^{\hat{\Lambda},\psi=0}$				0.10
				(0.25)
R^2	0.42	0.07	0.65	0.02
MAPE	4.29	3.28	3.85	3.69
$T^2(\chi^2)$	6.90	8.95	6.80	8.96
p-value	0.65	0.44	0.45	0.44

Notes: The dependent variable is the sample mean of annualized excess returns for the 10 portfolios constructed by sorting stocks by their estimated betas. See the note to Table A-3 in the paper for details on the statistics reported.

Table A-3: **Cross-sectional regression, 1999:Q1-2011:Q3 sample**

	Benchmark	CAPM	Three-factor FF	No Leverage
Intercept	-9.28	-0.85	-2.51	-6.60
	(0.26)	(0.66)	(0.66)	(0.31)
$\beta^{\hat{\Lambda}}$	12.33			
	(3.37)			
β^{MKT}		-6.43	-4.67	
		(6.06)	(6.09)	
β^{SMB}			0.31	
			(3.06)	
β^{HML}			5.14	
			(3.15)	
$\beta^{\hat{\Lambda},\psi=0}$				-0.05
				(0.25)
R^2	0.27	0.07	0.20	0.00
MAPE	5.23	3.94	4.25	4.37
$T^2(\chi^2)$	18.86	23.48	22.57	23.98
p-value	0.76	0.49	0.43	0.46

Notes: The dependent variable is the sample mean of annualized excess returns for the 25 portfolios. The sample period is 1999:Q1-2011:Q4. See the note to Table A-3 in the paper for details on the statistics reported.

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